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Pollaczek polynomials and hypergeometric representation

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Abstract This paper gives a solution, without the use of the three-term recurrence relation, of the problem posed in Ismail (Classical and Quantum Orthogonal Polynomials in One Variable, Cambridge University Press, Cambridge, 2005) (Problem 24.8.2, p. 658): that the hypergeometric representation of the general Pollaczek polynomials is a polynomial in $\cos(\theta)$ of degree n. Chu solved in (Ramanujan J. 13(1–3): 221–225, 2007) the problem in a particular case. We use elementary properties of functions of complex variables and Pfaff's transformation on hypergeometric ${}_2F_1$-series.

Keywords Orthogonal polynomials · Pollaczek polynomials · Hypergeometric functions · Pfaff–Euler transformation

Mathematics Subject Classification 33C45 · 33C47 · 33C50 · 42C05

1 Introduction

The general Pollaczek polynomials $P_n^\lambda(x,a,b)$, defined for $\lambda + a$ not a non-negative integer, satisfy the three-term recurrence relation (Szegö [6], and Chihara [2]),

$$(n+1)P_{n+1}^\lambda(x,a,b) = 2\big[(n+\lambda+a)x+b\big]P_n^\lambda(x,a,b) \\ - (n+2\lambda-1)P_{n-1}^\lambda(x,a,b), \quad n \geq 1,$$

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with the initial conditions

$$P_0^\lambda(x,a,b)=1,\qquad P_1^\lambda(x,a,b)=2(\lambda+a)x+2b.$$

Pollaczek [5] introduced these polynomials when $\lambda=\frac{1}{2}$ and Szegö [6] generalized them by introducing the parameter λ. The monic polynomials associated with the sequence $(P_n^\lambda(x,a,b))_n$ is the sequence $(Q_n^\lambda(x,a,b))_n$ defined by

$$\big(Q_n^\lambda(x,a,b)\big)_{n\geq 0}=\frac{n!}{2^n(\lambda+a)_n}\big(P_n^\lambda(x,a,b)\big)_{n\geq 0}.$$

The generating function associated with the sequence $(P_n^\lambda(x,a,b))_n$ (see [4]) is given by

$$\sum_{n=0}^{\infty}P_n^\lambda(x,a,b)t^n=\big(1-te^{i\theta}\big)^{-\lambda+i\Phi(\theta)}\big(1-te^{-i\theta}\big)^{-\lambda-i\Phi(\theta)},$$

where $\Phi(\theta)=\frac{a\cos\theta+b}{\sin\theta}$.

A classical computation leads to

$$\begin{aligned}\sum_{n=0}^{\infty}P_n^\lambda(x,a,b)t^n&=\big(1-te^{i\theta}\big)^{-\lambda+i\Phi(\theta)}\big(1-te^{-i\theta}\big)^{-\lambda-i\Phi(\theta)}\\&=\left(\sum_{n=0}^{\infty}\frac{(\lambda-i\Phi(\theta))_n}{n!}t^nz^n\right)\left(\sum_{n=0}^{\infty}\frac{(\lambda+i\Phi(\theta))_n}{n!}t^ne^{-in\theta}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(\lambda+i\Phi(\theta))_k(\lambda-i\Phi(\theta))_{n-k}}{k!(n-k)!}e^{-2ik\theta}\right)e^{in\theta}t^n.\end{aligned}$$

Since $(\lambda+x)_n=(\lambda+x)_{n-k}(\lambda+x+n-k)_k$ and $(\lambda+x+n-k)_k=(-1)^k(-\lambda-x-n+1)_k$, we have

$$\begin{aligned}P_n^\lambda(x,a,b)&=\sum_{k=0}^{n}\frac{(\lambda+i\Phi(\theta))_k(\lambda-i\Phi(\theta))_{n-k}}{k!(n-k)!}e^{-i(n-2k)\theta}\\&=e^{in\theta}\frac{(\lambda-i\Phi(\theta))_n}{n!}{}_2F_1\left(\begin{matrix}-n,\lambda+i\Phi(\theta)\\-\lambda+i\Phi(\theta)-n+1\end{matrix};e^{-2i\theta}\right).\end{aligned}\tag{1}$$

In [4], Problem 24.8.2, Mourad Ismail posed the problem to prove that the right-hand side of (1) is a polynomial in $\cos(\theta)$ of degree n without the use of the three-term recurrence relation of the Pollaczek polynomials. In 2007, Chu [3] solved the problem when $b=0$.

In this paper, we solve the problem in the general case and we prove the following result.

Theorem 1 *Let* $\Phi(\theta)=\frac{a\cos\theta+b}{\sin\theta}$ *and*

$$F(\theta)=e^{in\theta}\frac{(\lambda-i\Phi(\theta))_n}{n!}{}_2F_1\left(\begin{matrix}-n,\lambda+i\Phi(\theta)\\ -\lambda+i\Phi(\theta)-n+1\end{matrix};e^{-2i\theta}\right).$$

Then, F *is a polynomial of degree* n *of* $\cos(\theta)$ *and* $F(\theta)=P_n^{\lambda}(\cos(\theta),a,b)$.

2 Proof of the main theorem

Define the function A on $\mathbb{C}\setminus\{-1,1\}$ by $A(z)=\frac{az^2+2bz+a}{z^2-1}$. Thus $A(z)=-A(\frac{1}{z})$ and $-i\Phi(\theta)=A(e^{i\theta})$.

Let

$$\begin{aligned}f(z)&=z^n\frac{(\lambda+A)_n}{n!}{}_2F_1\left(\begin{matrix}-n,\lambda-A\\ -\lambda-A-n+1\end{matrix};z^{-2}\right)\\ &=\sum_{k=0}^{n}\frac{(\lambda-A)_k(\lambda+A)_{n-k}}{k!(n-k)!}z^{n-2k}.\end{aligned}\tag{2}$$

The function f is holomorphic in $\mathbb{C}\setminus\{0,1,-1\}$. The singularities $\{0,1,-1\}$ are either removable or poles of order at most n. We intend firstly to prove that -1 and 1 are removable.

We recall Pfaff's transformation formula for the hypergeometric function ${}_2F_1$ (see [1]), which is a consequence of the well-known Gauss formula.

Proposition 1 (Pfaff's transformation formula) *Let* $b,c,z\in\mathbb{C}$ *such that* $z\neq 1$, c *is not a negative integer. Then*

$$(1-z)^n{}_2F_1\left(\begin{matrix}-n,b\\ c\end{matrix};\frac{-z}{1-z}\right)={}_2F_1\left(\begin{matrix}-n,c-b\\ c\end{matrix};z\right).$$

It follows that

$$\begin{aligned}f(z)&=z^n\frac{(\lambda+A)_n}{n!}{}_2F_1\left(\begin{matrix}-n,\lambda-A\\ -\lambda-A-n+1\end{matrix};z^{-2}\right)\\ &=z^n\frac{(\lambda+A)_n}{n!}\frac{(z^2-1)^n}{z^{2n}}{}_2F_1\left(\begin{matrix}-n,-2\lambda-n+1\\ -\lambda-A-n+1\end{matrix};\tfrac{1}{1-z^2}\right)\\ &=\frac{(\lambda+A)_n(z^2-1)^n}{z^n}\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}\frac{(-2\lambda-n+1)_k}{(-\lambda-A-n+1)_k(1-z^2)^k}.\end{aligned}$$

It follows that 1 and -1 are removable singularities for the function f. Then, the only singularity of f is 0.

From (2), $f(z) = f(\frac{1}{z})$. Since 0 is a pole of order at most n, it follows that the Laurent series of f takes the form

$$f(z) = \sum_{k=-n}^{n} a_k z^k, \quad \text{where } a_k = a_{-k} \text{ for } k = 1, 2, \dots, n.$$

By the residue theorem

$$\begin{aligned} a_n &= \frac{1}{2i\pi} \int_{|z|=R} \frac{f(z)}{z^{n+1}} dz \\ &= \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \frac{1}{2\pi} \int_0^{2\pi} \big(\lambda - A\big(\mathrm{Re}^{i\theta}\big)\big)_k \big(\lambda + A\big(\mathrm{Re}^{i\theta}\big)\big)_{n-k} R^{-2k} e^{-2ik\theta} \, d\theta. \end{aligned}$$

But this integral is independent of R, therefore, letting $R \to \infty$, we have

$$a_n = \frac{(\lambda + a)_n}{n!}.$$

Therefore $f(e^{i\theta})$ is a polynomial of $\cos(\theta)$ of degree n. This finishes the proof.

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