Exercises. In the following exercise answer true or false

1. The point $x_0 = -1$ is a regular singular point for the differential equation

$$(1-x^2)y'' - 2xy' + 12y = 0.$$

2. The point $x_0 = 0$ is an ordinary point for the differential equation

$$xy'' + (1-x)y' + 2y = 0.$$

3. The point $x_0 = 0$ is a singular point for the differential equation

$$(1+x)y'' - 2y' + 2xy = 0.$$

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1) $\frac{a_1(x)}{a_2(x)} = \frac{-2x \to \text{Polynomial}}{1-x^2 \to \text{polynomial}}$, $1-x^2=0$

$\Rightarrow x=1,\ x=-1$

$\Rightarrow a_1(x)/a_2(x)$ not analytic at $x=-1$

$\Rightarrow x=-1$ is a singular point

Now : $(x-a_0) \cdot \frac{a_1(x)}{a_2(x)} = (x+1) \cdot \frac{-2x}{(1-x)(1+x)} = \frac{-2x}{1-x}$

is analytic at $x=-1$

and $(x-a_0)^2 \frac{a_1(x)}{a_2(x)} = (x+1)^2 \cdot \frac{-2x}{(1-x)(1+x)} = \frac{-2x(x+1)}{1-x}$

is analytic at $x=-1$

So $x=-1$ is a regular sigular point

The statement is true.

2) $\frac{a_1(x)}{a_2(x)} = \frac{1-x \to \text{Polynomial}}{x \to \text{Polynomial}}$, $x=0$

$\Rightarrow a_1(x)/a_2(x)$ is not analytic at $x=0$

$\Rightarrow x=0$ is a singular point

$\Rightarrow x=0$ is not an ordinary point

$\Rightarrow$ The statement is Flase.

Exercises

In exercises 1 through 9, locate the ordinary points, regular singular points and irregular singular points of the given differential equation

1) $xy'' - (2x+1)y' + y = 0.$

2) $(1-x)y'' - y' + xy = 0.$

3) $x^3(1-x^2)y'' + (2x-3)y' + xy = 0.$

4) $(1-x)^4 y'' - xy = 0.$

5) $2x^2y'' + (x - x^2)y' - y = 0.$

6) $x^2(x^2-9)y'' - (x^2-9)y' + xy = 0.$

7) $x^4 - 16)y'' + 2y = 0.$

8) $x(x^2+1)^3y'' + y' - 8xy = 0.$

9) $x^3 - 8)^3y'' - 2xy' + y = 0.$

In exercises 10 through 13 verify that all singular points of the differential equation are regular singular points

10) $x^2y'' + xy' + (x^2 - \nu^2)y = 0.$ (Bessel equation)

11) $(1-x^2)y'' - xy' + \nu^2 y = 0.$ (Chebyshev equation)

12) $xy'' + (1-x)y' + \nu y = 0.$ (Laguerre equation)

13) $(1-x^2)y'' - 2xy' + n(n+1)y = 0.$ (Legendre equation)

For the following equations, specify an interval around $x_0 = 0$ for which a power series solution converges

14) $y'' - xy' + 6y = 0.$

15) $(x^2-4)y'' - 2xy' + 9y = 0.$

In exercises 16 through 22 solve the initial value problems by using the method of power series about the given initial point x_0

16) $\begin{cases} (1-x^2)y'' - 2xy' + 6y = 0 \\ y(0) = 1, \; y'(0) = 0, \end{cases}$

17) $\begin{cases} y'' - 2(x+2)y' + 4y = 0 \\ y(-2) = 1, \; y'(-2) = 0, \end{cases}$

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3) $\frac{a_1(x)}{a_2(x)} = \frac{2x-3}{x^3(1-x^2)}$ $2x-3 \to$ Polynomial, $x^3(1-x^2) \to$ Polynomial

$x^3(1-x^2)=0$
$\Rightarrow x=0,\ x=1,\ x=-1$

$a_1(x)/a_2(x)$ is not analytic at $x=0,\ x=1,\ x=-1$

$x=0,\ x=1,\ x=-1$ are singular points of D.E

The ordinary points of D.E are $x \in \mathbb{R} - \{0,-1,1\}$

At $a_0=0$: $(x-a_0)\cdot\frac{a_1(x)}{a_2(x)} = x\cdot\frac{2x-3}{x^3(1-x^2)} = \frac{2x-3}{x^2(1-x^2)}$ not analytic at $x=0$

$(x-a_0)^2\cdot\frac{a_1(x)}{a_2(x)} = x^2\cdot\frac{2x-3}{x^3(1-x^2)} = \frac{2x-3}{x(1-x^2)}$ not analytic at $x=0$

So $x=0$ is irregular singular point

At $a_0=1$: $(x-a_0)\frac{a_1(x)}{a_2(x)} = (x-1)\cdot\frac{2x-3}{x^3(1-x)(1+x)} = \frac{3-2x}{x^3(1+x)}$ analytic at $x=1$

$(x-a_0)^2\frac{a_1(x)}{a_2(x)} = (x-1)^2\cdot\frac{2x-3}{x^3(1-x)(1+x)} = \frac{(3-2x)(x-1)}{x^3(1+x)}$ analytic at $x=1$

So $x=1$ is regular singular point.

At $a_0=-1$: $(x-a_0)\frac{a_1(x)}{a_2(x)} = (x+1)\cdot\frac{2x-3}{x^3(1-x)(1+x)} = \frac{2x-3}{x^3(1-x)}$ analytic at $x=-1$

$(x-a_0)^2\cdot\frac{a_1(x)}{a_2(x)} = (x+1)^2\cdot\frac{2x-3}{x^3(1-x)(1+x)} = \frac{(2x-3)(x+1)}{x^3(1-x)}$ analytic at $x=-1$

So $x=-1$ is regular singular point.

In exercises 16 through 22 solve the initial value problems by using the method of power series about the given initial point x_0

16) $\begin{cases} (1-x^2)y'' - 2xy' + 6y = 0 \\ y(0) = 1, \ y'(0) = 0, \end{cases}$.

(b) for $y=\sum_{n=0}^{\infty} a_n x^n$, $x_0=0$ is an ordinary point

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \Rightarrow y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

in D.E $\Rightarrow (1-x^2)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + 6\sum_{n=0}^{\infty} a_n x^n = 0$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

$\downarrow$ $n-2=k$ $\quad\downarrow$ $n=k$ $\quad\downarrow$ $n=k$ $\quad\downarrow$ $n=k$

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=2}^{\infty} k(k-1) a_k x^k - \sum_{k=1}^{\infty} 2k a_k x^k + \sum_{k=0}^{\infty} 6 a_k x^k = 0$$

$$2a_2 + 6a_3 x + \sum_{k=2}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=2}^{\infty} k(k-1) a_k x^k - 2a_1 x - \sum_{k=2}^{\infty} 2k a_k x^k$$
$$+ 6a_0 + 6a_1 x + \sum_{k=2}^{\infty} 6 a_k x^k = 0$$

$$(2a_2 + 6a_0) + (6a_3 + 4a_1)x + \sum_{k=2}^{\infty} \left[(k+2)(k+1) a_{k+2} - k(k-1) a_k + 6a_k - 2k a_k\right] x^k = 0$$

$2a_2 + 6a_0 = 0 \Rightarrow a_2 = -3a_0$, $6a_3 + 4a_1 = 0 \Rightarrow a_3 = -\frac{2}{3} a_1$

$(k+2)(k+1) a_{k+2} + (-k^2 - k + 6) a_k = 0 \Rightarrow a_{k+2} = \frac{k^2+k-6}{(k+2)(k+1)} a_k$, $k \geq 2$

For $k=2$: $a_4 = 0$

For $k=3$: $a_5 = \frac{3}{10} a_3 = -\frac{1}{5} a_1$

For $k=4$: $a_6 = \frac{7}{15} a_4 = 0$

For $k=5$: $a_7 = \frac{4}{7} a_5 = -\frac{4}{35} a_1$

$\vdots$ $a_n = 0$ $\forall$ n even, $n \geq 4$

$$y = a_0 + a_1 x - 3a_0 x^2 - \frac{2}{3} a_1 x^3 - \frac{1}{5} a_1 x^5 - \frac{4}{35} a_1 x^7 + \cdots$$
$$= a_0(1 - 3x^2) + a_1\left[x - \frac{2}{3}x^3 - \frac{1}{5}x^5 - \frac{4}{35}x^7 \ldots\right]$$

$y(0) = 1 \Rightarrow 1 = a_0$

$y' = a_0(-6x) + a_1\left(1 - 2x - x^4 - \frac{4}{5}x^6 - \ldots\right]$

$y'(0) = 0 \Rightarrow 0 = a_1$

Particular solution is $y = 1 - 3x^2$

20) $\begin{cases} y'' - 2(x-1)y' + 2y = 0 \\ y(1) = 0,\ y'(1) = 1, \end{cases}$

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20) $x_0 = 1$ is an ordinary point of the D.E:

Put the solution $y = \sum_{n=0}^{\infty} a_n (x-1)^n$

$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \rightarrow y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$

in D.E $\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - \sum_{n=1}^{\infty} 2 n a_n (x-1)^n + \sum_{n=0}^{\infty} 2 a_n (x-1)^n = 0$

$n-2=K \qquad K=n \qquad K=n$

$\sum_{K=0}^{\infty} (K+2)(K+1) a_{K+2} (x-1)^K - \sum_{K=1}^{\infty} 2K a_K (x-1)^K + \sum_{K=0}^{\infty} 2 a_K (x-1)^K = 0$

$2a_0 + \sum_{K=1}^{\infty} (K+2)(K+1) a_{K+2} (x-1)^K - \sum_{K=1}^{\infty} 2K a_K (x-1)^K + 2a_0 + \sum_{K=1}^{\infty} 2 a_K (x-1)^K = 0$

$4a_0 + \sum_{K=1}^{\infty} \left[(K+2)(K+1) a_{K+2} - 2K a_K + 2a_K\right](x-1)^K = 0$

$4a_0 = 0 \rightarrow a_0 = 0$

$(K+2)(K+1) a_{K+2} - 2K a_K + 2a_K = 0 \Rightarrow a_{K+2} = \dfrac{(2K-2)}{(K+2)(K+1)} a_K, \; K \geq 1$

For $K=1$: $a_3 = 0$

For $K=2$: $a_4 = \frac{1}{6} a_2$

For $K=3$: $a_5 = \frac{1}{5} a_3 = 0$

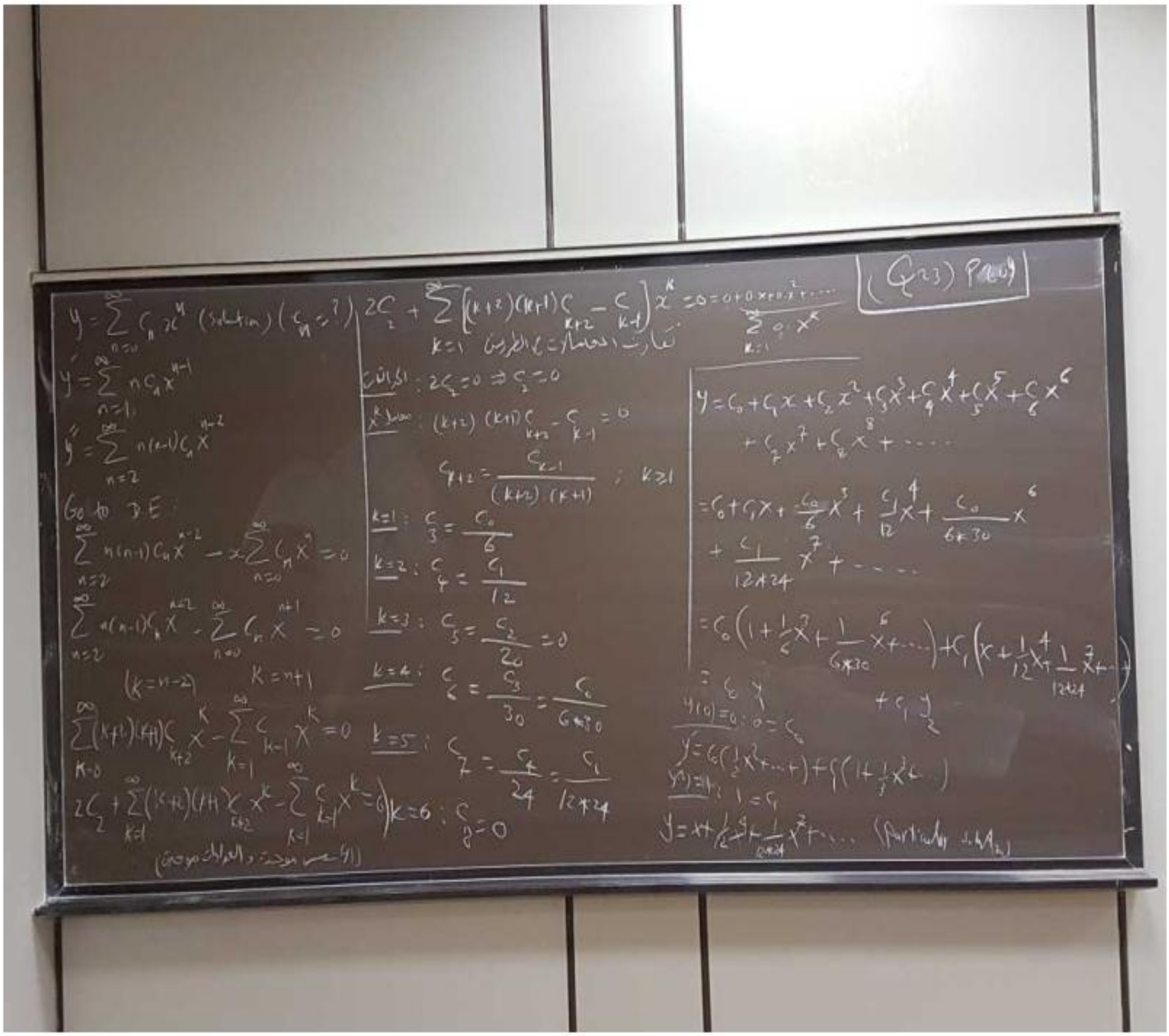
For $K=4$: $a_6 = \frac{1}{5} a_4 = \frac{1}{5 \cdot 6} a_2 = \frac{1}{30} a_2$

$a_n = 0 \;\; \forall\, n$ odd > 1

For $K=6$: $a_8 = \frac{1}{168} a_2$

G.S is $y = 0 + a_1(x-1) + a_2\left[(x-1)^2 + \frac{1}{6}(x-1)^4 + \frac{1}{30}(x-1)^6 + \frac{1}{168}(x-1)^8 + \dots\right]$

$$23)\quad \begin{cases} y'' - xy = 0 \\ y(0) = 0,\ y'(0) = 1, \end{cases}$$



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27) $\begin{cases} x^2y'' + xy' + 2y = 0 \\ y(1) = 1, \ y'(1) = 0. \end{cases}$

$$y=\sum_{n=0}^{\infty}a_n(x-1)^n \to y'=\sum_{n=1}^{\infty}na_n(x-1)^{n-1} \to y''=\sum_{n=2}^{\infty}n(n-1)a_n(x-1)^{n-2}$$

Go to D.E:

$$x^2\sum_{n=2}^{\infty}n(n-1)a_n(x-1)^{n-2}+x\sum_{n=1}^{\infty}na_n(x-1)^{n-1}+2\sum_{n=0}^{\infty}a_n(x-1)^n=0$$

$$(x-1+1)^2\cdot\sum_{n=2}^{\infty}n(n-1)a_n(x-1)^{n-2}+(x-1+1)\sum_{n=1}^{\infty}na_n(x-1)^{n-1}+\sum_{n=0}^{\infty}2a_n(x-1)^n=0$$

$$\left[(x-1)^2+2(x-1)+1\right]\cdot\sum_{n=2}^{\infty}n(n-1)a_n(x-1)^{n-2}+\sum_{n=1}^{\infty}na_n(x-1)^n+\sum_{n=1}^{\infty}na_n(x-1)^{n-1}$$
$$+\sum_{n=0}^{\infty}2a_n(x-1)^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n(x-1)^n \ (n=K)+\sum_{n=2}^{\infty}2n(n-1)a_n(x-1)^{n-1} \ (K=n-1)+\sum_{n=2}^{\infty}n(n-1)a_n(x-1)^{n-2} \ (K=n-2)$$
$$+\sum_{n=1}^{\infty}na_n(x-1)^n \ (K=n)+\sum_{n=1}^{\infty}na_n(x-1)^{n-1} \ (K=n-1)+\sum_{n=0}^{\infty}2a_n(x-1)^n=0 \ (K=n)$$

$$\sum_{K=2}^{\infty}K(K-1)a_K(x-1)^K+\sum_{K=1}^{\infty}2(K+1)Ka_{K+1}(x-1)^K+\sum_{K=0}^{\infty}(K+2)(K+1)a_{K+2}(x-1)^K$$
$$+\sum_{K=1}^{\infty}Ka_K(x-1)^K+\sum_{K=0}^{\infty}(K+1)a_{K+1}(x-1)^K+\sum_{K=0}^{\infty}2a_K(x-1)^K=0$$

$$\sum_{K=2}^{\infty}K(K-1)a_K(x-1)^K+4a_2(x-1)+\sum_{K=2}^{\infty}2(K+1)Ka_{K+1}(x-1)^K+2a_2+6a_3(x-1)$$
$$+\sum_{K=2}^{\infty}(K+2)(K+1)a_{K+2}(x-1)^K+a_1(x-1)+\sum_{K=2}^{\infty}Ka_K(x-1)^K+a_1+2a_2(x-1)$$
$$+\sum_{K=2}^{\infty}(K+1)a_{K+1}(x-1)^K+2a_0+2a_1(x-1)+\sum_{K=2}^{\infty}2a_K(x-1)^K=0$$

$$(2a_0+a_1+2a_2)+(3a_1+6a_2+6a_3)(x-1)+$$
$$\sum_{K=2}^{\infty}\left[K(K-1)a_K+2(K+1)Ka_{K+1}+(K+2)(K+1)a_{K+2}+Ka_K+(K+1)a_{K+1}+2a_K\right](x-1)^K=0$$

$2a_0+a_1+2a_2=0$ and $3a_1+6a_2+6a_3=0 \Rightarrow \boxed{a_3=a_0}$

$\Rightarrow \boxed{a_2=-a_0-\frac{1}{2}a_1}$

and: $$a_{K+2}=\frac{-(K^2+2)a_K-(2K^2+3K+1)a_{K+1}}{(K+2)(K+1)}\ ;(K\geq 2)$$

(recurrence formula)

$\underline{K=2}$: $a_4 = -\frac{3}{4}a_0 + \frac{1}{4}a_1$

$\underline{K=3}$: $a_5 = \frac{1}{2}a_0 - \frac{7}{20}a_1$

Solution B: $y = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + a_5(x-1)^5 + \cdots$

$$y = a_0 + a_1(x-1) + (-a_0 - \tfrac{1}{2}a_1)(x-1)^2 + a_0(x-1)^3 + (-\tfrac{3}{4}a_0 + \tfrac{1}{4}a_1)(x-1)^4 + (\tfrac{1}{2}a_0 - \tfrac{7}{20}a_1)(x-1)^5 + \cdots$$

$$= a_0\left[1 - (x-1)^2 + (x-1)^3 - \tfrac{3}{4}(x-1)^4 + \tfrac{1}{2}(x-1)^5 + \cdots\right] + a_1\left[(x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{4}(x-1)^4 - \tfrac{7}{20}(x-1)^5 + \cdots\right]$$

$\underline{y(1)=1}$: $\to 1 = a_0$

$y' = a_0[-2(x-1) + \cdots] + a_1[1 - (x-1) + \cdots]$

$\underline{y'(1)=0}$: $\to 0 = a_1$

$\Rightarrow$ particular solution is:

$$y = 1 - (x-1)^2 + (x-1)^3 - \tfrac{3}{4}(x-1)^4 + \tfrac{1}{2}(x-1)^5 + \cdots$$