Example (1):

A box contains five red balls, a ball is drawn at random, what is the possibility that the ball will be red?

$n(R) = 5 \qquad n(\Omega) = 5$

$$P(R) = \frac{X}{T} = \frac{5}{5} = 1$$ (Certain event)

Example (2):

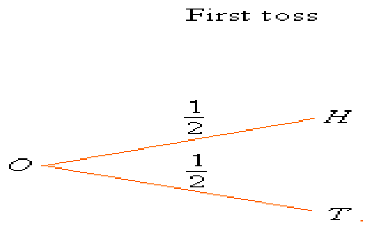
A box contain five red balls, a ball is drawn at random, what is the possibility that the ball will be blue?

$n(B) = 0 \qquad n(\Omega) = 5$

$$P(B) = \frac{X}{T} = \frac{0}{5} = 0$$ (Impossible event)

Example (3):

An experiment is consisting of tossing (flip) a fair coin once, what is the probability of getting a head?



Ω={ H,T}

1.

n(Ω)= 2 n(H)= 1

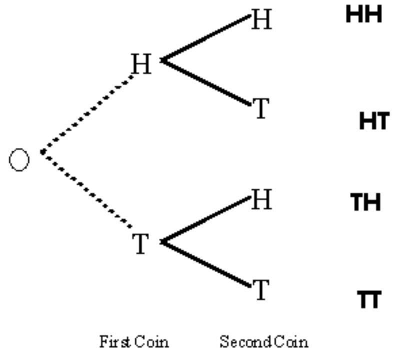
$$P(H) = \frac{X}{T} = \frac{1}{2} = 0.50$$

Example (4):

If an experiment is consisting of tossing a fair coin twice, find:

1. The Set of all possible outcomes of the experiment.
2. The probability of the event of getting at least one head.
3. The probability of the event of getting exactly one head in the two tosses.
4. The probability of the event of getting two heads.

Solution:



1.
Ω= {HH, HT, TH, TT}
Where,
$n(\Omega) = 2^2 = 4$
And since the coin is fair, then all of the elementary events are equally likely, i.e.
P(HH) = P(HT) = P(TH) = P(TT) = 0.25
2.
Let
E_1 = {HH, HT, TH} be the event of getting at least one head, then
$n(E_1) = 3$

And hence
$$P(E_1) = \frac{X}{T} = \frac{3}{4} = 0.75$$

3.
E_2 = {HT, TH} be the event of getting exactly one head, then
$n(E_2) = 2$ And hence

$$P(E_2) = \frac{X}{T} = \frac{2}{4} = 0.5$$

4.Let
E_3 = {HH} be the event of getting two heads, then $n(E_3) = 1$

$$P(E_3) = \frac{X}{T} = \frac{1}{4} = 0.25$$

Example (5):

For each of the following, indicate whether the type of probability involved is an example of a priori probability, empirical probability, or subjective probability.
a. According to weather forecasts comma there will be at least 18.6 inches of rainfall next year
b. A certain football team will win the trophy.
c. The next roll of a fair die will land on the six.
d. There is a 0.00001 probability that the ticket 6 3 8 9 5 will win a five-dash digit dash selection lottery.

Solution:

- **a.** empirical probability
- **b.** subjective probability
- **c.** priori probability
- **d.** priori probability

Example (6):

If the experiment is consisting of rolling a fair die once, find:

1. Set of all possible outcomes of the experiment.
2. The probability of the event of getting an even number.
3. The probability of the event of getting an odd number.
4. The probability of the event of getting a four or five.
5. The probability of the event of getting a number less than 5.

Solution:

1.
$\Omega=\{1,2,3,4,5,6\}$ $n(\Omega)=6$
Since the coin is fair, then all events are equally likely, i.e.

$$P(1)=P(2)=\cdots=P(6)=\frac{1}{6}$$

2. Let,
$E_1=\{2, 4, 6\}$ be the event of getting an even number, then

$$n(E_1)=3$$

$$P(E_1)=\frac{n(E_1)}{n(\Omega)}=\frac{3}{6}=0.50$$

3.
E_2= {1, 3, 5} be the event of getting an odd number, then

$n(E_2) = 3$

$$P(E_2) = \frac{n(E_2)}{n(\Omega)} = \frac{3}{6} = 0.50$$

4. Let,
E_3= {4, 5} be the event of getting a four or five, then

$n(E_3) = 2$

$$P(E_3) = \frac{n(E_3)}{n(\Omega)} = \frac{2}{6} = 0.33$$

5. Let,
E_4= {1, 2, 3, 4} be the event of getting a number less than 5, then

$$P(E_4) = \frac{n(E_4)}{n(\Omega)} = \frac{4}{6} = 0.67$$

Example (7): page (172)

Five coins are tossed.

a) Give an example of a simple event.
b) Give an example of a joint event.
c) What is the complement of a head on the first toss?
d) What does the sample space consist of?

Solution:

a) Give an example of a simple event.

Getting a tail on the first coin.
or
Getting a head on the first coin.

b) Give an example of a joint event.

Getting a head on the second coin and a tail on the first coin.
Or
Getting a head on the third coin and a head on the second coin.

c) What is the complement of a head on the first toss?
Getting a tail on the first coin.

d) What does the sample space consist of?

Getting a head or a tail on any of the five coins.

Example (8): page (172)

A box contains 12 red balls and 8 white balls. Two balls are to be selected from the box.

a) Give an example of a simple event.
b) What is the complement of a red ball?
c)What does the sample space consist of?

Solution:

a) Give an example of a simple event.
The first ball being red is a simple event.
b) What is the complement of the first ball being red?
The first ball being white.
c)What does the sample space consist of?
The sample space consists of the 12 red balls and the 8 white balls.

Example (9):

Use the contingency table to the right to determine the probability of events.

a. What is the probability of event A?
b. What is the probability of event A′?
c. What is the probability of event A and B?
d. What is the probability of event A' and B'?
e. What is the probability of event A or B?
f. What is the probability of event A' or B'?

	B	B'	Total
A	50	90	140
A'	80	10	90
Total	130	100	230

Solution:

a) What is the probability of event A?

$$p(A) = \frac{X}{T} = \frac{140}{230} = 0.609$$

b. What is the probability of event A′?

$$p(A') = \frac{X}{T} = \frac{90}{230} = 0.391$$

c. What is the probability of event A and B?

$$p(A \text{ and } B) = \frac{X}{T} = \frac{50}{230} = 0.217$$

d. What is the probability of event A' and B'?

$$p(A' \text{ and } B') = \frac{X}{T} = \frac{10}{230} = 0.043$$

e. What is the probability of event A or B?

$$p(A \text{ or } B) = p(A) + p(B) - p(A \text{ and } B)$$
$$= \frac{140}{230} + \frac{130}{230} - \frac{50}{230} = \frac{270}{230} - \frac{50}{230} = 1.174 - 0.217 = 0.957$$

f. What is the probability of event A' or B'?

$$p(A' \text{ or } B') = p(A') + p(B') - p(A' \text{ and } B')$$
$$= \frac{90}{230} + \frac{100}{230} - \frac{10}{230} = \frac{190}{230} - \frac{10}{230} = 0.826 - 0.043 = 0.783$$

Example (10):

When one card is drawn from a well-shuffled deck of 52 playing cards, what are the probabilities of getting?

Solution:

1. A black card**.**

$$P(Black\ card) = \frac{26}{52} = 0.5$$

2. Number 2.

$$P(Number\,2) = \frac{4}{52} = 0.08$$

3. A black king.

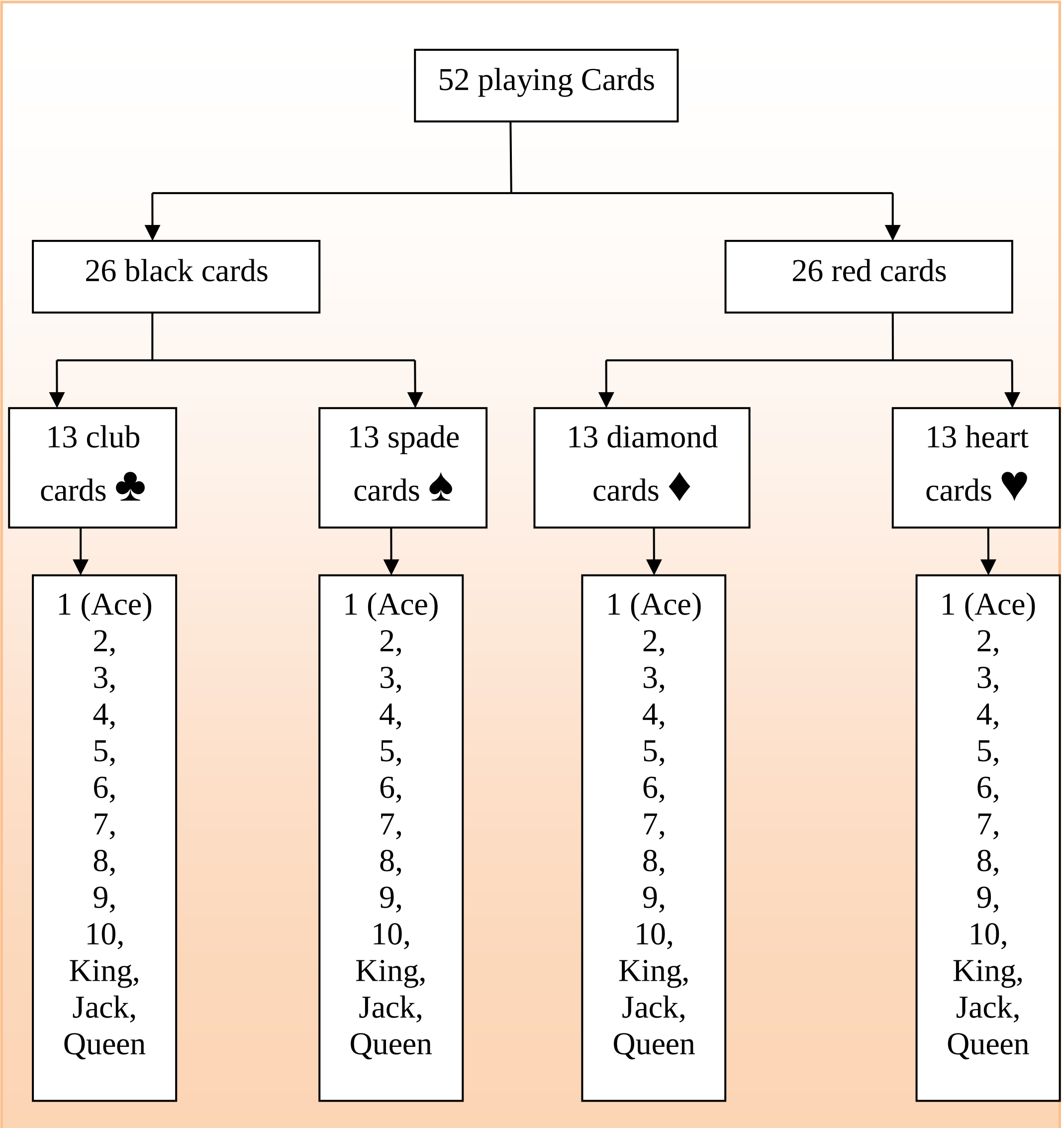
$$P(Black\,king) = \frac{2}{52} = 0.038$$

4. Number 3, 4, 5.

$$P(Number\,3,4,5) = \frac{12}{52} = 0.23$$

5. A heart card.

$$P(A\,heard\,card) = \frac{13}{52} = 0.25$$


52 playing Cards
26 black cards
26 red cards
13 club cards ♣
13 spade cards ♠
13 diamond cards ♦
13 heart cards ♥
1 (Ace) 2, 3, 4, 5, 6, 7, 8, 9, 10, King, Jack, Queen
1 (Ace) 2, 3, 4, 5, 6, 7, 8, 9, 10, King, Jack, Queen
1 (Ace) 2, 3, 4, 5, 6, 7, 8, 9, 10, King, Jack, Queen
1 (Ace) 2, 3, 4, 5, 6, 7, 8, 9, 10, King, Jack, Queen

Conditional probability:

$$p(B/A) = \frac{P(A\ \ and\ B)}{p(A)} = \frac{P(A \cap B)}{P(A)}$$

Multiplication:
General Multiplication
$p(A\ and\ B) = P(A)P(B/A)$
A and B not independent

Special Multiplication
$p(A\ and\ B) = P(A)P(B)$
A and B independent
or
$p(B/A) = P(B)$
A and B independent

Example (11)
A box contains eight red balls and five white balls, two balls are drawn at random, find:

1. The probability of getting both the balls white, when the first ball drawn is **replace**.
2. The probability of getting both the balls red, when the first ball drawn is **replace**
3. The probability of getting one of the balls red, when the first drawn ball is replaced back.

Solution:
Let W_1 be the event that the in the first draw is white and W_2. In a similar way define R_1 and R_2. Since the result of the first draw has no effects on the result of the second draw, it follows that W_1 and W_2 are independent and similarly R_1 and R_2 are independent.

1.

$$P(W_1 \cap W_2) = P(W_1)\,P(W_2) = \left(\frac{5}{13}\right)\left(\frac{5}{13}\right) = \frac{25}{169}$$

2.

$$P(R_1 \cap R_2) = P(R_1)\,P(R_2) = \left(\frac{8}{13}\right)\left(\frac{8}{13}\right) = \frac{64}{169}$$

3. Since the first drawn ball is replaced back, then the result of the first draw has no effect on the result of the second draw. Let E be the event that one of the ball is red, then:

$$P(E) = P(R_1)\,P(W_2) + P(W_1)\,P(R_2) = \left(\frac{8}{13}\right)\left(\frac{5}{13}\right) + \left(\frac{5}{13}\right)\left(\frac{8}{13}\right) = \frac{80}{169}$$

Example (12)
A box contains seven blue balls and five red balls, two balls are drawn at random without replacement, find:
1. The probability that both balls are blue.
2. The probability that both balls are red.
3. The probability that one of the balls is blue.
4. The probability that at least one of the balls is blue.
5. The probability that at most one of the balls is blue.

Solution:
Let B_1 denote the event that the ball in the first draw is blue and B_2 denote the event that the ball in the second draw is blue. In a similar way define R_1 and R_2.

1.

$$P(B_1 \ and \ B_2) = P(B_1 \cap B_2) = P(B_1)\, P(B_2|B_1)$$
$$= \left(\tfrac{7}{12}\right)\left(\tfrac{6}{11}\right) = \tfrac{42}{132} = 0.32$$

2.

$$P(R_1 \ and \ R_2) = P(R_1 \cap R_2) = P(R_1)\, P(R_2|R_1)$$
$$= \left(\tfrac{5}{12}\right)\left(\tfrac{4}{11}\right) = \tfrac{20}{132} = 0.15$$

3.

$$P(one\,ball\,is\,blue) =$$
$$P((B_1 and\, R_2)\, or\, (R_1 \ and\, B_2)) = P(B_1)\, P(R_2|B_1) + P(R_1) P(B_2|R_1)$$
$$= \left(\tfrac{7}{12}\right)\left(\tfrac{5}{11}\right) + \left(\tfrac{5}{12}\right)\left(\tfrac{7}{11}\right)$$
$$= \tfrac{35}{132} + \tfrac{35}{132} = \tfrac{70}{132} = 0.53$$

4. That at least one of the balls is blue

$$P(At\,least\,one\,ball\,is\,blue) = P((B_1 \ and\, B_2)\, or\, (B_1 and\, R_2)\, or\, (R_1 \ and\, B_2))$$
$$= P(B_1)\, P(B_2|B_1) + P(B_1)\, P(R_2|B_1) + P(R_1)\, P(B_2|R_1)$$
$$= \left(\tfrac{7}{12}\right)\left(\tfrac{6}{11}\right) + \left(\tfrac{7}{12}\right)\left(\tfrac{5}{11}\right) + \left(\tfrac{5}{12}\right)\left(\tfrac{7}{11}\right)$$
$$= \tfrac{42}{132} + \tfrac{35}{132} + \tfrac{35}{132} = \tfrac{112}{132} = 0.85$$

Another solution:

$$P(at\,least\,one\,blue\,is\,ball) = 1 - P(zero\,blue\,ball)$$
$$= 1 - P(R_1 and\, R_2)$$
$$= 1 - \left[\left(\tfrac{5}{12}\right)\left(\tfrac{4}{11}\right)\right]$$
$$= 1 - \tfrac{20}{132} = 1 - 0.15 = 0.85$$

5.

$$P(at\ most\ one\ ball\ is\ blue)$$

$$= ((B_1 and\ R_2)\ or\ (R_1 and\ B_2)\ or\ (R_1 and\ R_2))$$
$$= P(B_1)\,P(R_2|B_1) + P(R_1)P(B_2|R_1) + P(R_1)P(R_2|R_1)$$
$$= \left(\tfrac{7}{12}\right)\left(\tfrac{5}{11}\right) + \left(\tfrac{5}{12}\right)\left(\tfrac{7}{11}\right) + \left(\tfrac{5}{12}\right)\left(\tfrac{4}{11}\right)$$
$$= \tfrac{35}{132} + \tfrac{35}{132} + \tfrac{20}{132} = \tfrac{90}{132} = 0.68$$

Another solution:

$$P(at\ most\ one\ blue\ ball) = 1 - P(two\ \ blue\ balls)$$
$$= 1 - P(B_1 and\ B_2)$$
$$= 1 - \left[\left(\tfrac{7}{12}\right)\left(\tfrac{6}{11}\right)\right]$$
$$= 1 - \tfrac{42}{132} = 1 - 0.32 = 0.68$$

Example (13):

Use the contingency table to the right to determine the probability of events.

a. $p(A/B)$?
b. $p(A/B')$?
c. $p(A'/B')$?
d. Are event A and B independent?

	B	B'	Total
A	50	90	140
A'	80	10	90
Total	130	100	230

Solution:

a. $p(A/B) = \frac{P(A\ \ and\ B)}{p(B)} = \frac{P(A \cap B)}{P(B)} = \frac{50}{130} = 0.38$

b. $p(A/B') = \frac{P(A\ \ and\ B')}{p(B')} = \frac{P(A \cap B')}{P(B')} = \frac{90}{100} = 0.90$

c. $p(A'/B') = \frac{P(A'\ and\ B')}{p(B')} = \frac{P(A' \cap B')}{P(B')} = \frac{10}{100} = 0.10$

d. Are event A and B independent?

.$p(A/B) = \frac{P(A\ \ and\ B)}{p(B)} = \frac{P(A \cap B)}{P(B)} = \frac{50}{130} = 0.38$

$p(A) = \frac{140}{230} = 0.61$

$p(A/B) \neq p(A)$

The event A and B are not independent
or

$$p(A\ and\ B) = \frac{50}{230} = 0.217$$

$$P(A)P(B) = \frac{140}{230} \times \frac{130}{230} = \frac{18200}{52900} = 0.34$$

$$p(A\ and\ B) \neq P(A)P(B)$$

The event A and B are not independent

Example (14):

Each year, ratings are compiled concerning the performance of new cars during the first 90 days of use. Suppose that the cars have been categorized according to whether a car needs warranty-related repair (yes or no) and the country in which the company manufacturing a car is based (in some country X or not in country X). Based on the data collected, the probability that the new car needs a warranty repair is 0.08, the probability that the car is manufactured by a company based in country X is 0.70, and the probability that the new car needs a warranty repair and was manufactured by a company based in country X is 0.035.

Use this information to answer

(a) Suppose you know that a company based in country X manufactured a particular car. What is the probability that the car needs warranty?

(b) Suppose you know that a company based in country X did not manufactured a particular car. What is the probability that the car needs warranty?

(c) Are need for warranty repair and location of the company manufacturing the car independent?

warranty-related repair	The country in which the company manufacturing a car		
	country X	**Not in country X (X')**	**Total**
Yes	**0.035**	0.045	**0.08**
No	0.665	0.255	0.92
Total	**0.70**	0.30	**1**

Solution:

(a) Suppose you know that a company based in country X manufactured a particular car. What is the probability that the car needs warranty?

$$p(Yes/X) = \frac{P(Yes\ \ and\ X)}{p(X)} = \frac{P(Yes \cap X)}{P(X)} = \frac{0.035}{0.70} = 0.05$$

(b) Suppose you know that a company based in country X did not manufactured a particular car. What is the probability that the car needs warranty?

$$.p(Yes/X') = \frac{P(Yes\ \ and\ X')}{p(X')} = \frac{P(Yes \cap X')}{P(X')} = \frac{0.045}{0.30} = 0.15$$

(c)Are need for warranty repair and location of the company manufacturing the car independent?

$p(Yes/X) = 0.05$

$p(Yes) = 0.08$

$\therefore\ p(Yes/X) \neq p(Yes)$

Warranty repair and location of the company manufacturing the ca are not r independent?

$p(Yes/X') = 0.15$

$p(Yes) = 0.08$

$\therefore\ p(\text{Yes}/X') \neq p(Yes)$

Warranty repair and location of the company manufacturing the ca are not r independent?

Example (15):

If $p(Aand\ B) = 0.4\ and\ p(B) = 0.8 \quad ,\ find\ p(A/B)$

Solution:

$$p(A/B) = \frac{P(A\ \ and\ B)}{p(B)} = \frac{P(A \cap B)}{P(B)} = \frac{0.4}{0.8} = 0.50$$

Example (16):

If $p(A) = 0.7\ ,\ p(B) = 0.6\quad ,and\ A\ and\ B\ are\ independent\ ,$ $find\ p(A\ and\ B)$

Solution:

$\because\ A\ and\ B\ are\ independent$

$\therefore\ \ p(A\ and\ B) = p(A)p(B) = 0.7 \times 0.6 = 0.42$

Example (17):

If $p(A) = 0.3\ ,\ p(B) = 0.7\quad ,\ and\ p(A\ and\ B) = 0.21$

Are event A and B independent?

Solution:

$$p(B/A) = \frac{P(A\ \ and\ B)}{p(A)} = \frac{P(A \cap B)}{P(A)} = \frac{0.21}{0.3} = 0.7$$

$\because\ p(A\ and\ B) = 0.21$

$p(A\ )P(B) = 0.3 \times 0.7 = 0.21$

Or

$$p(B/A) = \frac{P(A\ \ and\ B)}{p(A)} = \frac{P(A \cap B)}{P(A)} = \frac{0.21}{0.3} = 0.7$$

$\because\ p(B/A) = P(B) = 0.7$

$\therefore\ \ A\ and\ B\ are\ independent$

Example (18):

Do Americans prefer Coke or Pepsi? A survey was conducted by Public Policy Polling in 2013; the results were as follows:

	Gender		
Preference	**Female**	**Male**	**Total**
Coke	**120**	**95**	**215**
Pepsi	**95**	**80**	**175**
Neither/Unsure	**65**	**45**	**110**
Total	**280**	**220**	**500**

a) Given that an American is a male, what is the probability that he prefers Pepsi?
b) Given that an American is a female, what is the probability that he prefers Pepsi?
c) Is preference independent of gender? Explain

Solution:

a) $p(\text{the probability that he prefers Pepsi}/\text{an American is a male}) = \frac{P(\text{Pepsi }\ and\ \text{male})}{p(\text{male})} = \frac{P(\text{Pepsi}\cap\text{male})}{P(\text{male})} = \frac{80}{220} = 0.36$

b) $p\ (\text{the probability that he prefers Pepsi}/\text{an American is a female}) = \frac{P(\text{Pepsi }\ and\ f\text{emale})}{p(\text{female})} = \frac{P(\text{Pepsi}\cap f\text{emale})}{P(\text{female})} = \frac{95}{280} = 0.34$

c.
$p(\text{Pepsi}) = 0.35$
$(\text{Pepsi/ male}) = 0.36$
$(\text{Pepsi/ male}) \neq P(Pepsi)$
No, the two are not independent events.

$p\ (\text{Pepsi}) = 0.35$

$p\ (\text{Pepsi/a female}) = 0.34$
$(\text{Pepsi/ female}) \neq P(Pepsi)$
No, the two are not independent events.

Counting Rules 1:
Counting rule 1 determines the number of possible outcomes for a set of mutually exclusive and collectively exhaustive events.
Counting Rules 1:
If any one of k different mutually exclusive and collectively exhaustive events can occur on each of n trials, the number of possible outcomes is equal to:
K^n

Example (19):
Suppose you roll a die twice. How many different possible outcomes can occur?

Solution:

$K^n = 6^2 = 36$

Example (20):
If there are 10 multiple-choice questions on an exam, each having three possible answers, how many different of answers are there?

Solution:

$$K^n = 3^{10} = 59049$$

Example (21):

a) If a coin is tossed seven times, how many different outcomes are possible?
b) If a die is tossed seven times, how many different outcomes are possible?
c) Discuss the differences in your answers to (a) and (b)

Solution:

a) $K^n = 2^7 = 128$
b) $K^n = 6^7 = 279936$
c) There are two mutually exclusive and collectively exhaustive outcomes in (a) and six in (b)

Counting Rules 2:
Counting rule 2 The second counting rule is more general version of the first counting rule and allows the number of possible events to differ from trail to trail.
Counting rule 2 If there are K_1 events on the first trail, K_2 event on the second trail ……., and K_n events on the nth trail ,then the number of possible outcomes is
$(K_1)(K_2)\dots\dots..(K_n)$

Example (22):

A restaurant menu has a price-fixed complete dinner that consists of an appetizer, an entrée, a beverage, and a dessert. You have a choice of 5 appetizers,10 entrees,3 beverages, and 6 desserts. Determine the total number of possible dinners?

Solution:

$(5)(10)(3)(6) = 900$

Example (23):

You would like to “build-your-own-burger” at a fast-food restaurant. There are 5 different breads,7different cheeses,4 different cold topping, and 5 different sauces on the menu. If you want to include one choice from each of these ingredient categories, how many different burgers can you build?

Solution:

$(5)(7)(4)(5) = 700$

Counting Rules 3:

The third counting rule involves computing the number of ways that a set of items can be arranged in order.

Counting Rules 3:

The number of ways that all n items can be arrange in order is

$n^! = (n)(n-1) \dots \dots$

Example (24):

If a set of six books is to be placed on a shelf, in how many ways can the six books be arranged?

Solution:

$6^! = (6)(5)(4)(3)(2)(1) = 720$

Example (25):

In Major League Baseball, there are 5 teams in the Eastern Division of the National League: Atlanta, Florida, New York, Philadelphia, and Washington. How many different orders of finish are there for these five teams? (Assume that there are no ties in the standings.) Do you believe that all these orders are equally likely? Discuss

Solution:

$5^! = (5)(4)(3)(2)(1) = 120$

Counting Rules 4:

In many instances you need to know the number of ways in which a subset of an entire group of items can be arranged in order. Each possible arrangement is called a permutation.

Counting Rules 4:

The number of ways of arranging x objects selected from n objects in order is

$$nP_x = \frac{n^!}{(n-x)^!}$$

Example (26):
If you have six books, but there is room for only four books on the shelf, in how many ways can you arrange these books on the shelf?

Solution:

$$6P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = \frac{720}{2} = 360$$

Example (27):
A gardener has ten rows available in his vegetable garden to place ten different vegetables. Each vegetable will be allowed one and only one row. How many ways are there to position these vegetables in his garden?

Solution:

$$10P_{10} = \frac{10!}{(10-10)!} = \frac{10!}{0!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{1} = \frac{3628800}{1}$$
$$= 3628800$$

There are 3628800 ways to position these vegetables in the garden

Counting Rules 5:
In many situations, you are not interested in the order of outcomes but only in the number of ways that x items can be selected from n items, irrespective of order. Each possible selection is called a combination.

Counting Rules 5:
The number of ways of selecting x objects, irrespective of order, is equal to

$$nC_x = \frac{n!}{x!(n-x)!}$$

Example (28):
If you have six books, but there is room for only four books on the shelf, if the order of the books on the shelf is irrelevant, in how many ways can you arrange these books on the shelf?

Solution:

$$6C_4 = \frac{6!}{4!(6-4)!} = \frac{6!}{4!2!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{(4 \times 3 \times 2 \times 1)(2 \times 1)} = \frac{720}{48} = 15$$

Example (29):
Four members of a group of 10 people are to be selected to a team. How many ways are there to selected these four members?

Solution:

$$10C_4 = \frac{10!}{4!(10-4)!} = \frac{10!}{4!6!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(4 \times 3 \times 2 \times 1)(6 \times 5 \times 4 \times 3 \times 2 \times 1)} = \frac{3628800}{17280}$$
$$= 210$$

Example (30):
A daily lottery is conducted in which 2 winning numbers are selected out of 100 numbers. How many different combinations of winning numbers are possible?

Solution:

$$100C_2 = \frac{100^!}{2^!(100-2)^!} = \frac{100^!}{2^!98^!} = 4950$$