Physics 201

Problem Set (6)

Problem (1)

Find the determinant of the following matrix using *Cofactor Expansion*

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ -1 & 0 & 3 & 1 \\ 3 & 1 & 2 & 0 \end{pmatrix}$$

Solution

$$det\, A = a_{14}C_{14} + a_{24}C_{24} + a_{34}C_{34} + a_{44}C_{44}$$

As a_{14} and a_{44} are zero, it is useless to find C_{14} and C_{44}. The cofactors C_{24} and C_{34} will be necessary…

$$C_{24} = (-1)^{2+4} M_{24} = 1 \begin{vmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$

$$C_{34} = (-1)^{3+4} M_{34} = -1 \begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

We let the reader verify that $C_{24} = 18$ et $C_{34} = -2$. Consequently, the determinant of A is

$$det\, A = a_{14}C_{14} + a_{24}C_{24} + a_{34}C_{34} + a_{44}C_{44}$$

$$det\, A = 0 \times C_{14} + 1 \times 18 + 1 \times (-2) + 0 \times C_{44} = 16$$

<u>Problem (2)</u>

<u>Solve the following linear system using *Cramer's Rule*</u>

$$\begin{cases} 4x - y + z = -5 \\ 2x + 2y + 3z = 10, \\ 5x - 2y + 6z = 1 \end{cases}$$

<u>Solution</u>

7. $\begin{cases} 4x - y + z = -5 \\ 2x + 2y + 3z = 10, \\ 5x - 2y + 6z = 1 \end{cases}$ $\quad D = \begin{vmatrix} 4 & -1 & 1 \\ 2 & 2 & 3 \\ 5 & -2 & 6 \end{vmatrix} = 55$

$$x = \frac{\begin{vmatrix} -5 & -1 & 1 \\ 10 & 2 & 3 \\ 1 & -2 & 6 \end{vmatrix}}{55} = \frac{-55}{55} = -1, \quad y = \frac{\begin{vmatrix} 4 & -5 & 1 \\ 2 & 10 & 3 \\ 5 & 1 & 6 \end{vmatrix}}{55} = \frac{165}{55} = 3, \quad z = \frac{\begin{vmatrix} 4 & -1 & -5 \\ 2 & 2 & 10 \\ 5 & -2 & 1 \end{vmatrix}}{55} = \frac{110}{55} = 2$$

Solution: $(-1, 3, 2)$

Problem (3)

Find the eigenvalues and eigenvectors of the following matrix

$$\mathbf{A} = \begin{bmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{bmatrix}.$$

Solution

First we compute $\det(\mathbf{A} - \lambda\mathbf{I})$ via a cofactor expansion along the second column:

$$\begin{aligned}
\begin{vmatrix} 7-\lambda & 0 & -3 \\ -9 & -2-\lambda & 3 \\ 18 & 0 & -8-\lambda \end{vmatrix} &= (-2-\lambda)(-1)^4 \begin{vmatrix} 7-\lambda & -3 \\ 18 & -8-\lambda \end{vmatrix} \\
&= -(2+\lambda)[(7-\lambda)(-8-\lambda)+54] \\
&= -(\lambda+2)(\lambda^2+\lambda-2) \\
&= -(\lambda+2)^2(\lambda-1).
\end{aligned}$$

Thus $\mathbf{A}$ has two distinct eigenvalues, $\lambda_1 = -2$ and $\lambda_3 = 1$. (Note that we might say $\lambda_2 = -2$, since, as a root, -2 has multiplicity two. This is why we labelled the eigenvalue 1 as λ_3.)

Now, to find the associated eigenvectors, we solve the equation $(\mathbf{A} - \lambda_j\mathbf{I})\mathbf{x} = \mathbf{0}$ for $j = 1, 2, 3$. Using the eigenvalue $\lambda_3 = 1$, we have

$$\begin{aligned}
(\mathbf{A} - \mathbf{I})\mathbf{x} &= \begin{bmatrix} 6x_1 - 3x_3 \\ -9x_1 - 3x_2 + 3x_3 \\ 18x_1 - 9x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
&\Rightarrow x_3 = 2x_1 \quad \text{and} \quad x_2 = x_3 - 3x_1 \\
&\Rightarrow x_3 = 2x_1 \quad \text{and} \quad x_2 = -x_1.
\end{aligned}$$

So the eigenvectors associated with $\lambda_3 = 1$ are all scalar multiples of

$$\mathbf{u_3} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Now, to find eigenvectors associated with $\lambda_1 = -2$ we solve $(\mathbf{A} + 2\mathbf{I})\mathbf{x} = \mathbf{0}$. We have

$$\begin{aligned} (\mathbf{A} + 2\mathbf{I})\mathbf{x} &= \begin{bmatrix} 9x_1 - 3x_3 \\ -9x_1 + 3x_3 \\ 18x_1 - 6x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ &\Rightarrow \; x_3 = 3x_1. \end{aligned}$$

Something different happened here in that we acquired no information about x_2. In fact, we have found that x_2 can be chosen arbitrarily, and independently of x_1 and x_3 (whereas x_3 cannot be chosen independently of x_1). This allows us to choose two linearly independent eigenvectors associated with the eigenvalue $\lambda = -2$, such as $\mathbf{u_1} = (1, 0, 3)$ and $\mathbf{u_2} = (1, 1, 3)$. It is a fact that all other eigenvectors associated with $\lambda_2 = -2$ are in the span of these two; that is, all others can be written as linear combinations $c_1\mathbf{u_1} + c_2\mathbf{u_2}$ using an appropriate choices of the constants c_1 and c_2.

Problem (4)

Consider the following matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

1) **Find the adjoint matrix**
2) **Use the adjoint to find the inverse A^{-1}**

Solution

To find the adjoint, we first find the cofactor matrix

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Then we take the *transpose* of the cofactor matrix to find the adjoint

$$\text{ad}\, A = C' = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Then we find the determinant of matrix A to find the inverse

det (A)= 2

Now, using the determinant and the adjoint, we can find the inverse by using the following Rule

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$