MATH 6310–COMPLEX ANALYSIS PROBLEMS AND THEOREMS

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1. Quiz #1 Problems

1. Prove that the function $f(z) = \overline{z}$ is not differentiable anywhere.

Proof. Let $z_0 \in \mathbb{C}$. Then

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\overline{z} - \overline{z_0}}{z - z_0} = \frac{\overline{z - z_0}}{z - z_0}. \tag{1}$$

Now if $z - z_0$ is real and non-zero, then (1) takes the value 1. If $z - z_0 = ik$ for $k \in \mathbb{R}$ (i.e. $z - z_0$ is purely imaginary), then (1) takes the value -1. Thus the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

does not exist and so $f(z)$ is not differentiable at z_0. We chose z_0 arbitrarily, so $f(z)$ is not differentiable anywhere. □

2. Prove that the function $f(z) = |z|$ is not analytic anywhere.

Proof. Note that $f(z) = \sqrt{x^2 + y^2}$, and so $u(x, y) = \sqrt{x^2 + y^2}$ and $v(x, y) = 0$. Note that

$$u_x = \frac{2x}{2\sqrt{x^2 + y^2}}, \quad v_x = 0, \quad u_y = \frac{2y}{2\sqrt{x^2 + y^2}} \quad \text{and} \quad v_y = 0.$$

Thus the Cauchy-Riemann equations can only be satisfied when $x = 0$ and $y = 0$, so f cannot be differentiable on any ϵ-disc. Hence f is not analytic anywhere. □

3. Give a definition of e^z, derive all its basic properties and prove that it is an entire function.

Proof. We define $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$. This suggests $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. Taking partial derivatives, we check that

$$u_x = e_x \cos y, \; u_y = -e^x \sin y, \; v_x = e^x \sin y, \quad \text{and} \quad v_y = e^x \cos y.$$

Thus $u_x = v_y$ and $u_y = -v_x$, so the Cauchy-Riemann equations hold and the partial derivatives are continuous. Note that this did not depend on choice of z, so e^z is differentiable everywhere; that is, it is entire.

Now note $f'(z) = u_x + iv_x$, so

$$(e^z)' = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y) = e^z.$$

Hence we have the familiar property that $(e^z)' = e^z$.

By Taylor's Theorem taking $z_0 = 0$ and using the fact that e^z is entire, we have

$$e^z = \sum_{n=0}^{\infty} \frac{(e^0)^{(n)}}{n!}(z-0)^n = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The last property we work toward is showing $e^{z+w} = e^z e^w$. Note that

$$\begin{aligned} e^{i(y_1+y_2)} &= \cos(y_1+y_2) + i\sin(y_1+y_2) \\ &= \cos(y_1)\cos(y_2) - \sin(y_1)\sin(y_2) + i\sin(y_1)\cos(y_2) + i\cos(y_1)\sin(y_2) \\ &= \cos(y_2)(\cos(y_1) + i\sin(y_1)) + i\sin(y_2)(\cos(y_1) + i\sin(y_1)) \\ &= (\cos(y_1) + i\sin(y_1))(\cos(y_2) + i\sin(y_2)) \\ &= e^{iy_1}e^{iy_2}. \end{aligned}$$

Thus we have

$$\begin{aligned} e^{z_1+z_2} &= e^{x_1+iy_1+x_2+iy_2} \\ &= e^{x_1+x_2}e^{i(y_1+y_2)} \quad \text{Are we assuming what we're trying to prove here?} \\ &= e^{x_1}e^{x_2}e^{iy_1}e^{iy_2} \\ &= e^{x_1+iy_1}e^{x_2+iy_2} \quad \text{And here?} \\ &= e^{z_1}e^{z_2} \end{aligned}$$

Finally, note that $|e^z| = |e^x||e^iy| = |e^x|$, and since $|e_x| \neq 0$ for any $x \in \mathbb{R}$, we see that $e^z \neq 0$ for any $z \in \mathbb{C}$. $\square$

4. Give a definition of $\sin z$, derive all its basic properties and prove that it is an entire function.

Proof. We define

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

The fact that $\sin z$ is entire follows from the fact that e^z is entire. Now we have

$$(\sin z)' = \frac{1}{2i}(ie^{iz} + ie^{-iz}) = \frac{1}{2}(e^{iz} + e^{-iz}) = \cos z,$$

where the definition of $\cos z$ is given in problem 6.

Now note that

$$e^{i(z+w)} = \cos(z+w) + i\sin(z+w)$$

and

$$\begin{aligned} e^{iz}e^{iw} &= (\cos(z) + i\sin(z))(\cos(w) + i\sin(w)) \\ &= \cos(z)\cos(w) - \sin(z)\sin(w) + i(\sin(z)\cos(w) + \cos(z)\sin(w)). \end{aligned}$$

Thus we have

$$\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w) \quad \text{and} \quad \sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w).$$

$\square$

5. Give a definition of $\cos z$, derive all its basic properties and prove that it is an entire function.

Proof. We define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Once again, the fact that e^z is entire guarantees that $\cos(z)$ is entire, and

$$(\cos z)' = \frac{1}{2}(ie^{iz} - ie^{-iz}) = -\frac{1}{2i}(e^{iz} - e^{-iz}) = \sin z.$$

This establishes the basis properties of $\sin z$ and $\cos z$. $\square$

6. Prove that if $f(z)$ is analytic in a domain D and Ref is constant, then $f(z)$ is constant.

Proof. The assumption states that $u(x, y)$ is constant. Thus we have

$$\frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial u}{\partial y} = 0.$$

Since f is analytic in D, the Cauchy-Riemann equations hold in D. Thus we have

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0 \text{ and } \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 0.$$

Thus f must have constant imaginary part $v(x, y)$, and so f must be constant in D. $\square$

7. Prove that if $f(z)$ is analytic in a domain D and Imf is constant, then $f(z)$ is constant.

Proof. The previous argument works exactly the same for this case, only this time we are assuming from the beginning that $v(x, y)$ is constant. $\square$

8. Prove that if $f(z)$ is analytic in a domain D and $|f|$ is constant, then $f(z)$ is constant.

Proof. Let $f = u + iv$. We are assuming that $u^2 + v^2 = c$ for some constant $c \geq 0$. Differentiating this with respect to x and y, we get the equations

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \text{ and } 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0.$$

Using $-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ in the first equation and $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ in the second equation, we get

$$2u\frac{\partial u}{\partial x} - 2v\frac{\partial u}{\partial y} = 0 \text{ and } 2v\frac{\partial u}{\partial x} + 2u\frac{\partial u}{\partial y} = 0.$$

We can write this as a homogeneous system of 2 equations in 2 unknowns

$$\begin{pmatrix} u & -v \\ v & u \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2}$$

Note that $\det \begin{vmatrix} u & -v \\ v & u \end{vmatrix} = u^2 + v^2 = c$. Thus if $c > 0$, then the matrix $\begin{pmatrix} u & -v \\ v & u \end{pmatrix}$ is invertible and hence the system (2) has a unique solution given by

$$\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0.$$

This implies $u(x, y)$ is constant, so by problem 6, it follows that f is constant. Now if $c = 0$, then clearly we have $u = v = 0$, so $f = 0$ in D. □

9. Prove that $\int_C p(z)\, dz = 0$ for any polynomial p and closed contour C.

Proof. Let $f(z) = z^n$ for $n \in \mathbb{N}$. We show that $f'(z)$ exists and is equal to nz^{n-1}. Let $z_0 \in \mathbb{C}$. Then

$$\begin{aligned} \frac{f(z) - f(z_0)}{z - z_0} &= \frac{z^n - z_0^n}{z - z_0} \\ &= \frac{(z - z_0)(z^{n-1} + z^{n-2}z_0 + z^{n-3}z_0^2 + \cdots + zz_0^{n-2} + z_0^{n-1})}{z - z_0} \\ &= z^{n-1} + z^{n-2}z_0 + z^{n-3}z_0^2 + \cdots + zz_0^{n-2} + z_0^{n-1}. \end{aligned}$$

Thus

$$f'(z_0) = \lim_{z \to z_0} z^{n-1} + z^{n-2}z_0 + z^{n-3}z_0^2 + \cdots + zz_0^{n-2} + z_0^{n-1} = nz_0^{n-1}.$$

This is the standard proof that a polynomial is differentiable.

Now $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$. Consider $P(z) = \frac{a_n}{n+1}z^{n+1} + \frac{a_{n-1}}{n}z^{n-1} + \cdots + \frac{a_1}{2}z^2 + a_0 z + c$, where $c \in \mathbb{C}$ is some constant. The fact previously proven shows that P is differentiable and $P'(z) = p(z)$. Furthermore, there was no restriction of the value z_0 chosen when computing the derivative, so P is entire. We have by the Fundamental Theorem of Calculus for complex-valued functions that

$$\int_C p(z)\, dz = P(z_2) - P(z_1).$$

Since we are working with a closed contour, we have $z_2 = z_1$, so this integral is 0, as desired. □

10. Evaluate the integral

$$\int_C \frac{R + z}{(R - z)z}\, dz$$

where $C = \{z : |z| = r\}$, $r < R$.

Proof. We have by the method of partial fractions that

$$\frac{R + z}{(R - z)z} = \frac{1}{z} + \frac{2}{R - z},$$

so that

$$\int_C \frac{R+z}{(R-z)z}\ dz = \int_C \frac{1}{z}\ dz + \int_C \frac{2}{R-z}\ dz.$$

Note that $\frac{2}{R-z}$ is analytic in the star domain $D = \{z : |z| < R\}$, so since C is a closed contour, we have $\int_C \frac{2}{R-z}\ dz = 0$. Thus

$$\int_C \frac{R+z}{(R-z)z}\ dz = \int_C \frac{1}{z}\ dz = 2\pi i.$$

□

11. Let C be any countour, $I(z) = \int_C \frac{1}{\zeta - z}\ d\zeta$, $z \notin C$. Show that $I(z)$ is continuous at every $z \notin C$.

Proof. Since $z \notin C$, there is an $\epsilon > 0$ such that the ϵ-neighborhood of z does not meet C. Suppose $h \in \mathbb{C}$ and $|h| < \epsilon/2$. Then we have

$$I(z+h) - I(z) = \int_C \left(\frac{1}{\zeta - z - h} - \frac{1}{\zeta - z} \right)\ d\zeta = \int_C \frac{h}{(\zeta - z - h)(\zeta - z)}\ d\zeta.$$

For any $\zeta \in C$, we have $|\zeta - z| > \epsilon$ and $|\zeta - z - h| > \epsilon/2$. Thus

$$\frac{1}{|(\zeta - z - h)||(\zeta - z)|} < \frac{2}{\epsilon^2}.$$

Thus using the ML-inequality, we have

$$\left| \int_C \frac{h}{(\zeta - z - h)(\zeta - z)}\ d\zeta \right| \leq |h| \int_C \frac{1}{|(\zeta - z - h)||(\zeta - z)|}\ d\zeta \leq \frac{2L|h|}{\epsilon^2},$$

where L is the length of C. This tends to 0 as $h \to 0$. □

Key Points

(1) Suppose $|h| < \epsilon/2$ and form an estimate (using ML-inequality) for $I(z+h) - I(z)$.

12. Let D be a domain which is obtained from $\mathbb{C}$ by deleting a half ray (from the origin). Show that D is a star domain and there is an analytic function $F(z)$ in D such that $F'(z) = \frac{1}{z}$.

Proof. We show that the function $f(z) = \frac{1}{z}$ is analytic in D. Let $z \in D$. Then we can choose $\epsilon > 0$ such that $B(z, \epsilon)$ does not meet ℓ, the ray deleted from D. Choose h such that $|h| < \epsilon$ (which we may freely do since we will be taking the limit as h tends to 0 anyway). Thus $z + h \neq 0$. Then we have

$$\frac{f(z+h) - f(z)}{h} = \frac{1}{h}\left(\frac{1}{z+h} - \frac{1}{z} \right) = \frac{1}{h}\left(\frac{-h}{(z+h)(z)} \right) = \frac{-1}{(z+h)(z)}.$$

Thus

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = -\frac{1}{z^2},$$

which shows that $f'(z)$ exists at z. But our choice of z was arbitrary, so it follows that $f'(z)$ exists everywhere on D and hence f is analytic on D.

We needed to delete ℓ so that D would be a star domain. Indeed, if ℓ were extended in the opposite direction, any point on this extension would be a star center. Thus, we may use Theorem 5 below to show that there is an analytic function $F(z)$ in D such that $F'(z) = f(z)$. This completes the proof and $F(z)$ is one possible definition of the logarithm. $\square$

13. Let $g(\zeta)$ be a continuous function on a circle C. Show that there is an analytic function $f(z)$ in the disk D with boundary C such that $f(z)$ has continuous extension to C and $f|_C = g$.

Proof. This is apparently a famous result with a very difficult proof. Dr. Vu used the entire class period trying to show it today (October 9, 2013) and didn't get all of it done. So he said this would not appear on the test. $\square$

2. Theorems for Quiz #1

Theorem 1: If $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z = x + iy$, then u and v satisfy the Cauchy-Riemann equations: $u_x = v_y$, $u_y = -v_x$; moreover, $f'(z) = u_x + iv_x = v_y - iu_y$.

Proof. Use the formula
$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$
Let $h \to 0$ through real values. Then
$$\begin{aligned}\lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} &= \lim_{h \to 0} \frac{u(x+h, y) - u(x, y)}{h} + i \lim_{h \to 0} \frac{v(x+h, y) - v(x, y)}{h} \\ &= u_x(x, y) + iv_x(x, y).\end{aligned}$$
By going through purely imaginary values ($h \in \mathbb{R}$), we also have
$$\begin{aligned}\lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{ih} &= \lim_{h \to 0} \frac{u(x, y+h) - u(x, y)}{ih} + i \lim_{h \to 0} \frac{v(x, y+h) - v(x, y)}{ih} \\ &= v_y(x, y) - iu_y(x, y).\end{aligned}$$
By equating real and imaginary parts, we have $u_x = v_y$ and $u_y = -v_x$. This completes the proof. $\square$

Theorem 2: If $f(z)$ is a continuous function in a domain D and if there exists an analytic function $F(z)$ in D such that $F'(z) = f(z)$, then for any contour C in D with initial point z_1 and terminal point z_2, we have
$$\int_C f(z)\ dz = F(z_2) - F(z_1).$$

Proof. Let $\psi(t) = F(\zeta(t))$ where $\zeta : [A, B] \to \mathbb{C}$ is a parametrization of C. Then by the chain rule, $\psi'(t) = F'(\zeta(t))\zeta'(t) = f(\zeta(t))\zeta'(t)$. Thus
$$\begin{aligned}\int_C f(z)\ dz &= \int_A^B f(\zeta(t))\zeta'(t)\ dt \\ &= \int_A^B F'(\zeta(t))\zeta'(t)\ dt \\ &= \int_A^B \psi'(t)\ dt \\ &= \psi(B) - \psi(A) \quad \text{(Fundamental Theorem of Calculus)} \\ &= F(\zeta(B)) - F(\zeta(A)) \\ &= F(z_2) - F(z_1).\end{aligned}$$
$\square$

Theorem 3: If $f(z)$ is continuous on a contour C, $|f(z)| \leq M$ for all $z \in C$, and L is the length of C, then

$$\left| \int_C f(z)\ dz \right| \leq ML.$$

Proof. Let $\zeta : [A, B] \to \mathbb{C}$ be the parametrization of C. We have

$$\begin{aligned} \left| \int_C f(z)\ dz \right| &= \left| \int_A^B f(\zeta(t))\zeta'(t)\ dt \right| \\ &\leq \int_A^B |f(\zeta(t))||\zeta'(t)|\ dt \\ &\leq M \int_A^B |\zeta'(t)|\ dt \\ &= M \int_A^B \sqrt{(x'(t))^2 + (y'(t))^2}\ dt \\ &= ML. \end{aligned}$$

□

Theorem 4 (Cauchy Integral Theorem): If $f(z)$ is analytic in D, T is a triangle in D with boundary C, then $\int_C f(z)\ dz = 0$.

Proof. Write $I(T) = \int_C f(z)\ dz$. Denote T into four triangular regions by joining the midpoints of the three sides of T. Note when you divide the triangle this way, and orient each of the sub triangles in the positive direction, we get sides canceling out so that

$$I(T) = I(T_1) + I(T_2) + I(T_3) + I(T_4),$$

where $I(T_j) = \int_{C_j} f(z)\ dz$. Now at least one of these integrals satisfies

$$|I(T_j)| \geq \frac{1}{4}|I(T)|.$$

Let this triangle be denoted T_1. We can now repeat the process with T_1 to find a triangle $T_2 \subset T_1$ such that

$$|I(T_2)| \geq \frac{1}{4}|I(T_1)| \geq \frac{1}{16}|I(T)|.$$

Continuing in this fashion, we have a sequence of nested triangles

$$T = T_0 \supset T_1 \supset T_2 \supset \cdots \supset T_k \supset \cdots$$

such that

$$|I(T_k)| \geq 4^{-k}|I(T)|.$$

Now these triangles collapse to a single point, $z^* \in D$. Let $\epsilon > 0$. Since f is analytic at z^*, there exists a δ-neighborhood Δ of z^* contained in D such that

$$\left| \frac{f(z) - f(z^*)}{z - z^*} - f'(z^*) \right| < \epsilon \tag{3}$$

whenever $|z - z^*| < \delta$. We can also choose k large enough such that $T_k \subset \Delta$.

Now since $\int_{C_k} dz = 0$ and $\int_{C_k} z\ dz = 0$, we have

$$\begin{aligned} I(T_k) &= \int_{C_k} f(z)\ dz \\ &= \int_{C_k} f(z)\ dz - f(z^*) \int_{C_k} dz - f'(z^*) \int_{C_k} z\ dz + z^* f'(z^*) \int_{C_k} dz \\ &= \int_{C_k} (f(z) - f(z^*) - (z - z^*) f'(z^*))\ dz. \end{aligned}$$

Now by (3) we have

$$|f(z) - f(z^*) - (z - z^*) f'(z^*)| \le \epsilon |z - z^*| \le \epsilon d_k,$$

where $d_k = \operatorname{diam}(T_k)$. By the ML-inequality, we have

$$|I(T_k)| \le \epsilon d_k L_k,$$

where L_k is the perimeter of T_k. Now we have

$$|I(T)| \le 4^k |I(T_k)| \le 4^k \epsilon d_k L_k = 4^k \epsilon 2^{-k} d 2^{-k} L = \epsilon d L,$$

where d and L are the diameter and perimeter of T, respectively, and using the fact that $d_k = 2^{-k} d$ and $L_k = 2^{-k} L$. Since ϵ was arbitrary, we have $I(T) = 0$, as desired. $\square$

Key Points

(1) Subdivide triangles. Let $d_k = \operatorname{diam}(T_k)$, $L_k =$ length of C_k. Note that $d_k = \frac{1}{2^k} d$ and $L_k = \frac{1}{2^k} L$.
(2) Note that $\bigcap T_k = z^*$.
(3) $I(T_k) = \int_{C_k} (f(z) - f(z^*) - (z - z^8) f'(z^*))\ dz$.
(4) Note $|f(z) - f(z^*) - (z - z^*) f'(z^*)| \le \epsilon |z - z^*|$.

Theorem 5: If $f(z)$ is analytic in a star domain D, then there is an analytic function $F(z)$ in D such that $F'(z) = f(z)$.

Proof. Let $z_0 \in D$ be a star center. For each $z \in D$, define

$$F(z) = \int_{[z_0, z]} f(\zeta)\ d\zeta,$$

where $[z_1, z_2]$ denotes the directed lien segment from z_1 to z_2. Since $z \in D$, there exists an ϵ-neighborhood of z which is contained in D. Choose $h \in D$ such that $|h| < \epsilon$. Then $z + h$ is contained in the ϵ-neighborhood of z. Because D is a star domain, all three points z_0, z, and $z + h$ lie in D, and furthermore the triangle formed by these three points is in D. By the Cauchy Integral Theorem for triangles, we have

$$\int_{[z_0, z]} f(\zeta)\ d\zeta + \int_{[z, z+h]} f(\zeta)\ d\zeta + \int_{[z+h. z_0]} f(\zeta)\ d\zeta = 0.$$

We may re-write this, changing signs (orientations) as necessary to get

$$\int_{[z_0,z+h]} f(\zeta)\ d\zeta - \int_{[z_0,z]} f(\zeta)\ d\zeta = \int_{[z,z+h]} f(\zeta)\ d\zeta.$$

It follows from how we defined $F(z)$ that

$$F(z+h) - F(z) = \int_{[z,z+h]} f(\zeta)\ d\zeta.$$

If $h \neq 0$ then we have

$$\begin{aligned}
\frac{F(z+h)-F(z)}{h} - f(z) &= \frac{1}{h}\int_{[z,z+h]} f(\zeta)\ d\zeta - f(z) \\
&= \frac{1}{h}\int_{[z,z+h]} f(\zeta)\ d\zeta - \frac{1}{h} f(z)\int_{[z,z+h]} d\zeta \\
&= \frac{1}{h}\int_{[z,z+h]} f(\zeta)\ d\zeta - \frac{1}{h}\int_{[z,z+h]} f(z)\ d\zeta \\
&= \frac{1}{h}\int_{[z,z+h]} (f(\zeta) - f(z))\ d\zeta.
\end{aligned}$$

Now since f is differentiable at z, it follows that f is continuous at z and hence, given $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(\zeta) - f(z)| < \epsilon$ whenever $|\zeta - z| < \delta$. Thus if $|h| < \delta$, then $|f(\zeta) - f(z)| < \epsilon$ holds for all $\zeta \in [z, z+h]$. We now have by Theorem 3 that

$$\left| \int_{[z,z+h]} (f(\zeta) - f(z))\ d\zeta \right| \leq \epsilon |h|.$$

Hence we have

$$\begin{aligned}
\left| \frac{F(z+h)-F(z)}{h} - f(z) \right| &= \left| \frac{1}{h}\int_{[z,z+h]} (f(\zeta) - f(z))\ d\zeta \right| \\
&= \frac{1}{|h|}\left| \int_{[z,z+h]} (f(\zeta) - f(z))\ d\zeta \right| \\
&\leq \frac{1}{|h|}\epsilon|h| \\
&= \epsilon
\end{aligned}$$

We have successfully shown that

$$\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = f(z),$$

which establishes our result. $\square$

Theorem 6 (Cauchy Integral Formula): If $f(z)$ is analytic in a domain D, $\alpha \in D$, and $C = \{z : |z-\alpha| = r\}$, then

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

for all $z \in B(\alpha, r)$.

Proof. Suppose γ is a circle of positive orientation and of radius ρ and centered at z. Suppose also that ρ is small enough so that γ is in the interior of C. Then we may perform "surgery" to construct two new contours, C^+ and C^-, such that $C^+ \cup C^- = C \cup -\gamma$. Furthermore, since C^+ and C^- are closed contours in D, and since z is not in the interior of either, we have

$$\int_{C^+} \frac{f(\zeta)}{\zeta - z}\, d\zeta = 0 \quad \text{and} \quad \int_{C^-} \frac{f(\zeta)}{\zeta - z}\, d\zeta = 0.$$

It follows that

$$\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int_{C^+} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \int_{C^-} \frac{f(\zeta)}{\zeta - z}\, d\zeta = 0,$$

so

$$\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

We can then write

$$\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z) \int_\gamma \frac{d\zeta}{\zeta - z} + \int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta.$$

Now $\int_\gamma \frac{d\zeta}{\zeta - z} = 2\pi i$, so we may write

$$\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z) 2\pi i + \int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta.$$

We focus on the second integral in the right hand side. Since f is continuous at z, it follows that for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(\zeta) - f(z)| < \epsilon$. Thus if we choose $\rho < \delta$, we have

$$\left| \frac{f(\zeta) - f(z)}{\zeta - z} \right| < \frac{\epsilon}{\rho}$$

for every $\zeta \in \gamma$, and so we have from Theorem 3 that

$$\left| \int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta \right| \le \frac{\epsilon}{\rho} \cdot 2\pi\rho = 2\pi\epsilon.$$

Since ϵ was chosen arbitrarily, we have that

$$\int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta = 0,$$

which yields our result. $\square$

Theorem 7 (Cauchy Integral Formula for Derivatives): If $f(z)$ is analytic in D, then $f^{(n)}(z)$ is analytic in D for all $n \in \mathbb{N}$. Moreover, if $\overline{B}(\alpha, r) \subset D$ and $C = \{z : |z - \alpha| = r\}$, then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta. \tag{4}$$

Proof. We first show that f' exists and is analytic in D. Note that for every $z \in D$, since D is an open domain we see that z is inside some circle C centered at some α, where C is contained in D. Hence by the Cauchy Integral Formula, we have

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

Now suppose that $h \in \mathbb{C}$ has sufficiently small hodulus so that $z + h \in C$. Then by the Cauchy Integral Formula we have

$$\begin{aligned} \frac{f(z+h) - f(z)}{h} &= \frac{1}{2\pi i h} \int_C \left(\frac{f(\zeta)}{\zeta - z - h} - \frac{f(\zeta)}{\zeta - z} \right) \, d\zeta \\ &= \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z - h)(\zeta - z)} \, d\zeta \\ &= \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta + \frac{h}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2 (\zeta - z - h)} \, d\zeta. \end{aligned}$$

Now since z is inside the circle C, the number

$$\delta = \inf_{\zeta \in C} |\zeta - z| > 0,$$

so if $|h| < \frac{\delta}{2}$, then for every $\zeta \in C$ we have

$$|\zeta - z - h| \geq |\zeta - z| - |h| > \delta - \frac{\delta}{2} = \frac{\delta}{2}.$$

On the other hand, the circle C is closed and bounded, so there is some real constant M such that $|f(\zeta)| \leq M$ for every $\zeta \in C$. Also, recall the circle C has radius r. It follows from the ML-inequality that

$$\left| \frac{h}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2 (\zeta - z - h)} \, d\zeta \right| \leq \frac{|h|}{2\pi} \frac{2M}{\delta^3} 2\pi r = \frac{2Mr|h|}{\delta^3} \to 0$$

as $h \to 0$. Hence f' exists in D and

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta. \tag{5}$$

Note that this satisfies (4) for $n = 1$. We then take the derivative with the same argument we just used to show that f'' exists in D. Hence f' is analytic in D, as claimed. What we have essentially shown in (5) is the base case for an inductive argument to establish (4). Now suppose $f^{(n)} = g$ is analytic in D. What we have just shown is that $g' = (f^{(n)})' = f^{(n+1)}$ is analytic in D. Thus the derivative exists and is analytic for all $n \in \mathbb{N}$ by induction.

Now, to establish (4), we apply the Cauchy Integral Formula to the function $f^{(n)}$ to get

$$f^{(n)}(z) = \frac{1}{2\pi i} \int_C \frac{f^{(n)}(\zeta)}{\zeta - z} \, d\zeta.$$

Integrating by parts n times gives us (4). $\square$

Theorem 8 (Liouville Theorem): If $f(z)$ is an entire function which is bounded, then $f(z)$ is constant.

Proof. Since f is bounded, there exists an $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Thus by Cauchy's estimate (taking $n = 1$), we have

$$|f'(z)| \leq \frac{M}{r},$$

where r is the radius of a disc centered at z contained in the domain D for which f is analytic. But because f is entire, we have $D = \mathbb{C}$, so r is unbounded. Thus

$$|f'(z)| \leq \lim_{r \to \infty} \frac{M}{r} = 0,$$

so we must have $f'(z) = 0$ for all $z \in \mathbb{C}$. Hence f is constant. $\square$

Theorem 9 (Taylor's Theorem): If $f(z)$ is analytic in $\{z : |z - z_0| < R\}$, then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n,$$

where the series converges uniformly in $\{z : |z - z_0| < r\}$ for every $r < R$. Moreover,

$$f(z) = \sum_{k=0}^{n-1} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k + (z - z_0)^n f_n(z),$$

where $f_n(z) = \frac{1}{2\pi i} \int_C \frac{1}{(\zeta - z_0)^n} \frac{f(\zeta)}{\zeta - z}\, d\zeta$, and $C = \{z : |z - z_0| = \rho, r < \rho < R\}$.

Proof. Without loss of generality we may assume $z_0 = 0$. To see why this is not a problem, let $g(z) = f(z + z_0)$. Then $g(z)$ is analytic in $\{z : |z| < R\}$ and $g^{(n)}(0) = f^{(n)}(z_0)$, so we are really just needing to prove the result about g. That is, to show that

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n$$

uniformly in $\{z : |z| < r\}$. Let ρ be chosen such that $r < \rho < R$, and let C denote the circle $\{\zeta : |\zeta| = \rho\}$, oriented positively. By the Cauchy Integral Formula, we have

$$g(z) = \frac{1}{2\pi i} \int_C \frac{g(\zeta)}{\zeta - z}\, d\zeta$$

for every z such that $|z| \leq r$. Now note that

$$1 - w^n = (1 - w)(w^{n-1} + w^{n-2} + \cdots + w + 1),$$

which implies that

$$1 = (1 - w)\left(w^{n-1} + \cdots + w + 1 + \frac{w^n}{1 - w}\right),$$

or

$$\frac{1}{1-w} = 1 + w + \cdots + w^{n-1} + \frac{w^n}{1-w}.$$

Now if we substitute $w = \frac{z}{\zeta}$ and divide by ζ, we get

$$\frac{1}{\zeta - z} = \frac{1}{\zeta} + \frac{z}{\zeta^2} + \frac{z^2}{\zeta^3} + \cdots + \frac{z^{n-1}}{\zeta^n} + \frac{z^n}{\zeta^n}\frac{1}{\zeta - z}.$$

Thus we have

$$g(z) = \frac{1}{2\pi i}\int_C \frac{g(\zeta)}{\zeta}\, d\zeta + \cdots + \frac{z^{n-1}}{2\pi i}\int_C \frac{g(\zeta)}{\zeta^n}\, d\zeta + z^n g_n(z), \tag{6}$$

where

$$g_n(z) = \frac{1}{2\pi i}\int_C \frac{g(\zeta)}{\zeta^n(\zeta - z)}\, d\zeta.$$

By the Cauchy Integral Formula for Derivatives, we have

$$\frac{1}{2\pi i}\int_C \frac{g(\zeta)}{\zeta^{n+1}}\, d\zeta = \frac{g^{(k)}(0)}{k!}$$

for all $k \in \{0, \ldots, n-1\}$, so that (6) becomes

$$\begin{aligned} g(z) &= g(0) + g'(0)z + \cdots + \frac{g^{(n-1)}(0)}{(n-1)!}z^{n-1} + z^n g_n(z) \\ &= \sum_{k=0}^{n-1} \frac{g^{(k)}(0)}{k!}z^k + z^n g_n(z). \end{aligned}$$

This establishes the second part of the conclusion after we perform the reverse substitution.

To complete the proof, we need to show that

$$|z^n g_n(z)| \to 0$$

uniformly in $\{z : |z| \leq r\}$ as $n \to \infty$. Now $C = \{\zeta : |\zeta| = \rho\}$ is closed and bounded, and g is continuous on C. Thus there is a positive real constant M such that $|g(\zeta)| \leq M$ for every $\zeta \in C$. Hence, for every $\zeta \in C$ and every z such that $|z| \leq r$, we have

$$\left|\frac{g(\zeta)}{\zeta^n(\zeta - z)}\right| \leq \frac{M}{\rho^n(\rho - r)}.$$

(This is because $|\zeta^n| = \rho^n$ and $|\zeta - z| \geq \rho - r$. Drawing a picture helps to see this fact.) We then have by the ML-inequality

$$\left|\int_C \frac{g(\zeta)}{\zeta^n(\zeta - z)}\, d\zeta\right| \leq \frac{M}{\rho^n(\rho - r)}2\pi\rho,$$

and so since $|z| \leq r$, we have

$$|z^n g_n(z)| \leq \frac{r^n}{2\pi}\frac{M}{\rho^n(\rho - r)}2\pi\rho = \frac{M\rho}{\rho - r}\left(\frac{r}{\rho}\right)^n.$$

Since $r < \rho$, the right hand side converges to 0 as $n \to \infty$, independently of the choice of z in $\{z : |z| \leq r\}$. This completes the proof after we perform the reverse substitution. $\square$

Theorem 10: If $f(z)$ and $g(z)$ are analytic in a domain D, $B(z_0, r) \subset D$, and $f(z) = g(z)$ for all $z \in B(z_0, r)$, then $f(z) = g(z)$ for all $z \in D$.

Proof. For every $z \in D$, write $h(z) = f(z) - g(z)$. Then clearly h is analytic in D since f and g are. Let

$$S_1 = \{z_1 \in D : h(z) = 0 \text{ in some neighborhood of } z_1\} \ \text{ and } \ S_2 = D \setminus S_1.$$

We aim to show that both S_1 and S_2 are open. With this in mind, let $z_1 \in S_1$. Then there exists $\epsilon_1 > 0$ such that $h(z) = 0$ in $B(z_1, \epsilon_1) = \{z : |z - z_1| < \epsilon_1\}$. Now choose $z' \in B(z_1, \epsilon_1)$, which implies $|z_1 - z'| < \epsilon_1$. Then choose $\delta > 0$ such that $\delta < \epsilon_1 - |z_1 - z'|$. Then we have $B(z', \delta) \subset B(z_1, \epsilon_1)$, which implies $h(z) = 0$ in $B(z', \delta) \subset B(z_1, \epsilon_1)$, which in turn implies $z' \in S_1$. Thus we have shown that $B(z_1, \epsilon_1) \subset S_1$, which implies that S_1 is open.

Now let $z_2 \in S_2$. Thus $(*)$ there is no neighborhood V of z_2 such that $h(z) = 0$ in V. Since $z_2 \in D$, it follows that there is an $R > 0$ such that the disc $B(z_2, R) \subset D$, which means h is analytic in $B(z_2, R)$. It follows from Taylor's Theorem that the Taylor series expansion

$$(**) \qquad h(z) = \sum_{n=0}^{\infty} \frac{h^{(n)}(z_2)}{n!}(z - z_2)^n$$

is valid in $\overline{B}(z_2, r)$ for every $r < R$. Combining $(*)$ and $(**)$, we see that $n = \min\{k : h^{(k)}(z_2) \neq 0\}$ exists. Thus $h^{(n)}(z_2) \neq 0$ but $h^{(j)}(z_2) = 0$ for all $j \in \{0, \ldots, n-1\}$. Now by Taylor's formula, this implies

$$h(z) = \sum_{k=0}^{n-1} \frac{h^{(k)}(z_2)}{k!}(z - z_2)^k + h_n(z)(z - z_2)^n = h_n(z)(z - z_2)^n,$$

where $h_n(z) = \frac{h^{(n)}(z_2)}{n!} \neq 0$ is analytic in $B(z_2, R)$. Now analyticity implies continuity, so since h_n is continuous on $B(z_2, R)$ it follows that there is an $\epsilon_2 > 0$ such that $h(z) \neq 0$ on the punctured disc $\{z : 0 < |z - z_2| < \epsilon_2\}$. Hence this punctured disc is contained in S_2, and since $z_2 \in S_2$, we have $B(z_2, \epsilon_2) \subset S_2$. This shows at last that S_2 is open.

Now we assumed that $z_0 \in S_1$, so $S_1 \neq \emptyset$. Since $D = S_1 \cup S_2$, $S_1 \cap S_2 = \emptyset$, S_1 and S_2 are open, and D is connected, it follows that $S_2 = \emptyset$. In other words, $D = S_1$, so we have $h(z) = 0$ for all $z \in D$; that is, $f \equiv g$ on D, as desired to complete the proof. □

Theorem 11: If $f(z)$ is analytic in D, $z_0 \in D$, $f(z_0) = 0$, and $f(z) \not\equiv 0$, then there is $n \in \mathbb{N}$ such that

$$f(z) = (z - z_0)^n g(z),$$

where $g(z)$ is analytic in D and $g(z_0) \neq 0$.

Proof. Since $f(z) \not\equiv 0$, it follows that $f(z) \not\equiv 0$ in any $B(z_0, R)$ (otherwise, by the previous Theorem 10, it would follow that $f(z) = g(z)$ on all of D, where $g(z) \equiv 0$, contrary to assumption). Thus the number $n = \min\{k : f^(k)(z_0) \neq 0\}$ exists. Using the Taylor formula, we have

$$f(z) = \sum_{k=0}^{n-1} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k + (z - z_0)^n f_n(z) = (z - z_0)^n f_n(z),$$

where $f_n(z_0) = \frac{f^{(n)}(z_0)}{n!} \neq 0$ is analytic in $B(z_0, R)$. Now we define

$$g(z) = \begin{cases} f_n(z) & \text{in } B(z_0, R) \\ f(z)(z - z_0)^{-n} & \text{for all } z \in D \setminus B(z_0, R), \end{cases}$$

which completes the proof since $g(z)$ is analytic in D and $g(z_0) \neq 0$. □

Theorem 12: If $f(z)$ is analytic in D, $f(z_0) = 0$ and $f(z) \not\equiv 0$, then there is $r > 0$ such that $f(z) \neq 0$ for all z with $0 < |z - z_0| < r$.

Proof. By Theorem 11 there is an $n \in \mathbb{N}$ such that $f(z) = (z - z_0)^n g(z)$, where $g(z)$ is analytic in D and $g(z_0) \neq 0$. Since analyticity implies continuity, there is an $\epsilon > 0$ such that $g(z) \neq 0$ in $B(z_0, \epsilon)$. Thus $f(z) \neq 0$ in $\{z : 0 < |z - z_0| < \epsilon\}$, as desired. □

Theorem 13: If $f(z)$ and $g(z)$ are analytic in a domain D, $z_n \in D$, $z_n \to z_0 \in D$, and $f(z_n) = g(z_n)$ for all $n \in \mathbb{N}$, then $f(z) = g(z)$ for all $z \in D$.

Proof. Let $h(z) = f(z) - g(z)$. Note that h is analytic in D. Since $h(z_n) = 0$ for all $n \in \mathbb{N}$, and since h is continuous, we have

$$h(z_0) = h\left(\lim_{n\to\infty} z_n\right) = \lim_{n\to\infty} h(z_n) = \lim_{n\to\infty} 0 = 0.$$

Now note that $h(z_n) = 0$ for all $n \in \mathbb{N}$, and since $z_n \to z_0$ as $n \to \infty$, we see that the zero z_0 is not isolated. Hence, by Theorem 12, we must have that $h(z) \equiv 0$ on D, so $f(z) \equiv g(z)$ on D. □

Theorem 14 (The Maximum Principle): If $f(z)$ is analytic in D, $\alpha \in D$ and $|f(\alpha)| = \max\{|f(z)| : z \in D\}$, then $f(z)$ is constant.

Proof. Suppose to the contrary that $f(z)$ is not constant and there is an $\alpha \in D$ such that $|f(\alpha)| = \max\{|f(z)| : z \in D\}$. Since D is open, we may choose an ϵ-neighborhood S of α which is contained in D. Now if $|f(z)| = |f(\alpha)|$ for all $z \in S$, then $|f(z)|$ is constant and hence f is constant in S by Problem 8 from the exercise list, and hence constant in D by Theorem 10. But we assumed f was not constant; hence, we may choose a $z_1 \in S$ such that $|f(z_1)| < |f(\alpha)|$.

Now let $r = |z_1 - \alpha|$ and note that $r < \epsilon$. Let $C = \{z : |\alpha - z| = r\}$ oriented in the positive direction. We have by the Cauchy Integral Formula (Theorem 6) that

$$f(\alpha) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - \alpha}\, d\zeta.$$

Writing $\zeta = \alpha + re^{it}$ and noting $d\zeta = ire^{it}\, dt$, we have

$$f(\alpha) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\alpha + re^{it})}{(\alpha + re^{it}) - \alpha}\, ire^{it}\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + re^{it})\, dt.$$

This implies that

$$|f(\alpha)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(\alpha + re^{it})|\, dt. \tag{7}$$

Note also that since $|f(\alpha)| \geq |f(z)|$ for every $z \in D$, we have

$$|f(\alpha)| \geq \frac{1}{2\pi} \int_0^{2\pi} |f(\alpha + re^{it}| \; dt.$$

The right hand side of this inequality can be thought of as the "average" value of $|f|$ while integrating along the circle, and so the average is going to be less than or equal to the maximum.

Now note that since $z_1 \in C$, we have $z_1 = \alpha + re^{it_1}$ for some $t_1 \in [0, 2\pi]$. Since f is continuous, there is an interval $I \subset [0, 2\pi]$ such that $|f(\alpha + re^{it})| < |f(\alpha)|$ for every $t \in I$. (That is, since there is one value that falls below the maximum, and since f is continuous, there must be an interval of values that fall below the maximum.) Thus,

$$|f(\alpha)| > \frac{1}{2\pi} \int_0^{2\pi} |f(\alpha + re^{it}| \; dt. \tag{8}$$

That is, the interval falling below the maximum guarantees the average will be below the maximum. The inequalities (7) and (8) contradict each other, so we are done. $\square$

3. Quiz #2 Problems

1. Show that the function $f(z) = \exp\left(\frac{1}{z}\right)$ has an essential singularity at $z = 0$.

Proof. Suppose the Laurent Series of $g(z)$ about z_0 is given as

$$g(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n.$$

Recall that g has an essential singularity at z_0 if an only if an infinite number of coefficients a_n with $n < 0$ are non-zero. Knowing that the exponential function is entire, we can use the Taylor series expansion

$$f(z) = \exp\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = \sum_{n=-\infty}^{0} \frac{z^n}{(-n)!} = \sum_{n=-\infty}^{0} \frac{1}{(-n)!}(z - 0)^n.$$

Therefore, for every $n < 0$, it follows that $a_n = \frac{1}{(-n)!} \neq 0$. Thus $\exp\left(\frac{1}{z}\right)$ has an essential singularity at $z = 0$. □

Note 3.1. Let $f(z)$ be a function and let $g(z) = \frac{1}{z}$. Suppose f is analytic in a neighborhood of ∞ (i.e. in a domain $D = \{z : R < |z| < \infty\}$ for some $R > 0$). Then $f \circ g$ is analytic in $\{z : 0 < |z| < R^{-1}\}$. We note that $f(z)$ has a singularity at ∞ if an only if $f \circ g(z)$ has a singularity at 0. Thus, to show that $f(z)$ has a certain type of singularity at ∞, we must show that $f \circ g(z)$ has a singularity of that type at 0.

2. Suppose $f(z)$ is analytic in a neighborhood of ∞ (i.e. in a domain $D = \{z : R < |z| < \infty\}$ for some $R > 0$), such that $\lim_{z\to 0} zf\left(\frac{1}{z}\right) = 0$. Prove that $f(z)$ has a removable singularity at ∞.

Proof. By Note 3.1, we know that $f \circ g(z)$ is analytic in the punctured disc $\{z : 0 < |z| < R^{-1}\}$. We wish to show that $f \circ g$ has a removable singularity at $z = 0$. By Riemann's Theorem on removable singularities, since

$$\lim_{z\to 0}(z - 0)(f \circ g)(z) = \lim_{z\to 0} zf\left(\frac{1}{z}\right) = 0$$

by assumption, it follows that $f \circ g$ has a removable singularity at 0, and hence f has a removable singularity at ∞. □

3. Suppose $f(z)$ is analytic in domain $D = \{z : R < |z| < \infty\}$ for some $R > 0$, such that $|f(z)| \to \infty$ as $|z| \to \infty$. Prove that $f(z)$ has a pole at ∞, i.e., there exists $n > 0$ such that $f(z) = z^n h(z)$, where $h(z)$ is analytic in domain $\{z : R < |z| < \infty\}$ and $h(\infty) \neq 0$.

Proof. By Note 3.1, we note that $f \circ g$ is analytic in a puncture disc centered at 0; we want to show that $f \circ g$ has a pole at 0. By Theorem 8B in the textbook, this happens if and only if

$$\lim_{z\to 0} |f \circ g(z)| = \infty.$$

Note that
$$\lim_{z\to 0}|f\circ g(z)| = \lim_{z\to 0}\left|f\left(\frac{1}{z}\right)\right| = \lim_{|z|\to\infty}|f(z)| = \infty,$$
where the last equality follows by assumption. Thus f has a pole at ∞, as desired. $\square$

4. Suppose $f(z)$ is analytic in domain $D = \{z : R < |z| < \infty\}$ for some $R > 0$, such that $f(z)$ has an essential singularity at ∞. Prove that for every $\omega \in \mathbb{C}$ and real numbers $\epsilon > 0$, $N > 0$, there exists z such that $|z| > N$ and $|f(z) - \omega| < \epsilon$.

Proof. Since $f(z)$ is analytic in $D = \{z : R < |z| < \infty\}$ for some $R > 0$ with an essential singularity at ∞, it follows from Note 3.1 that $f \circ g(z)$ is analytic in $\{z : 0 < |z| < R^{-1}\}$ and has an essential singularity at $z = 0$. Thus, by Theorem 8C, for $\omega \in \mathbb{C}$ and $\epsilon > 0$ and $\frac{1}{N} > 0$, there is a z in the punctured disc satisfying $0 < |z| < \frac{1}{N}$ and $|f \circ g(z) - \omega| < \epsilon$.

Let $\mathfrak{z} = \frac{1}{z}$, which is well-defined since $|z| > 0$. Then $z = \frac{1}{\mathfrak{z}}$, and so by the previous paragraph we have
$$\left|\frac{1}{\mathfrak{z}}\right| < \frac{1}{N} \quad\text{and}\quad \left|f\circ g\left(\frac{1}{\mathfrak{z}}\right) - \omega\right| < \epsilon,$$
or
$$|\mathfrak{z}| > N \quad\text{and}\quad |f(\mathfrak{z}) - \omega| < \epsilon,$$
as desired to complete the proof. $\square$

5. Show that the function $f(z) = \dfrac{e^z - 1}{z(z-1)}$ has a removable singularity at $z = 0$, a simple pole at $z = 1$, and an essential singularity at ∞.

Proof. First of all, the fact that f has isolated singularities at 0 and 1 is clear, but we must show what form they are. For $z_1 = 0$, note that
$$\lim_{z\to 0} zf(z) = \lim_{z\to 0}\frac{e^z - 1}{z - 1} = \frac{e^0 - 1}{0 - 1} = 0,$$
and so by Riemann's Theorem on removable singularities, we see that $f(z)$ has a removable singularity at $z_1 = 0$.

Now note that
$$f(z) = \frac{e^z - 1}{z(z-1)} = \frac{g(z)}{z-1}, \text{ where } g(z) = \frac{e^z - 1}{z}.$$
Now g is analytic in any neighborhood of $z_2 = 1$ not including the origin, and furthermore, $g(1) = e - 1 \neq 0$, so by definition, $f(z)$ has a pole of order 1 (i.e. simple pole) at $z_2 = 1$.

Note that, in the style of Note 3.1,
$$f\circ g(z) = \frac{e^{1/z} - 1}{\frac{1}{z}\left(\frac{1-z}{z}\right)} = \frac{z^2(e^{1/z} - 1)}{1 - z},$$
which clearly has a singularity at $z = 0$; hence, $f(z)$ has a singularity at ∞. Note that $\lim_{|z|\to\infty} f(z)$ does not exist since
$$\lim_{x\to+\infty} f(x) = \infty \text{ and } \lim_{x\to-\infty} f(x) = 0.$$

Thus, the singularity is not removable. Now note that f has a pole of order n at infinity if and only if $f \circ g$ has a pole of order n at 0, which happens if and only if $\frac{1}{f \circ g}$ has a zero of order n at 0. But

$$\frac{1}{f \circ g(z)} = \frac{1-z}{z^2(e^{1/z}-1)},$$

which clearly has no zero of any order at 0. Hence the singularity is not a pole of any order, and hence must be essential. $\square$

6. Find poles and residues at the poles of the function $f(z) = \frac{e^{2iz}}{1+4z^2}$.

Proof. Factoring the denominator, we have

$$f(z) = \frac{e^{2iz}}{1+4z^2} = \frac{e^{2iz}}{4(z^2+\frac{1}{4})} = \frac{e^{2iz}}{4\left(z+\frac{i}{2}\right)\left(z-\frac{i}{2}\right)}.$$

This tells us that there are simple poles at $z_1 = \frac{i}{2}$ and $z_2 = -\frac{i}{2}$. Since these poles are simple, we can easily calculate the residues as follows.

$$\operatorname{res}(f, z_1) = \lim_{z\to\frac{i}{2}} \left(z - \frac{i}{2}\right) f(z) = \lim_{z\to\frac{i}{2}} \frac{e^{2iz}}{4\left(z+\frac{i}{2}\right)} = \frac{e^{2i(i/2)}}{4(2i/2)} = \frac{e^{-1}}{4i},$$

and

$$\operatorname{res}(f, z_2) = \lim_{z\to -i/2} \left(z + \frac{i}{2}\right) f(z) = \lim_{z\to -i/2} \frac{e^{2iz}}{4(z-i/2)} = \frac{e^{2i(-i/2)}}{4(-2i/2)} = -\frac{e}{4i}.$$

$\square$

7. Find poles and residues at the poles of the function $f(z) = \frac{e^z}{z^4}$.

Proof. It is immediate that $f(z)$ has a pole of order 4 at $z = 0$, and this is the only singularity. We use the formula

$$a_{-1} = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}((z-z_0)^m f(z))$$

to find the residue of the pole of order m. Thus, for $f(z)$ given above, we have

$$\operatorname{res}(f, 0) = \frac{1}{3!} \lim_{z\to 0} \frac{d^3}{dz^3}\left(z^4 \cdot \frac{e^z}{z^4}\right) = \frac{1}{3!} \lim_{z\to 0} e^z = \frac{1}{3!} = \frac{1}{6}.$$

$\square$

8. Suppose $f(z)$ is analytic in a domain D, $z_0 \in D$, $f(z_0) = 0$ and $F(z) = \frac{f(z)}{z-z_0}$. Show that $F(z)$ has a removable singularity at z_0.

Proof. Recall from Riemann's Theorem for Removable Singularities that if a function g is analytic in a punctured disc $\{z : 0 < |z - z_0| < R\}$ and $\lim_{z\to z_0}(z - z_0)g(z) = 0$, then g has a removable singularity at z_0. The result we are asked to prove follows immediately from this theorem for the following reason. First, we note that since $f(z)$ is analytic in D, there exists an $\epsilon > 0$ such that $f(z)$ is analytic in $B(z_0, \epsilon)$. It follows that $F(z)$ is analytic in the punctured disc $\{z : 0 < |z - z_0| < \epsilon\}$.

Furthermore,

$$\lim_{z\to z_0}(z-z_0)F(z)=\lim_{z\to z_0}f(z)=f(z_0)=0$$

by assumption, so the hypotheses of Riemann's Theorem are satisfied; hence $F(z)$ has a removable singularity at z_0. □

9. Use the principle of argument to find the number of zeros of the function $f(z) = z^4+z^3-2z^2+2z+4$ in the first quadrant.

Proof. Let $R>0$ and let $C=C_1\cup C_2\cup C_3$ where $C_1=[0,R]$, $C_2=Re^{i\theta}$ for $0\le\theta\le\frac{\pi}{2}$, and $C_3=[iR,0]$. Let us consider C_1 first.

On C_1, we have

$$\begin{aligned} f(z)=f(x)&=x^4+x^3-2x^2+2x+4\\ &=\begin{cases} x^4+x^3+4-2x(x-1)\\ x^2(x+2)(x-1)+2x+4\end{cases} \geq \begin{cases} x^4+x^3+4 & \text{if } 0\le x\le 1\\ 2x+4 & \text{if } x\ge 1\end{cases} >0. \end{aligned}$$

Hence on C_1, we have $f(z)=\operatorname{Re}(f(z))>0$, and so $\arg(f(R))=0$ and $\arg(f(0))=0$; hence, $\operatorname{var}(\arg(f(z)),C_1)=\arg(f(R))-\arg(f(0))=0$.

We consider C_2 next. Note that

$$f(z)=z^4\left(1+\frac{z^3-2z^2+2z+4}{z^4}\right). \tag{9}$$

Let $\omega=\frac{z^3-2z^2+2z+4}{z^4}$, and note that since $|z|=R$ we have

$$\begin{aligned} |\omega|=\left|\frac{z^3-2z^2+2z+4}{z^4}\right| &\le \frac{|z|^3+2|z|^2+2|z|+4}{|z|^4}\\ &=\frac{R^3+2R^2+2R+4}{R^4}\\ &<\frac{2R^3}{R^4}\\ &=\frac{2}{R} \end{aligned}$$

for sufficiently large R. Hence, $|\omega|\to 0$ as $R\to\infty$. Noting that $z=Re^{it}$ where $0\le t\le \pi/2$ on C_2 and using (9), we have

$$f(z)=R^4e^{4it}(1+\omega).$$

Since $|\omega|<2/R$ for sufficiently large R, it follows that $|\arg(1+\omega)|$ can be made arbitrarily small as $R\to\infty$. Since the argument of a product is equal to the sum of the arguments, we have

$$\arg(f(z))=4t+\arg(1+\omega)$$

and hence $\operatorname{var}(\arg(f(z),C_2)=4(\pi/2-0)+\epsilon_1=2\pi+\epsilon_1$, where $\epsilon_1\to 0$ as $R\to\infty$.

We now consider C_3. Note that on C_3, we have $z=iy$ for some $0\le y\in\mathbb{R}$. Thus

$$\begin{aligned} f(z)=f(iy)&=(iy)^4+(iy)^3-2(iy)^2+2(iy)+4\\ &=y^4-iy^3+2y^2+2iy+4 \end{aligned}$$

$$= (y^4 + 2y^2 + 4) + i(2y - y^3)$$
$$= (y^2 + 1)^2 + 3 + i(2y - y^3),$$

from which we note that $\mathrm{Re}(f(iy)) > 0$. Thus, $f(iy)$ is either in the first or fourth quadrant. Let $\theta = \arg(f(z))$, and note that $\tan\theta = \mathrm{Im}(f(z))/\mathrm{Re}(f(z))$. It follows that

$$\tan\theta = \frac{2y - y^3}{y^4 + 2y^2 + 4} \to 0$$

as y (i.e. R) $\to \infty$. Thus $\theta \to 0$ as $y \to \infty$. Since $f(0) = 4$ (and hence $\arg(f(0)) = 0$), it follows that $\mathrm{var}(\arg(f(z)), C_3) = \epsilon_2 \to 0$ as $R \to \infty$.

Hence, we have $\mathrm{var}(\arg(f(z)), C) = 2\pi + \epsilon_1 + \epsilon_2$, where $\epsilon_1, \epsilon_2 \to 0$ as $R \to \infty$. But C is a closed contour, and so $\mathrm{var}(\arg(f(z)), C)$ must be an integer multiple of 2π, and so $\mathrm{var}(\arg(f(z)), C) = 2\pi$. By the argument principle, we have

$$N - P = \frac{1}{2\pi}\mathrm{var}(\arg(f(z)), C) = 1,$$

and since $P = 0$, it follows that $N = 1$. $\square$

10. Use Rouché's Theorem to determine the number of solutions of equation $e^z = 2z + 1$ such that $|z| < 1$.

Proof. Write $f(z) = -2z$ and $g(z) = e^z - 1$. Note that $f(z) + g(z) = e^z - 2z - 1$, and so to find the number of solutions with $|z| < 1$ of $e^z = 2z + 1$, it suffices to find the number of zeros of $f + g$ inside the unit circle $C = \{z : |z| = 1\}$. In order to use Rouché's Theorem, we must show that f and g are analytic in a domain $D \supset C$, and that $|f(z)| > |g(z)|$ on C.

Clearly f and g are entire, and so they are analytic in a domain $D \supset C$. It remains to show that $|f| > |g|$ on C.

First of all, note that $|f(z)| = |-2z| = |-2||z| = 2$ for any $z \in C$. We calculate $|g(z)|$ on C in the following way. Note that

$$e^z - 1 = \int_{[0,z]} e^{\zeta}\, d\zeta = \int_0^1 e^{zt} z\, dt,$$

where the first equality follows from the fundamental theorem of calculus for complex functions (Theorem 4A in Chen), and the second equality follows using the parametrization of $[0, z]$ of $\zeta = zt$ for $0 \le t \le 1$, and hence $d\zeta = z\, dt$.

Now note that if $z = x + iy \in C$, then $x \le 1$ and hence

$$|e^{zt}| = |e^{tx+ity}| = e^{tx} \le e^t.$$

Thus we have

$$|g(z)| = |e^z - 1| = \left|\int_0^1 e^{zt} z\, dt\right| \le \int_0^1 |e^{zt} z|\, dt = \int_0^1 |e^{zt}||z|\, dt \le \int_0^1 e^t\, dt = e - 1.$$

Thus, for every $z \in C$, we have $|f(z)| = 2 > e - 1 \ge |g(z)|$, and thus can use Rouché's Theorem to assert that f and $f + g$ have the same number of zeros inside C. Clearly f has one zero inside

C; the zero $z = 0$. Thus $f + g$ has one zero inside C as well, and hence the equation has one solution. $\square$

11. Evaluate the integral $\displaystyle\int_0^\pi \frac{d\theta}{a + \cos\theta}$ $(a > 1)$.

Proof. Note that

$$\int_0^\pi \frac{d\theta}{a+\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}. \qquad \text{Needs proof.} \tag{10}$$

Now substitute $z = e^{i\theta} = \cos\theta + i\sin\theta$, so that $dz = ie^{i\theta}\ d\theta$, or $d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz} = -i\frac{dz}{z}$. Also, note that $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + z^{-1})$. Now let C be the unit circle. Thus

$$\int_0^{2\pi} \frac{d\theta}{a+\cos\theta} = -i\int_C \frac{dz}{z(a + \frac{1}{2}(z+z^{-1}))} = -i\int_C \frac{2\ dz}{z^2+2az+1}.$$

By the original equality (10), it follows that

$$\int_0^\pi \frac{d\theta}{a+\cos\theta} = -i\int_C \frac{dz}{z^2+2az+1}. \tag{11}$$

Now note by the quadratic formula that the roots of $z^2 + 2az + 1$ are

$$z = \frac{-2a \pm \sqrt{(2a)^2-4}}{2} = \frac{-2a \pm \sqrt{4a^2-4}}{2} = -a \pm \sqrt{a^2-1}.$$

Let $\alpha = -a + \sqrt{a^2-1}$ and $\beta = -a - \sqrt{a^2-1}$. Note that $\alpha\beta = 1$, and since $|\beta| > 1$, it follows that $|\alpha| < 1$. Hence $f(z) = \frac{1}{z^2+2az+1}$ is analytic in some domain containing C, except for a simple pole at α inside C. Now

$$\operatorname{res}(f,\alpha) = \lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha}\frac{1}{z-\beta} = \frac{1}{\alpha-\beta} = \frac{1}{2\sqrt{a^2-1}}.$$

It follows from Cauchy's Residue Theorem (Theorem 10B) that

$$\int_C f(z)\ dz = 2\pi i\ \operatorname{res}(f,\alpha) = \frac{2\pi i}{2\sqrt{a^2-1}},$$

and so by (10) and (11), we have

$$\int_0^\pi \frac{d\theta}{a+\cos\theta} = -i\int_C f(z)\ dz = \frac{\pi}{\sqrt{a^2-1}},$$

completing the computation. $\square$

12. Evaluate the integral $\displaystyle\int_{-\infty}^\infty \frac{x^2}{(x^2+a^2)^3}\ dx$ $(a > 0)$.

Proof. Let $f(z) = \dfrac{z^2}{(z^2+a^2)^3} = \dfrac{z^2}{(z+ia)^3(z-ia)^3}$. Thus f has poles of order 3 at $z = \pm ia$. Consider the Jordan contour $C = [-R, R] \cup C_R$, where $R > a$ and C_R is the half-circle of radius R in the upper half-plane. The only pole inside C is $z = ia$. By the Cauchy Residue Theorem, we have

$$2\pi i \ \mathrm{res}(f, ia) = \int_C f(z)\ dz = \int_{-R}^{R} f(x)\ dx + \int_{C_R} f(z)\ dz.$$

We now calculate (rather miserably) $\mathrm{res}(f, ia)$. We have

$$\begin{aligned}
\mathrm{res}(f, ia) &= \frac{1}{2} \lim_{z\to ia} \frac{d^2}{dz^2}\left((z-ia)^3 \frac{z^2}{(z^2+a^2)^3}\right) \\
&= \frac{1}{2} \lim_{z\to ia} \frac{d^2}{dz^2}\left(\frac{z^2}{(z+ia)^3}\right) \\
&= \frac{1}{2} \lim_{z\to ia} \frac{d}{dz}\left(\frac{2z(z+ia)^3 - 3z^2(z+ia)^2}{(z+ia)^6}\right) \\
&= \frac{1}{2} \lim_{z\to ia} \frac{d}{dz}\left(\frac{2z(z+ia) - 3z^2}{(z+ia)^4}\right) \\
&= \frac{1}{2} \lim_{z\to ia} \frac{d}{dz}\left(\frac{2z}{(z+ia)^3} - \frac{3z^2}{(z+ia)^4}\right) \\
&= \frac{1}{2} \lim_{z\to ia} \left(\frac{2(z+ia)^3 - 6z(z+ia)^2}{(z+ia)^6} - \frac{6z(z+ia)^4 - 12z^2(z+ia)^3}{(z+ia)^8}\right) \\
&= \frac{1}{2} \lim_{z\to ia} \left(\frac{2}{(z+ia)^3} - \frac{12z}{(z+ia)^4} + \frac{12z^2}{(z+ia)^5}\right) \\
&= \frac{1}{2} \left(\frac{2}{(2ia)^3} - \frac{12ia}{(2ia)^4} - \frac{12a^2}{(2ia)^5}\right) \\
&= \frac{1}{2} \left(\frac{2i}{8a^3} - \frac{6i}{8a^3} + \frac{3i}{8a^3}\right) \\
&= -\frac{i}{16a^3}.
\end{aligned}$$

It follows that

$$\int_{-R}^{R} f(x)\ dx + \int_{C_R} f(z)\ dz = 2\pi i\left(-\frac{i}{16a^3}\right) = \frac{\pi}{8a^3}.$$

Now suppose $z \in C_R$ so that $|z| = R$. We want to show

$$\left|\frac{z^2}{(z^2+a^2)^3}\right| \leq \frac{R^2}{(R^2-a^2)^3}.$$

Since $|z^2| = R^2$, this is equivalent to showing $|z^2+a^2|^3 \geq (R^2-a^2)^3$, or that $|z^2+a^2| \geq R^2 - a^2$. Let $z = x + iy$ and remember that $x^2 + y^2 = R^2$. Note, then, that

$$\begin{aligned}
|z^2+a^2| &= |z+ia||z-ia| \\
&= |x+i(y+a)||x+i(y-a)| \\
&= \sqrt{x^2+(y+a)^2} \cdot \sqrt{x^2+(y-a)^2} \\
&= \sqrt{x^2+y^2+2ay+a^2} \cdot \sqrt{x^2+y^2-2ay+a^2} \\
&= \sqrt{R^2+2ay+a^2} \cdot \sqrt{R^2-2ay+a^2} \\
&= \sqrt{R^4+2a^2R^2-4a^2y^2+a^4}
\end{aligned}$$

and

$$\begin{aligned} R^2 - a^2 &= \sqrt{(R^2 - a^2)^2} \\ &= \sqrt{R^4 - 2a^2R^2 + a^4} \end{aligned}$$

Thus, our problem reduces to checking that $R^4 - 2a^2R^2 + a^4 \leq R^4 + 2a^2R^2 - 4a^2y^2 + a^4$. After canceling common summands, we have left to check

$$-2a^2R^2 \overset{?}{\leq} 2a^2R^2 - 4a^2y^2 \;\Rightarrow\; 4a^2y^2 \overset{?}{\leq} 4a^2R^2.$$

Now since $4a^2 > 0$, we can cancel this factor from both sides without changing the inequality sign. Thus our problem reduces to determining

$$y^2 \overset{?}{\leq} R^2.$$

This is clearly true, though, since $x^2 + y^2 = R^2$ and $x^2 \geq 0$. Thus $R^2 - a^2 \leq |z^2 + a^2|$, as claimed, and hence

$$|f(z)| \leq \frac{R^2}{(R^2 - a^2)^3}$$

for every $z \in C_R$.

Hence, by the $\left|\int f\right| \leq \int |f|$ and ML-inequalities, we have

$$\left|\int_{C_R} f(z)\ dz\right| \leq \frac{R}{(R^2 - a^2)^3}\pi R \to 0$$

as $R \to \infty$. Hence, letting $R \to \infty$ we have

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)^3}\ dx = \int_{-\infty}^{\infty} f(x)\ dx = \frac{\pi}{8a^3},$$

which completes the computation. □

13. Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 + 3}{x^4 + 5x^2 + 4}\ dx$.

Proof. Consider the function $f(z) = \dfrac{z^2 + 3}{z^4 + 5z^2 + 4} = \dfrac{z^2 + 3}{(z^2 + 1)(z^2 + 4)} = \dfrac{z^2 + 3}{(z + i)(z - i)(z + 2i)(z - 2i)}$. This function has simple poles at $\pm i$ and $\pm 2i$. Let $R > 2$. Consider the Jordan contour $C = [-R, R] \cup C_R$, where C_R is the upper semi-circle centered at the origin with radius R. Then by the residue theorem, it follows that

$$\int_{-R}^{R} f(x)\ dx + \int_{C_R} f(z)\ dz = \int_C f(z)\ dz = 2\pi i\left(\operatorname{res}(f, i) + \operatorname{res}(f, 2i)\right).$$

Hence, we must calculate these residues. We have

$$\operatorname{res}(f, i) = \lim_{z \to i}(z - i)f(z) = \lim_{z \to i} \frac{z^2 + 3}{(z + i)(z + 2i)(z - 2i)} = \frac{2}{(2i)(3i)(-i)} = -\frac{2}{6i^3} = \frac{1}{3i}$$

and

$$\operatorname{res}(f, 2i) = \lim_{z \to 2i}(z - 2i)f(z) = \lim_{z \to 2i} \frac{z^2 + 3}{(z + i)(z - i)(z + 2i)} = \frac{-1}{(3i)(i)(4i)} = \frac{1}{12i}.$$

Thus, we have

$$\int_{-R}^{R} f(x)\ dx + \int_{C_R} f(z)\ dz = 2\pi i \left(\frac{1}{3i} + \frac{1}{12i}\right) = \frac{5\pi}{6}.$$

Now since $\left|\int_{C_R} f(z)\ dz\right| \to 0$ as $R \to \infty$, we have

$$\int_{-\infty}^{\infty} f(x)\ dx = \frac{5\pi}{6},$$

completing the calculation. □

14. Evaluate the integral $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\ dx \quad (a > 0)$.

Proof. Consider the function $F(z) = \frac{e^{iz}}{z^2+a^2} = \frac{e^{iz}}{(z+ia)(z-ia)}$, which has simple poles at $\pm ia$. Consider the Jordan contour $C = C_R \cup [-R, R]$, where $R > a$. By the residue theorem, we have

$$\int_{-R}^{R} F(x)\ dx + \int_{C_R} F(z)\ dz = \int_C F(z)\ dz = 2\pi i \operatorname{res}(f, ia).$$

Note that

$$\operatorname{res}(f, ia) = \lim_{z\to ia} (z-ia)F(z) = \lim_{z\to ia} \frac{e^{iz}}{z+ia} = \frac{e^{-a}}{2ia},$$

so that

$$\int_{-R}^{R} F(x)\ dx + \int_{C_R} F(z)\ dz = 2\pi i\ \frac{e^{-a}}{2ia} = \frac{\pi e^{-a}}{a}.$$

Now by Jordan's Lemma, we have $\left|\int_{C_R} F(z)\ dz\right| \to 0$ as $R \to \infty$. Thus we have

$$\int_{-\infty}^{\infty} F(x)\ dx = \int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\ dx + i\int_{-\infty}^{\infty} \frac{\sin x}{x^2+a^2}\ dx = \frac{\pi e^{-a}}{a};$$

hence, equating real parts gives us that $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\ dx = \frac{\pi e^{-a}}{a}$. □

15. Evaluate the integral $\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2+a^2)(x^2+b^2)}\ dx \quad (a > 0, b > 0)$.

Proof. Consider the function $F(z) = \frac{z^3 e^{iz}}{(z^2+a^2)(z^2+b^2)}$, which has simple poles at $\pm ia$ and $\pm ib$. Consider as usual the Jordan contour $C = [-R, R] \cup C_R$, where $R > \max\{a, b\}$. By the residue theorem, we have

$$\int_{-R}^{R} F(x)\ dx + \int_{C_R} F(z)\ dz = \int_C F(z)\ dz = 2\pi i\ (\operatorname{res}(F, ia) + \operatorname{res}(F, ib)).$$

We can calculate

$$\operatorname{res}(F, ia) = \lim_{z\to ia} ((z-ia)F(z)) = \lim_{z\to ia} \frac{z^3 e^{iz}}{(z+ia)(z^2+b^2)} = \frac{(ia)^3 e^{-a}}{(2ia)(b^2-a^2)} = \frac{a^2 e^{-a}}{2(a^2-b^2)}$$

and

$$\operatorname{res}(F, ib) = \lim_{z \to ib}((z - ib)F(z)) = \lim_{z \to ib} \frac{z^3 e^{iz}}{(z^2 + a^2)(z + ib)} = \frac{(ib)^3 e^{-b}}{(a^2 - b^2)(2ib)} = \frac{b^2 e^{-b}}{2(b^2 - a^2)}.$$

Hence we have

$$\int_{-R}^{R} F(x)\ dx + \int_{C_R} F(z)\ dz = \int_C F(z)\ dz = \pi i \left(\frac{a^2 e^{-a} - b^2 e^{-b}}{a^2 - b^2} \right).$$

Note that by the Jordan Lemma and the fact that the degree of the polynomial part of the denominator exceeds the degree of the polynomial part of the numerator, we have that $\left| \int_{C_R} F(z)\ dz \right| \to 0$ as $R \to \infty$. Hence, we have

$$\int_{-\infty}^{\infty} F(x)\ dx = \frac{\pi i (a^2 e^{-a} - b^2 e^{-b})}{a^2 - b^2}.$$

Equating imaginary parts gives us

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2 + a^2)(x^2 + b^2)}\ dx = \frac{\pi (a^2 e^{-a} - b^2 e^{-b})}{a^2 - b^2},$$

which completes the computation. $\square$

16. Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\ dx$.

Proof. Consider the function $F(z) = \dfrac{e^{iz}}{z}$. We cannot use the same technique as before since the singularity $z = 0$ is on the contour. Thus we must "bend around the contour." Consider the contour $C = [-R, -\delta] \cup K(\delta) \cup [\delta, R] \cup C_R$, where everything is as before except for $K(\delta)$, which is the semicircular arc in the lower half-plane centered at 0 with radius $\delta < R$. By the residue theorem, we have

$$\int_{-R}^{-\delta} F(x)\ dx + \int_{K(\delta)} F(z)\ dz + \int_{\delta}^{R} F(x)\ dx + \int_{C_R} F(z)\ dz = \int_C F(z)\ dz = 2\pi i \operatorname{res}(F, 0).$$

Now observe that

$$F(z) = \frac{e^{iz}}{z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{n=-1}^{\infty} \frac{z^n}{(n+1)!},$$

so that $\operatorname{res}(F, 0) = 1$ and $F(z) = \frac{1}{z} + G(z)$, where $G(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$ is an entire function.

Notice that $K(\delta)$ can be parametrized as follows. Let $z = \delta e^{it}$ for $\pi \leq t \leq 2\pi$. Then $dz = i\delta e^{it} dt$, and hence

$$\int_{K(\delta)} \frac{dz}{z} = \int_{\pi}^{2\pi} \frac{i\delta e^{it}}{\delta e^{it}}\ dt = \pi i.$$

Hence, we have $\int_{K(\delta)} F(z)\ dz = \pi i + \int_{K(\delta)} G(z)\ dz$, and thus we have

$$\int_{-R}^{-\delta} F(x)\ dx + \int_{K(\delta)} G(z)\ dz + \int_{\delta}^{R} F(x)\ dx + \int_{C_R} F(z)\ dz = \pi i. \tag{12}$$

We also make the observation that for $|z| \leq 1$, we have

$$|G(z)| = \left|\sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}\right| \leq \sum_{n=0}^{\infty} \left|\frac{z^n}{(n+1)!}\right| = \sum_{n=0}^{\infty} \frac{|z|^n}{(n+1)!} \leq \sum_{n=0}^{\infty} \frac{1}{(n+1)!} < \infty,$$

and so there exists some number $M > 0$ such that $|G(z)| < M$ whenever $|z| \leq 1$. Hence, whenever $\delta < 1$, we have

$$\left|\int_{K(\delta)} G(z)\ dz\right| \leq M\pi\delta. \tag{13}$$

We also have

$$\left|\int_{C_R} F(z)\ dz\right| \leq \int_{C_R} \left|\frac{e^{iz}}{z}\right|\ |dz| \leq \frac{1}{R}\int_{C_R} |e^{iz}|\ |dz| < \frac{\pi}{R}, \tag{14}$$

where the last inequality follows from Jordan's Lemma. Combining (12), (13), and (14), if $\delta < 1$ we have

$$\begin{aligned}\left|\int_{-R}^{-\delta} F(x)\ dx + \int_{\delta}^{R} F(x)\ dx - \pi i\right| &= \left|-\int_{K(\delta)} G(z)\ dz - \int_{C_R} F(z)\ dz\right| \\ &\leq \left|\int_{K(\delta)} G(z)\ dz\right| + \left|\int_{C_R} F(z)\ dz\right| < M\pi\delta + \frac{\pi}{R}.\end{aligned}$$

Hence, letting $\delta \to 0$ and $R \to \infty$, we have

$$\int_{-\infty}^{\infty} F(x)\ dx = \pi i.$$

Equating imaginary parts, we have

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\ dx = \pi,$$

which completes the computation. $\square$

17. Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2}\ dx \quad (a > 0)$**.**

Proof. Consider the function $F(z) = \dfrac{e^{iz}}{a^2 - z^2}$, which has simple poles at $\pm a$. Consider the Jordan Contour

$$C = [-R, -a - \delta_1] \cup J_1(\delta_1) \cup [-a + \delta_1, a - \delta_2] \cup J_2(\delta_2) \cup [a + \delta_2, R] \cup C_R.$$

where $R < 2a$ and $0 < \delta_1, \delta_2 < a$. By Cauchy's Integral Theorem, we have

$$\int_{-R}^{-a-\delta_1} + \int_{J_1(\delta_1)} + \int_{-a+\delta_1}^{a-\delta_2} + \int_{J_2(\delta_2)} + \int_{a+\delta_2}^{R} + \int_{C_R} = 0. \tag{15}$$

Now we have

$$\operatorname{res}(F, a) = \lim_{z \to a} (z - a)F(z) = -\frac{e^{ia}}{2a}$$

and

$$\operatorname{res}(F, -a) = \lim_{z \to -a} (z + a)F(z) = \frac{e^{-ia}}{2a}.$$

Hence $F(z) = \dfrac{e^{-ia}}{2a(z+a)} + G_1(z) = -\dfrac{e^{ia}}{2a(z-a)} + G_2(z)$, where G_1 and G_2 are analytic in neighborhoods of $-a$ and a, respectively. We now wish to find $\int_{J_1(\delta_1)} \frac{e^{-ia}}{2a(z+a)}\ dz$ and $\int_{J_2(\delta_2)} -\frac{e^{ia}}{2a(z-a)}\ dz$. First, note that on $J_1(\delta_1)$, we have $z = -a + \delta_1 e^{it}$ for $t \in [\pi, 0]$. Hence we have

$$\int_{J_1(\delta_1)} \frac{e^{-ia}}{2a(z+a)}\ dz = \frac{e^{-ia}}{2a} \int_{\pi}^{0} i\ dz = -\frac{i\pi e^{-ia}}{2a}.$$

Likewise, we have

$$-\int_{J_2(\delta_2)} \frac{e^{ia}}{2a(z-a)} = -\frac{e^{ia}}{2a} \int_{\pi}^{0} i\ dz = \frac{i\pi e^{ia}}{2a}.$$

Thus by (15) we have

$$\int_{-R}^{-a-\delta_1} + \int_{J_1(\delta_1)} G_1(z)\ dz + \int_{-a+\delta_1}^{a-\delta_2} + \int_{J_2(\delta_2)} G_2(z)\ dz + \int_{a+\delta_2}^{R} + \int_{C_R} = \frac{i\pi(e^{-ia} - e^{ia})}{2a},$$

and perhaps more usefully,

$$(16) \qquad \int_{-R}^{-a-\delta_1} + \int_{-a+\delta_1}^{a-\delta_2} + \int_{a+\delta_2}^{R} - \frac{i\pi(e^{-ia} - e^{ia})}{2a} = -\int_{J_1(\delta_1)} G_1(z)\ dz - \int_{J_2(\delta_2)} G_2(z)\ dz - \int_{C_R}.$$

Now the right hand side goes to 0 as $\delta_1, \delta_2 \to 0$ and $R \to \infty$, and so we have

$$\int_{-\infty}^{\infty} F(x)\ dx = \frac{i\pi(e^{-ia} - e^{ia})}{2a} = \frac{\pi}{a}\frac{(e^{ia} - e^{-ia})}{2i} = \frac{\pi \sin a}{a};$$

equating real parts, we have

$$\int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2}\ dx = \frac{\pi \sin a}{a},$$

completing the computation. □

18. Evaluate the integral $\displaystyle\int_0^{\infty} \frac{x^{\alpha-1}}{1+x}\ dx \quad (0 < \alpha < 1)$.

Proof. Consider the substitution $x = u^2$. Then we have

$$\int_0^{\infty} \frac{x^{\alpha-1}}{1+x}\ dx = \int_0^{\infty} \frac{u^{2\alpha-2}}{1+u^2}\ 2u\ du = 2\int_0^{\infty} \frac{u^{2\alpha+1}}{u^2+u^4}\ du.$$

Thus, consider the function

$$F(z) = \frac{z^{2\alpha+1}}{z^2+z^4} = \frac{z^{2\alpha+1}}{z^2(1+z^2)},$$

which has a pole of order 2 at $z = 0$ and simple poles at $z = \pm i$. Consider the Jordan contour

$$C = [-R, -\delta] \cup J(\delta) \cup [\delta, R] \cup C_R,$$

where $J(\delta)$ is the (negatively oriented) upper-half circle centered at the origin with radius $0 < \delta < 1 < R$, and where C_R is the standard curve we have been considering. By the residue theorem,

$$(17) \qquad \int_C F(z)\ dz = \int_{-R}^{\delta} F(z)\ dz + \int_{J(\delta)} F(z)\ dz + \int_{\delta}^{R} F(z)\ dz + \int_{C_R} F(z)\ dz = 2\pi i\ \mathrm{res}(F, i)$$

since i is the only singularity inside C. Computing this residue, we have

$$\operatorname{res}(F,i) = \lim_{z\to i}(z-i)F(z) = \frac{i^{2\alpha+1}}{i^2(2i)} = -\frac{1}{2}i^{2\alpha} = -\frac{1}{2}(e^{i\pi/2})^{2\alpha} = -\frac{1}{2}e^{\pi i\alpha}.$$

It follows from (17) that

$$\int_{-R}^{\delta} F(z)\ dz + \int_{J(\delta)} F(z)\ dz + \int_{\delta}^{R} F(z)\ dz + \int_{C_R} F(z)\ dz = -\pi i e^{\pi i\alpha}.$$

Now note that

$$\left|\int_{J(\delta)} F(z)\ dz\right| \le \frac{\delta^{2\alpha+1}}{\delta^2-\delta^4}\pi\delta = \frac{\delta^{2\alpha}\delta^2}{\delta^2(1-\delta^2)}\pi \to 0$$

and

$$\left|\int_{C_R} F(z)\ dz\right| \le \frac{R^{2\alpha+1}}{R^4-R^2}\pi R = \frac{R^{2\alpha}R^2}{R^2(R^2-1)}\pi \to 0$$

as $\delta \to 0$ and $R \to \infty$, respectively (where the second one holds since $\alpha < 1$). Hence,

$$\int_{-\infty}^{\infty} \frac{z^{2\alpha+1}}{z^2+z^4}\ dz = \int_{-\infty}^{\infty} F(z)\ dz = -\pi i e^{\pi i\alpha}.$$

If we let $f(z) = \frac{1}{z+z^2}$, then we have $f(z^2) = \frac{1}{z^2+z^4}$, and hence

$$\int_{-\infty}^{\infty} F(z)\ dz = \int_{-\infty}^{\infty} z^{2\alpha+1} f(z^2)\ dz.$$

We make an estimate to help with this. Note in the following computation that, going from line 1 to line 2, we made the substitution (in the second integral only) of $z = -\zeta$ and so $dz = -d\zeta$.

$$\begin{aligned}
\int_{-\infty}^{\infty} z^{2\alpha+1} f(z^2)\ dz &= \int_{0}^{\infty} z^{2\alpha+1} f(z^2)\ dz + \int_{-\infty}^{0} z^{2\alpha+1} f(z^2)\ dz \\
&= \int_{0}^{\infty} z^{2\alpha+1} f(z^2)\ dz - \int_{\infty}^{0} (-\zeta)^{2\alpha+1} f((-\zeta)^2)\ d\zeta \\
&= \int_{0}^{\infty} z^{2\alpha+1} f(z^2)\ dz + \int_{0}^{\infty} (-\zeta)^{2\alpha+1} f((-\zeta)^2)\ d\zeta \\
&= \int_{0}^{\infty} z^{2\alpha+1} f(z^2)\ dz + \int_{0}^{\infty} (-z)^{2\alpha+1} f((-z)^2)\ dz \\
&= \int_{0}^{\infty} z^{2\alpha+1} f(z^2)\ dz - \int_{0}^{\infty} (-z)^{2\alpha+1}(z) f(z^2)\ dz \\
&= \int_{0}^{\infty} z^{2\alpha+1} f(z^2)\ dz - \int_{0}^{\infty} e^{2\pi i\alpha} z^{2\alpha}(z) f(z^2)\ dz \\
&= (1-e^{2\pi i\alpha}) \int_{0}^{\infty} z^{2\alpha+1} f(z^2)\ dz.
\end{aligned}$$

The sixth equality above follows from the identity $(-z)^{2\alpha} = e^{2\pi i\alpha}z^{2\alpha}$, which is an easy identity since $-1 = e^{i\pi}$. Hence we have

$$(1-e^{2\pi i\alpha}) \int_{0}^{\infty} \frac{z^{2\alpha+1}}{z^2+z^4}\ dz = \int_{-\infty}^{\infty} F(z)\ dz = -\pi i e^{\pi i\alpha},$$

from which it follows from the opening substitution that

$$\frac{1}{2}\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx = \int_0^\infty \frac{u^{2\alpha+1}}{u^2+u^4}\,du = -\frac{\pi i e^{\pi i\alpha}}{(1-e^{2\pi i\alpha})} = -\frac{\pi i}{e^{-\pi i\alpha}-e^{\pi i\alpha}} = -\frac{\pi i}{-2\sin(\pi\alpha)} = \frac{\pi}{2\sin(\pi\alpha)}.$$

Thus we have

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx = \frac{\pi}{\sin(\pi\alpha)}.$$

This completes the computation. □

19. Show that the real part of $z^{1/2}$ ($z \neq 0$) is always positive.

Proof. We show that the real part of $z^{1/2}$ is non-negative, for if $z = -c$ for $c \in \mathbb{R}^+$, then $z^{1/2} = i\sqrt{c}$, and hence $\mathrm{Re}(z) = 0$.

This is most easily seen through polar coordinates. Let $z = re^{i\theta}$ for $\theta \in [-\pi, \pi)$. (Here we are defining the range of values θ so as to give the principal value of the square root.) Then $z^{1/2} = r^{1/2}e^{i\theta/2}$. Notice that $\frac{\theta}{2} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right)$, and so

$$\mathrm{Re}(z) = r^{1/2}\cos\left(\frac{\theta}{2}\right) \geq 0,$$

as desired. □

20. Let $D = \mathbb{C} \setminus \{z \in \mathbb{R} : z \leq 0\}$. Describe all analytic functions $f(z)$ in D such that $z = [f(z)]^n$ (branches of the n^{th} root).

Proof. To have $z = [f(z)]^n$, we must have $f(z) = z^{1/n}$. Let $z = re^{i(\theta+2\pi k)}$ for $r > 0$ and $\theta \neq \pi$ and $k \in \mathbb{N}$. Hence $f(z) = z^{1/n} = e^{2\pi k/n}r^{1/n}e^{i\theta/n}$. Let $\lambda_k = r^{1/n} \cdot e^{2\pi k/n}$, so that $f(z) = \lambda_k e^{i\theta/n}$, and note that $k = 0, \ldots, n-1$ gives you different values, but the values repeat after that. Hence,

$$f(z) = \lambda_k e^{i\theta/n} \quad \text{where} \quad k \in \{0, \ldots, n-1\}$$

describes all such functions. □

21. Prove that there is no branch of $\log$ defined in $\mathbb{C} \setminus \{0\}$.

Proof. Any definition of $\ln(z)$ must be continuous. Hence, a branch of $\ln z$ defined in $\mathbb{C} \setminus \{0\}$ must be continuous; that is, for any sequence $\{z_n\} \subset \mathbb{C} \setminus \{0\}$ such that $z_n \to z \in \mathbb{C} \setminus \{0\}$, we have

$$\lim_{n\to\infty} \ln(z_n) = \ln(z).$$

But note that for $z = 1$, we have two sequences $\{s_n\}$ and $\{t_n\}$ given by

$$s_n = e^{i(1/n)} \quad \text{and} \quad t_n = e^{i(2\pi - 1/n)}.$$

We have

$$\lim_{n\to\infty} s_n = e^0 = 1 \quad \text{and} \quad \lim_{n\to\infty} t_n = e^{2\pi i} = 1.$$

Then note that

$$\lim_{n\to\infty} \ln(s_n) = \lim_{n\to\infty} i \cdot \frac{1}{n} = 0$$

and

$$\lim_{n\to\infty} \ln(t_n) = \lim_{n\to\infty} i \cdot \left(2\pi - \frac{1}{n}\right) = 2\pi i$$

and so since these limits do not agree, it is not a continuous function. Hence, there is no branch of log defined on $\mathbb{C} \setminus \{0\}$. $\square$

Theorem 15 (Riemann's Theorem on Removable Singularities): If $f(z)$ is analytic in $\{z : 0 < |z - z_0| < R\}$ and $\lim_{z \to z_0} (z - z_0) f(z) = 0$, then z_0 is a removable singularity.

Proof. Suppose z is a point in the punctured disc $\{z : 0 < |z - z_0| < R\}$. Let r_1 and r_2 satisfy $0 < r_1 < |z - z_0| < r_2 < R$ and let C_1 and C_2 denote two circles, oriented positively, centered at z_0 with radius r_1 and r_2, respectively. Let

$$g(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z},$$

which is clearly analytic in the punctured disc $\{\zeta : 0 < |\zeta - z_0| < R\}$. Then, performing "surgery" (as we did in the proof of the Cauchy Integral Theorem), we have $\int_{C_1} g(\zeta)\, d\zeta - \int_{C_2} g(\zeta)\, d\zeta = 0$, and so

$$\int_{C_1} g(\zeta)\, d\zeta = \int_{C_2} g(\zeta)\, d\zeta.$$

Hence, by how we defined g, we have

$$\underbrace{\int_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta}_{=A} - \underbrace{f(z) \int_{C_1} \frac{d\zeta}{\zeta - z}}_{=B} = \underbrace{\int_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta}_{=C} - \underbrace{f(z) \int_{C_2} \frac{d\zeta}{\zeta - z}}_{=D}.$$

We consider the pieces A, B, C, and D separately. Since $C_1 \subset G = \{\zeta : |\zeta - z_0| < |z - z_0|\}$, it follows that $\frac{1}{\zeta - z}$ is analytic in G, and so $B = 0$. However, since z is in the interior of the region bounded by C_2, it follows from a simple argument (using polar coordinates, for instance) that $D = f(z) 2\pi i$.

Now by the assumption made in the statement of the theorem, given any $\epsilon > 0$ there exists a $\delta > 0$ such that $|(\zeta - z_0) f(\zeta)| < \epsilon$ whenever $|\zeta - z_0| < \delta$. Without loss of generality since we are making δ arbitrarily small anyway, assume $\delta < \frac{1}{2}|z - z_0|$. Now let $r_1 = \delta$ and note

$$\left| \int_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta \right| = \left| \int_{C_1} \frac{(\zeta - z_0) f(\zeta)}{(\zeta - z_0)(\zeta - z)}\, d\zeta \right| \le \frac{\epsilon}{(\inf_{\zeta \in C_1}\{\zeta - z_0\})(\inf_{\zeta \in C_1}\{\zeta - z\})} 2\pi\delta$$
$$= \frac{\epsilon}{\delta(|z - z_0| - \delta)} 2\pi\delta = \frac{2\pi\epsilon}{|z - z_0| - \delta} \le \frac{4\pi\epsilon}{|z - z_0|},$$

by the ML-inequality. Since $\epsilon > 0$ was arbitrary, it follows that $A = 0$. Thus, combining this information gives

$$f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

which holds for every $z \in \{z : 0 < |z - z_0| < r_2\}$. Now this shows that f is analytic in every punctured disc $\{z : 0 < |z - z_0| < r_2\}$. To show that z_0 is a removable singularity, all we must do is define $f(z_0)$, which we may do so as

$$f(z_0) = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z_0}\, d\zeta,$$

which makes f analytic in the disc $\{z : |z - z_0| < r_2\}$. Since r_2 can be chosen arbitrarily close to R, the result follows by the definition of a removable singularity. $\square$

Theorem 16: Suppose $f(z)$ is analytic in $\{z : 0 < |z - z_0| < R\}$. Then z_0 is a pole if and only if $\lim_{z\to z_0} |f(z)| = \infty$.

Proof. Recall that, by definition, z_0 is a pole of order n if and only if we can write

$$f(z) = \frac{g(z)}{(z - z_0)^n},$$

where $n \in \mathbb{N}$ and g is analytic in some neighborhood of z_0 and where $g(z_0) \neq 0$. Thus, by the definition of z_0 being a pole implies that

$$\lim_{z\to z_0} |f(z)| = \lim_{z\to z_0} \left| \frac{g(z)}{(z - z_0)^n} \right| = \infty$$

since $g(z_0) \neq 0$ (i.e. so we don't have to worry about an indeterminate case $0/0$).

Now suppose $\lim_{z\to z_0} |f(z)| = \infty$. This implies that $f(z) \neq 0$ is some punctured disc $\{z : 0 < |z - z_0| < r\}$, where $r \leq R$. This implies that the function $F(z) = \frac{1}{f(z)}$ is analytic in $\{z : z < |z - z_0| < r\}$, and has an isolated singularity at z_0. Now by assumption we have

$$\lim_{z\to z_0} (z - z_0)F(z) = \frac{0}{\infty} = 0,$$

and so by Riemann's Theorem on Removable Singularities, we have shown that F has a removable singularity at z_0. Hence, we define $F(z_0) = 0$ to make F analytic in the disc $\{z : |z - z_0| < r\}$. Now $F \not\equiv 0$ in $\{z : |z - z_0| < r\}$, and so by one of the uniqueness theorems (7F) that

$$F(z) = (z - z_0)^n h(z),$$

where h is analytic in $\{z : |z - z_0| < r\}$ and where $h(z_0) \neq 0$. Now we define

$$g(z) = \frac{1}{h(z)}.$$

Since $h(z_0) \neq 0$ and since h is continuous, there is some neighborhood D of z_0 for which $h(z) \neq 0$ for all $z \in D$. Hence, g is analytic in D. Furthermore, $g(z_0) = \frac{1}{h(z_0)} \neq 0$. This all combines to tell us that

$$f(z) = \frac{1}{F(z)} = \frac{1}{(z - z_0)^n h(z)} = \frac{g(z)}{(z - z_0)^n}$$

where $g(z_0) \neq 0$, which implies that z_0 is a pole of order n by the definition of a pole. □

Theorem 17 (Casorati-Weierstrass): Suppose $f(z)$ is analytic in $\{z : 0 < |z - z_0| < R\}$ and z_0 is an essential singularity. Then for all $\omega \in \mathbb{C}$ and for all $\epsilon > 0$ and $\delta > 0$, there exists a $z \in \mathbb{C}$ such that $0 < |z - z_0| < \delta$ and $|f(z) - \omega| < \epsilon$.

Proof. Suppose to the contrary that the theorem does not hold. Then there exists an $\omega \in \mathbb{C}$ and some real numbers $\epsilon > 0$ and $\delta > 0$ such that for all $z \in \mathbb{C}$, we have $|f(z) - \omega| \geq \epsilon$ whenever $0 < |z - z_0| < \delta$. Consider the function

$$g(z) = \frac{1}{f(z) - \omega}.$$

Since $|f(z)-\omega| \geq \epsilon$ whenever $z \in \{z : 0 < |z-z_0| < \delta\}$, it follows that on this punctured disc $g(z)$ is bounded and also analytic. Furthermore,

$$\lim_{z \to z_0} (z-z_0)g(z) = 0,$$

and hence by Riemann's Theorem on Removable Singularities, it follows that g has a removable singularity at z_0. Hence, we may define $g(z_0)$ appropriately to make g analytic in the disc $\{z : |z-z_0| < \delta\}$, and note that $g \not\equiv 0$ in this disc. By simple algebra we have

$$f(z) = \omega + \frac{1}{g(z)},$$

and this makes sense since $g \not\equiv 0$ on the disc $\{z : |z-z_0| < \delta\}$. If $g(z_0) \neq 0$, then f is analytic at z_0, and if $g(z_0) = 0$, then $\lim_{z \to z_0} |f(z)| = \infty$, and hence f has a pole at z_0 by the previous theorem. Either case contradicts that z_0 is an essential singularity, completing the proof. $\square$

Theorem 18: Suppose $f(z)$ is analytic in $\{z : 0 < |z-z_0| < R\}$ with an isolated singularity at z_0. Then there exist unique functions $f_1(z)$ and $f_2(z)$ such that
(a) $f(z) = f_1(z) + f_2(z)$ in $\{z : 0 < |z-z_0| < R\}$,
(b) f_1 is analytic in $\mathbb{C} \setminus \{z_0\}$,
(c) $f_1(z) \to 0$ as $|z| \to \infty$, and
(d) f_2 is analytic in the disc $D = \{z : |z-z_0| < R\}$.

Proof. Let us consider the same setup as in Theorem 15 (Riemann's Theorem on Removable Singularities). Since $B = 0$ and $D = f(z)2\pi i$, it follows that

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Let

$$f_1(z) = -\frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \text{and} \quad f_2(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Thus $f(z) = f_1(z) + f_2(z)$, as required for part (a). For part (d), note that f_2 represents the same function given in Riemann's Theorem on Removable Singularities, which we showed was analytic. Now f_1 is analytic in every annulus $\{z : |z-z_0| > r_1\}$, and since $r_1 > 0$ can be made arbitrarily small, (b) follows as a result. By part (a), since f_2 and f are independent of the choice of r_1, it follows that f_1 is independent of the choice of r_1. Similarly, f_2 is independent of the choice of r_2. Since f is continuous on the punctured disc centered at z_0, and since C_1 is a compact subset of the punctured disc, it follows that f is bounded on C_1. Thus

$$\lim_{\substack{|z| \to \infty \\ \zeta \in C_1}} \frac{f(\zeta)}{\zeta - z} = 0,$$

and hence

$$\lim_{|z| \to \infty} \int_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta = 0,$$

which establishes (c).

We now argue uniqueness. Suppose g_1 and g_2 are two function satisfying the same conclusions as f_1 and f_2. Then $f_1 + f_2 = g_1 + g_2$, which implies $f_1 - g_1 = g_2 - f_2$ in the punctured disc

$\{z : 0 < |z - z_0| < R\}$. Let

$$F(z) = \begin{cases} g_2(z) - f_2(z) & \text{if } |z - z_0| < R, \\ f_1(z) - g_1(z) & \text{if } |z - z_0| > 0. \end{cases}$$

(This is an odd way to define a function, seeing as how the domains overlap, possibly resulting in different outputs for a given input. However, by the previous comment, that is, $f_1 - g_1 = g_2 - f_2$, we see this definition works. We are just emphasizing that since $f_1 = g_1$ might not be analytic at z_0, then at that point we will simply use $g_2 - f_2$.) This makes F entire. Furthermore, we can use the bottom "piece" along with part (c) to show $\lim_{|z|\to\infty} F(z) = 0$. It follows that F is bounded. By Liouville's Theorem, it follows that F is constant on $\mathbb{C}$, and so we must have $F(z) = 0$ for all $z \in \mathbb{C}$. This completes the proof. □

Theorem 19: Suppose $f(z)$ is analytic in $\{z : 0 < |z-z_0| < R\}$ and let $C = \{z : |z-z_0| = r\}$ for some $0 < r < R$, where z_0 is an isolated singularity. Let

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta. \tag{18}$$

Then the series $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ converges to $f(z)$ uniformly in each annulus $\{z : r_1 < |z - z_0| < r_2\}$ where $0 < r_1 < r_2 < R$.

Proof. First, suppose $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ does converge to $f(z)$ uniformly in the circe C centered at z_0 and of radius r, where $0 < r < R$. We show the coefficients must be the ones given in (18). Let $n \in \mathbb{Z}$ be chosen and fixed, and we show that a_n is as it is in (18). Since we are assuming the uniform convergence of $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$, for any $\epsilon > 0$ we can choose $N_1, N_2 \in \mathbb{N}$ so large so that $-N_1 \le n \le N_2$ and

$$\left| f(z) - \sum_{j=-N_1}^{N_2} a_j(z - z_0)^j \right| < \epsilon$$

for every $z \in C$. It follows from the ML-inequality that

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_C \left(f(z) - \sum_{j=-N_1}^{N_2} a_j(z - z_0)^j \right) \frac{dz}{(z - z_0)^{n+1}} \right| &\le \frac{\epsilon}{2\pi} \int_C \left| \frac{1}{(z - z_0)^{n+1}} \right| \, dz \\ &= \frac{\epsilon}{2\pi} \int_C \frac{1}{|z - z_0|^{n+1}} \, dz \\ &\le \frac{\epsilon}{2\pi r^{n+1}} 2\pi r \\ &= \frac{\epsilon}{r^n} \end{aligned} \tag{19}$$

Now since $\int_C (z - z_0)^k \, dz = 2\pi i$ if $k = 1$ and 0 otherwise, it follows that

$$\frac{1}{2\pi i} \int_C (z - z_0)^k \, dz = \begin{cases} 1 & \text{if } k = -1 \\ 0 & \text{otherwise} \end{cases}$$

from whence we have

$$\begin{aligned}\frac{1}{2\pi i}\int_C\left(\sum_{j=-N_1}^{N_2} a_j(z-z_0)^j\right)\frac{dz}{(z-z_0)^{n+1}} &= \sum_{j=-N_1}^{N_2}\frac{1}{2\pi i}\int_C\left(a_j(z-z_0)^j\frac{dz}{(z-z_0)^{n+1}}\right)\\ &= \frac{1}{2\pi i}\int_C a_n\frac{dz}{(z-z_0)}\\ &= a_n.\end{aligned}$$

This allows us to simplify (19) as follows:

$$\left|\frac{1}{2\pi i}\int_C\left(f(z)-\sum_{j=-N_1}^{N_2} a_j(z-z_0)^j\right)\frac{dz}{(z-z_0)^{n+1}}\right| = \left|\frac{1}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^{n+1}}\,dz - a_n\right| \le \frac{\epsilon}{r^n}.$$

Letting $\epsilon\to 0$, we have

$$a_n = \frac{1}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^{n+1}}\,dz,$$

as required.

Now we must only show that $f(z)$ can be written as $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ in $\{z : 0<|z-z_0|<R\}$, and that the convergence is uniform in each annulus $\{z : r_1<|z-z_0|<r_2\}$ where $0<r_1<r_2<R$. Suppose $0<r_1<r<r_2<R$; by Theorem 18, we can write $f(z)=f_1(z)+f_2(z)$, where f_1 and f_2 are unique and satisfy conditions (b)-(d) of Theorem 18. Since f_2 is analytic in the disc $\{z : |z-z_0|<R\}$ it follows from Taylor's Theorem that

$$f_2(z)=\sum_{n=0}^{\infty} A_n(z-z_0)^n \tag{20}$$

converges in the disc $\{z : |z-z_0|<R\}$ and uniformly in the closed disc $\{z : |z-z_0|\le r_2\}$. Let us consider the function f_1 now, which is analytic in $\mathcal{C}\setminus\{z_0\}$. Let

$$w=\frac{1}{z-z_0}\quad\text{so that}\quad z=\frac{1}{w}+z_0.$$

Note that $w\neq 0$ and also that $\frac{1}{w}\neq 0$; it follows that

$$f_1(z)=f_1\left(\frac{1}{w}+z_0\right),$$

which is analytic in $\mathcal{C}$; that is, entire. Thus, by Taylor's Theorem we can write

$$f\left(\frac{1}{w}+z_0\right)=\sum_{m=1}^{\infty} B_m w^m, \tag{21}$$

which converges in $\mathbb{C}$ and hence uniformly on the closed disc $\{w : |w|\le 1/r_1\}$. The term B_0 is missing because this corresponds to when $w=0$, or equivalently when $z=\infty$, and in view of Theorem 18(c), this function value is 0. However, (21) is equivalent to saying

$$f_1(z)=\sum_{m=1}^{\infty} B_m(z-z_0)^{-m} \tag{22}$$

converges in $\mathcal{C}\setminus\{0\}$, uniformly in $\{z : |z-z_0|\ge r_1\}$. Putting (20) and (22) together yields our result. $\square$

Note 4.1. Dr. Vu said he may state a question (referencing Theorem 19) similarly to the following: State and prove the theorem for exapansion into Laurent series (for functions analytic in $\{z : 0 < |z - z_0| < R\}$.)

Theorem 20: Suppose $f(z)$ is analytic in $\{z : 0 < |z - z_0| < R\}$ and $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ is its Laurent series. Then

(a) z_0 is a removable singularity if and only if $a_n = 0$ for all $n < 0$,
(b) z_0 is a pole if and only if there exists an $m \in \mathbb{N}$ such that $a_n = 0$ for all $n < -m$, and
(c) z_0 is an essential singularity if neither (a) nor (b) holds.

Proof. First of all, f has a removable singularity at z_0 if and only if we can choose $f(z_0)$ to make f analytic in $\{z : |z - z_0| < R\}$, and f is analytic in this disc if and only if it equals it's Taylor series in this disc. Note that a Laurent series with no principal part is a Taylor series. This establishes (a).

For (b), note that the "only if" conditions happen if and only if

$$f(z) = \sum_{n=-m}^{\infty} a_n(z - z_0)^n,$$

which happens if and only if

$$f(z) = \frac{1}{(z - z_0)^m}\left(a_n + a_{n-1}(z - z_0) + a_{n-2}(z - z_0)^2 + \cdots\right) = \frac{g(z)}{(z - z_0)^m},$$

where g is analytic in some neighborhood of z_0 and $g(z_0) = a_{-m} \neq 0$, which happens if and only if f has a pole of order m at z_0. This establishes (b). By the definition of essential singularity, (i.e. z_0 is an essential singularity if and only if it is neither removable nor a pole), it follows that z_0 is an essential singularity. □

Theorem 21: If $f(z)$ is analytic in a simply connected domain D and C is a simple closed polygon contour in D, then $\int_C f(z)\ dz = 0$.

Proof. Every closed polygonal contour can be triangulated. Suppose $C = C_1 \cup C_2 \cup \cdots \cup C_k$ for triangular regions C_i (technically we are adding many sides on the interior of C, but these sides will all cancel out during the integration). It follows that

$$\int_C f(z)\ dz = \int_{C_1} f(z)\ dz + \int_{C_2} f(z)\ dz + \cdots + \int_{C_k} f(z)\ dz = 0,$$

where the last equality follows from k applications of the triangular version of the Cauchy Integral Theorem. □

Theorem 22: If $f(z)$ is analytic in a simply connected domain D, then there is an analytic function $F(z)$ such that $F'(z) = f(z)$.

Proof. Let $z_0 \in D$ be fixed. Let z be any other point in D and let C_1 and C_2 be two different polygonal contours lying entirely in D. We wish to show that

$$\int_{C_1} f(z)\ dz = \int_{C_2} f(z)\ dz, \tag{23}$$

which will establish that the integral is independent of the choice of polygonal contour. Let $C = C_1 \cup -C_2$. Showing $\int_C f(z)\ dz = 0$ will establish (23). Now if C self-intersects, then it really just divides into smaller polygonal contours, still oriented correctly. Suppose C self-intersects k times. Then we can write $C = B_1 \cup \cdots \cup B_k$ where each B_i is a closed simple polygonal contour. It follows that $\int_C f(z)\ dz = 0$ by k applications of Theorem 21. Hence, the integral is independent of choice of polygonal curve.

Knowing this, define

$$F(z) = \int_{z_0}^{z} f(\zeta)\ d\zeta.$$

where the integral is taken over any polygonal curve from z_0 to z. Now we can choose $h \in \mathbb{C}$ such that $|h|$ is small enough that $[z, z+h]$ is entirely in D. It follows that

$$F(z+h) - F(z) = \int_{z_0}^{z+h} f(\zeta)\ d\zeta - \int_{z_0}^{z} f(\zeta)\ d\zeta = \int_{z_0}^{z+h} f(\zeta)\ d\zeta + \int_{z}^{z_0} f(\zeta)\ d\zeta = \int_{[z,z+h]} f(\zeta)\ d\zeta.$$

If $h \neq 0$ then we have

$$\begin{aligned}
\frac{F(z+h)-F(z)}{h} - f(z) &= \frac{1}{h}\int_{[z,z+h]} f(\zeta)\ d\zeta - f(z) \\
&= \frac{1}{h}\int_{[z,z+h]} f(\zeta)\ d\zeta - \frac{1}{h} f(z) \int_{[z,z+h]} d\zeta \\
&= \frac{1}{h}\int_{[z,z+h]} f(\zeta)\ d\zeta - \frac{1}{h}\int_{[z,z+h]} f(z)\ d\zeta \\
&= \frac{1}{h}\int_{[z,z+h]} (f(\zeta) - f(z))\ d\zeta.
\end{aligned}$$

Now since f is differentiable at z, it follows that f is continuous at z and hence, given $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(\zeta) - f(z)| < \epsilon$ whenever $|\zeta - z| < \delta$. Thus if $|h| < \delta$, then $|f(\zeta) - f(z)| < \epsilon$ holds for all $\zeta \in [z, z+h]$. We now have by Theorem 3 that

$$\left| \int_{[z,z+h]} (f(\zeta) - f(z))\ d\zeta \right| \leq \epsilon |h|.$$

Hence we have

$$\begin{aligned}
\left| \frac{F(z+h)-F(z)}{h} - f(z) \right| &= \left| \frac{1}{h}\int_{[z,z+h]} (f(\zeta) - f(z))\ d\zeta \right| \\
&= \frac{1}{|h|} \left| \int_{[z,z+h]} (f(\zeta) - f(z))\ d\zeta \right| \\
&\leq \frac{1}{|h|} \epsilon |h| \\
&= \epsilon
\end{aligned}$$

We have successfully shown that

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z),$$

which establishes our result. $\square$

Theorem 23: (Cauchy Integral Theorem): If $f(z)$ is analytic in a simply connected domain D and C is a simple closed contour in D, then $\int_C f(z)\ dz = 0$.

Proof. By Theorem 22, it follows that there is a function, analytic in D, such that $F' = f$. By the Fundamental Theorem of Calculus for Complex Valued Functions (Theorem 2 above), it follows that

$$\int_C f(z)\ dz = F(z_2) - F(z_1),$$

where z_1 and z_2 are the initial point and terminal points of C, respectively. But since C is closed, $z_1 = z_2$ and hence $\int_C f(z)\ dz = 0$, as desired. $\square$

Theorem 24 (Cauchy Integral Formula): If $f(z)$ is analytic in a simply connected domain D and C is a closed contour in D where $z \notin C$, then

$$\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\ d\zeta = n(C, z) f(z),$$

where $n(C, z)$ is the number of times C winds around z.

Proof. Consider the function

$$g(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z},$$

which is analytic in D apart from a singularity at z. Now this singularity is removable because

$$\lim_{\zeta \to z} (\zeta - z) g(\zeta) = \lim_{\zeta \to z} [f(\zeta) - f(z)] = 0$$

and so Theorem 15 applies. Now this singularity is removed by defining $g(z) = \lim_{\zeta \to z} \frac{f(\zeta) - f(z)}{\zeta - z} = f'(z)$. Hence the function

$$g(\zeta) = \begin{cases} \frac{f(\zeta) - f(z)}{\zeta - z} & \text{if } \zeta \neq z \\ f'(z) & \text{if } \zeta = z \end{cases}$$

is an analytic function in D. It now follows from Theorem 23 that $\int_C g(\zeta)\ d\zeta = 0$. Since we are assuming that $z \notin C$, this gives us

$$\int_C \frac{f(\zeta) - f(z)}{\zeta - z}\ d\zeta = 0 \quad \text{and so} \quad \int_C \frac{f(\zeta)}{\zeta - z}\ d\zeta = f(z) \int_C \frac{d\zeta}{\zeta - z}.$$

We now consider that

$$\frac{1}{2\pi i} \int_C \frac{d\zeta}{\zeta - z} = n(C, z)$$

represents the number of times that C winds around z, acknowledging the possibility that this number could be 0 if z lies outside C. (Think of this as the variation of the argument of $\zeta - z$ as ζ follows C. We have done similar things before; e.g. if C is a circle centered at the origin and we are asked to find $\int_C \frac{1}{z}\ dz$, we use polar coordinates and note that $z = re^{i\theta}$ for $0 \leq \theta \leq 2pik$ for some integer k that represents the number of times C winds around the origin. Then $\int_C \frac{1}{z}\, dz = i\int_C dz = 2\pi ik$. Returning, since C is closed, this must be an integer multiple of $2\pi i$, and so when we divide by $2\pi i$ out front, this gives the number of times C winds around z.) It follows that

$$\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta = 2\pi i f(z) n(C, z),$$

which finishes the proof. $\square$

Theorem 25: If $f(z)$ is analytic in a simlpy connected domain D except an isolated singularity at z_0 and $f_1(z)$ is the principal part of $f(z)$ at z_0, and C is a simple closed contour in D with $z_0 \notin C$, then

$$\frac{1}{2\pi i}\int_C f_1(z)\; dz = \begin{cases} a_{-1} & \textbf{if } z_0 \textbf{ is inside } C \\ 0 & \textbf{if } z_0 \textbf{ is outside } C \end{cases}. \tag{24}$$

Proof. First consider when z_0 is outside C. Since C is a simple closed contour, we can form a simple polygonal contour ℓ with initial point z_0 and terminal point ∞ such that $C \cap \ell = \emptyset$. Then take $D_0 = D \setminus \ell$, which is a simply connected domain containing C, and furthermore, f is assumed to be analytic in D_0, and since f_2 is analytic in D_0, it follows that $f_1 = f - f_2$ is analytic in D_0. By Cauchy's Integral Theorem,

$$\int_C f_1(z)\; dz = 0,$$

as desired for the second part of (24).

Now suppose z_0 is inside C. We perform a similar trick as in the proof of the Cauchy Integral Formula (Theorem 6). Since z_0 is in the interior of C, there exists a $r > 0$ such that $\overline{B(z_0, r)}$ is in the interior of C. Let $\gamma = \{z : |z - z_0| = r\}$ be oriented positively. Furthermore, as in Theorem 6, let C^- and C^+ denote the two pieces of C after performing "surgery." (A picture is worth a thousand words here.) As in Theorem 6, we have

$$\int_{C^-} f(z)\; dz - \int_\gamma f(z)\; dz = \int_{C_1} f(z)\; dz + \int_{C_1} f(z)\; dz. \tag{25}$$

Now since z_0 is not in the interior of either C^- or C^+ and since f is analytic in D except at z_0, it follows that the right hand side of (25) is equal to 0. Thus,

$$\int_C f(z)\; dz = \int_\gamma f(z)\; dz, \tag{26}$$

similarly to the proof of Theorem 6. Recall from Theorem 19 that

$$a_{-1} = \frac{1}{2\pi i}\int_\gamma f(z)\; dz,$$

and so by (26) we have

$$a_{-1} = \frac{1}{2\pi i}\int_C f(z)\; dz = \frac{1}{2\pi i}\int_C f_1(z)\; dz + \frac{1}{2\pi i}\int_C f_2(z)\; dz.$$

Since f_2 is analytic in D, the right integral is equal to 0 by the Cauchy Integral Theorem and so we have

$$\frac{1}{2\pi i}\int_C f_1(z)\ dz = a_{-1},$$

as desired to finish the proof. $\square$

Theorem 26: If $f(z)$ is analytic in a simply connected domain D except at isolated singularities z_i where $1 \leq i \leq k$, and if C is a simple closed contour in D which does not contain the singularites, then

$$\frac{1}{2\pi i}\int_C f(z)\ dz = \sum_{\substack{j=1 \\ z_j \textbf{ inside } C}}^{k} \mathbf{res}(f, z_j).$$

Proof. For each $j \in \{1, \dots, k\}$, let $f_j(z)$ denote the principal part of $f(z)$ at z_j. By Theorem 18 above, it follows that f_j is analytic in $\mathbb{C} \setminus \{z_j\}$. Now consider the function

$$g(z) = f(z) - \sum_{j=1}^{k} f_j(z).$$

Then we claim g is analytic in D. Clearly g is analytic everywhere in D except for the possibility of z_i for some $i \in \{1, \dots, k\}$. But note that

$$g(z_i) = f(z_i) - f_i(z_i) - \sum_{\substack{j=1 \\ j\neq i}}^{k} f_j(z_i) = f_i^{(2)}(z_i) - \sum_{\substack{j=1 \\ j\neq i}}^{k} f_j(z_i),$$

where $f_i^{(2)}$ is the non-principal (and analytic) part of f at z_i. This shows g is analytic in D. Thus,

$$\int_C g(z)\ dz = 0 \quad \text{and so} \quad \int_C f(z)\ dz = \int_C \sum_{j=1}^{k} f_j(z)\ dz = 2\pi i \sum_{j=1}^{k} \frac{1}{2\pi i}\int_C f_j(z)\ dz.$$

Using Theorem 25, we have

$$\frac{1}{2\pi i}\int_C f(z)\ dz = \sum_{\substack{j=1 \\ z_j \text{ inside } C}}^{k} \text{res}(f, z_j),$$

as desired to finish the proof. $\square$

Theorem 27: If $f(z)$ has a pole of order m at z_0, then

$$\mathbf{res}(f(z), z_0) = \lim_{z\to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right].$$

Proof. Since f has a pole of order m at z_0, we can write

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z-z_0} + g(z),$$

where $g(z)$ is analytic at z_0. Hence,

$$(z-z_0)^m f(z) = a_{-m} + a_{-m+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{m-1} + (z-z_0)^m g(z).$$

Differentiating with respect to z a total of $m-1$ times gives us

$$\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right] = a_{-1}(m-1)! + \underbrace{\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m g(z)\right]}_{\to 0 \text{ as } z\to z_0}.$$

Hence we have

$$a_{-1} = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right],$$

as desired to finish the proof. □

Theorem 28: The following hold.

(1) If $f(z)$ is analytic in a neighborhood of z_0 and z_0 is a zero of order m, then $f'(z)/f(z)$ is analytic in a punctured neighborhood of z_0, has a simple pole at z_0 and res$(f'/f, z_0) = m$.

(2) If $f(z)$ is analytic in a punctured neighborhood of z_0 and z_0 is a pole of order m, then $f'(z)/f(z)$ is analytic in a punctured neighborhood of z_0, has a simple pole at z_0, and res$(f'/f, z_0) = -m$.

Proof. (1) Since f is analytic in a neighborhood of z_0 and since z_0 is a zero of order m, it follows that $f(z) = (z-z_0)^m g(z)$, where g is analytic in a neighborhood of z_0 and $g(z_0) \neq 0$. Thus we have

$$\frac{f'(z)}{f(z)} = \frac{m(z-z_0)^{m-1}g(z) + (z-z_0)^m g'(z)}{(z-z_0)^m g(z)} = \frac{m}{z-z_0} + \frac{g'(z)}{g(z)}.$$

Now since $g(z_0) \neq 0$, it follows that g'/g is analytic in a neighborhood of z_0. Thus z_0 is a simple pole of f'/f and the residue is m, as desired. □

(2) Since f has a pole of order m at z_0, we can write

$$f(z) = \frac{g(z)}{(z-z_0)^m}$$

where g is analytic in some neighborhood of z_0 and $g(z_0) \neq 0$. Hence we have

$$\frac{f'(z)}{f(z)} = \frac{(z-z_0)^m}{g(z)} \cdot \frac{(z-z_0)^m g'(z) - m(z-z_0)^{m-1}g(z)}{(z-z_0)^{2m}} = \frac{g'(z)}{g(z)} + \frac{-m}{z-z_0}.$$

Similarly to part (1), the function g'/g is analytic in a neighborhood of z_0 since $g(z_0) \neq 0$. Thus f'/f is analytic in a punctured neighborhood of z_0 and has a simple pole at z_0 with residue $-m$. □

Theorem 29 (The Principle of Argument): If $f(z)$ is meromorphic in a simply connected domain D, C is a (positively oriented) Jordan contour in D which does not contain zeros of poles of $f(z)$, then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\, dz = N - P,$$

where $N(P)$ is the number of zeros (poles) of $f(z)$ inside C, counted with multiplicities (orders).

Proof. By Theorem 28, the poles of f'/f are precisely that zeros and poles of f, and these are the only singularities of f'/f. Thus, by Theorem 26, we have

$$\int_C \frac{f'(z)}{f(z)}\, dz = \sum_{\substack{j=1 \\ z_j \text{ inside } C}}^{k} \operatorname{res}\left(\frac{f'}{f}, z_j\right) = \sum_{\substack{\alpha \text{ zero of } f \\ \alpha \text{ inside } C}} \operatorname{res}\left(\frac{f'}{f}, \alpha\right) + \sum_{\substack{\beta \text{ pole of } f \\ \beta \text{ inside } C}} \operatorname{res}\left(\frac{f'}{f}, \beta\right) = N - P,$$

where the last equality follows because zeros of f of multiplicity m yield residues of f'/f of m and poles of order m of f yield residues of f'/f of $-m$ by Theorem 28. $\square$

Theorem 30 (Rouché): If $f(z)$ and $g(z)$ are analytic in a simply connected domain D, C is a Jordan contour in D and $|g(z)| < |f(z)|$ for all $z \in C$, then f and $f+g$ have the same number of zeros inside C.

Proof. Consider the function

$$F(z) = \frac{f(z) + g(z)}{f(z)}.$$

Since $|f(z)| > |g(z)|$, it follows that f and $f+g$ have no zeros on C. Also, note that for every $z \in C$, we have

$$|F(z) - 1| = \left|\frac{g(z)}{f(z)}\right| < 1, \tag{27}$$

also using the condition that $|g(z)| < |f(z)|$ on C. Using the substitution $w = F(z)$, we have

$$\frac{1}{2\pi i}\int_C \frac{F'(z)}{F(z)}\, dz = \frac{1}{2\pi i}\int_{F(C)} \frac{dw}{w} = n(F(C), 0), \tag{28}$$

the number of windings of $F(C)$ around 0. Now by (27), it follows that $F(C) \subset \{w : |w-1| < 1\} = D_1$. Now $0 \notin D_1$, and so $n(F(C), 0) = 0$. Hence, by the Principle of Argument and combining $n(F(C), 0)$ and (28), we have

$$N - P = \frac{1}{2\pi i}\int_C \frac{F'(z)}{F(z)}\, dz = 0,$$

where N and P are as in the Principle of Argument applied to the function F. Hence $N = P$, and so Chen states that it follows that F has the same number of zeros and poles inside C. (But N and P are counted with multiplicities, so wouldn't this only hold if we knew the multiplicities were the same also?) The poles of F are precisely the zeros of f, and the zeros of F are precisely the zeros of $f + g$. $\square$

The remaining items are not theorems, but are rather methods to evaluate integrals using the method of residue.

Evaluate $\int_0^{2\pi} f(\cos\theta, \sin\theta)\ d\theta$, where $f(x,y)$ is a rational function with real coefficients.

Method. Use the substitution

$$z = e^{i\theta} = \cos\theta + i\sin\theta \quad \text{and} \quad dz = ie^{i\theta}\, d\theta.$$

Then

$$\frac{1}{z} = e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta \quad \text{and} \quad d\theta = -i\frac{dz}{z}.$$

Putting these two lines together gives us

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \quad \text{and} \quad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

Thus using this substitution, we have

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = -i\int_C f\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)\, \frac{dz}{z},$$

where C is the unit circle $\{z : |z| = 1\}$ oriented positively. □

Show that $\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum_{\mathbf{Im}\alpha>0} \mathbf{res}(f, \alpha)$, where $f(x) = p(x)/q(x)$ is a rational function (with real coefficients), $\mathbf{deg}(p) + 2 \le \mathbf{deg}(q)$, and $q(x) \neq 0$ on $\mathbb{R}$.

Method. Consider the integral

$$\int_{-R}^{R} f(x)\, dx \quad \text{where} \quad R > 0.$$

We extend the function to the complex plane and also consider the integral

$$\int_{C_R} f(z)\, dz,$$

where C_R is best shown by a picture (the upper half-circle centered at the origin of radius R). Now consider the Jordan contour $C = [-R, R] \cup C_R$. By the residue theorem, we have

$$\int_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz = 2\pi i \sum_{z_i \text{ inside } C} \text{res}(f, z_i)$$

where the sum is taken over all poles inside C. It is easily shown that

$$\int_{C_R} f(z)\, dz \to 0$$

as $R \to 0$ since the degree of the denominator exceeds the degree of the numerator by at least 2. Hence, letting $R \to \infty$, we have

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum_{\text{Im}(z_i)>0} \text{res}(f, z_i),$$

where this sum is taken over all poles of f in the upper-half plane. □

Show $\int_{-\infty}^{\infty} f(x)e^{ix}\, dx = 2\pi i \sum_{\mathbf{Im}\alpha>0} \mathbf{res}(f(z)e^{iz}, \alpha)$, where $f(x) = p(x)/q(x)$ is a rational function (with real coefficients), and where $\mathbf{deg}(p) + 2 \le \mathbf{deg}(q)$, where $q(x) \neq 0$ on $\mathbb{R}$.

Method. Note that we can write

$$\int_{-\infty}^{\infty} f(x)e^{ix}\ dx = \int_{-\infty}^{\infty} f(x)\cos x\ dx + i\int_{-\infty}^{\infty} f(x)\sin x\ dx.$$

Now consider the integral extended to the complex plane

$$\int_C f(z)e^{iz}\ dz = \int_{-R}^{R} f(x)e^{ix}\ dx + \int_{C_R} f(z)e^{iz}\ dz,$$

where $C = [-R, R] \cup C_R$ is broken up in the usual way. By the residue theorem, we have

$$\int_{-R}^{R} f(x)e^{ix}\ dx + \int_{C_R} f(z)e^{iz}\ dz = 2\pi i \sum_{z_i \text{ inside } C} \operatorname{res}(f(z)e^{iz}, z_i), \tag{29}$$

where the sum is taken over all the poles of $f(z)e^{iz}$ inside C.

Lemma 4.2 (Jordan's Lemma)**.** *Suppose that $R > 0$ and C_R is the semi-circular arc given by $z = Re^{it}$ for $t \in [0, \pi]$. Then*

$$\int_{C_R} |e^{iz}||dz| < \pi.$$

Using Jordan's Lemma, it follows that

$$\left|\int_{C_R} f(z)e^{iz}\ dz\right| \leq \int_{C_R} \left|f(z)e^{iz}\ dz\right| \to 0$$

as $R \to \infty$ since the degree of the denominator exceeds the degree of the numerator by at least 2. Hence, letting $R \to \infty$, we have from (29)

$$\int_{-\infty}^{\infty} f(x)e^{ix}\ dx = 2\pi i \sum_{\mathrm{Im}\alpha > 0} \operatorname{res}(f(z)e^{iz}, \alpha).$$

$\square$