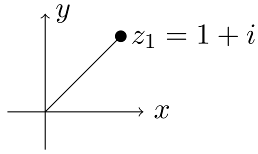
18.04 Problem Set 1, Spring 2017 Solutions

Problem 1. (25: 5,5,10,5 points)

(a) $z_1 = 1+i$, $z_2 = 1+3i$: $\boxed{z_1 z_2 = -2+4i}$, $\dfrac{z_1}{z_2} = \dfrac{1+i}{1+3i}\cdot\dfrac{1-3i}{1-3i} = \dfrac{4-2i}{10} = \boxed{\dfrac{2-i}{5}}$.

$|z_1| = \sqrt{2}$, $\text{Arg}(z_1) = \pi/4$ (principal branch).
So $\log(z_1) = \log(\sqrt{2}) + i\frac{\pi}{4} = \frac{1}{2}\log(2) + i\frac{\pi}{4}$ (principal branch).
Thus,



$$z_1^{z_2} = e^{z_2\log(z_1)} = e^{(1+3i)(\log(2)/2+i\pi/4)} = \boxed{e^{\log(2)/2-3\pi/4}e^{i(3\log(2)/2+\pi/4)}} \text{ (not pretty!)}$$

(b) $|i| = 1$ and $\text{Arg}(i) = \pi/2$, so $\log(i) = \dfrac{\pi}{2}i + 2n\pi i$, where n is any integer. Thus

$$\boxed{i^i = e^{i\log(i)} = e^{-\pi/2-2n\pi}}$$

On the principal branch $i^i = e^{-\pi/2}$. I'm surprised that it's real!

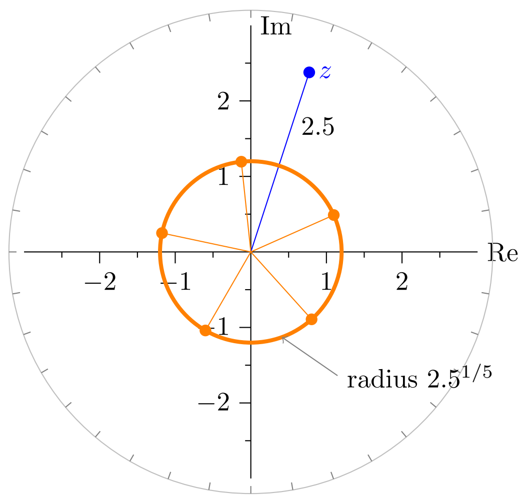
(c) (i) $z = 1+\sqrt{3}\,i = 2e^{i\pi/3}$, so

$$z^8 = 256e^{i8\pi/3} = 256e^{i2\pi/3} = 128(1+\sqrt{3}\,i)$$

(ii) $z^{1/4} = (2e^{i\pi/3+2n\pi i})^{1/4} = 2^{1/4}e^{i\pi/12}$, $2^{1/4}e^{i7\pi/12}$, $2^{1/4}e^{i13\pi/12}$, $2^{1/4}e^{i19\pi/12}$

Since $\pi/12 = 15°$ this doesn't have a pretty standard form, so we'll leave them as is. We can always get numerical approximations if needed.

(d) The roots are shown in orange. The first one has argument $\text{Arg}(z)/5$. The others are spaced at intervals of $2\pi/5$ around the circle of radius $|z|^{1/5} = (2.5)^{1/5}$.



Problem 2. (10: 5,5 points)

(a) Let $z = x+iy$, then $e^z = e^x e^{iy} = e^x\cos(y) + ie^x\sin(y)$. So,

$$\overline{(e^z)} = e^x\cos(y) - ie^x\sin(y) = e^x e^{-iy} = e^{x-iy} = e^{\bar{z}} \quad \blacksquare$$

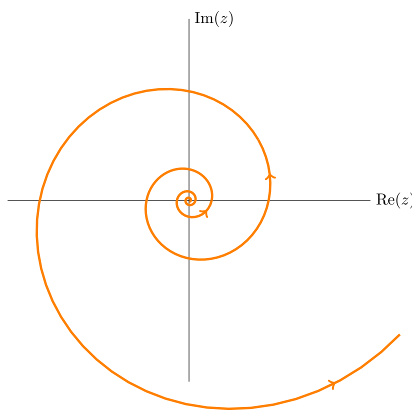
(b) We'll give two ways to say this:

Method 1: $\dfrac{1}{z} = \dfrac{\overline{z}}{z \cdot \overline{z}} = \dfrac{\overline{z}}{|z|^2} = \dfrac{\overline{z}}{1} = \overline{z}$.

Method 2: $z = \mathrm{e}^{i\theta}$, so by part (a) $\overline{z} = \mathrm{e}^{-i\theta} = z^{-1}$.

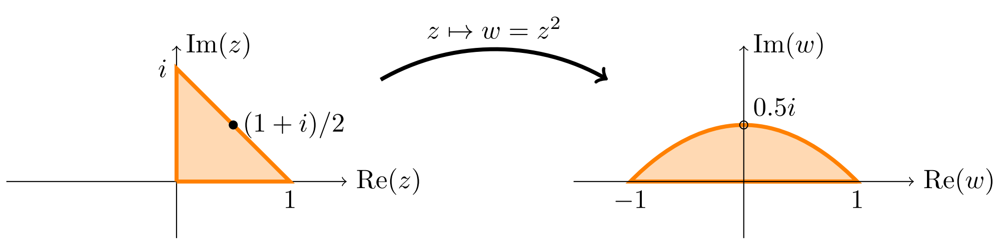
Problem 3. (5 points)
We know that $|z_1/z_2| = |z_1|/|z_2|$ (think polar form). So since $|x+iy| = \sqrt{x^2+y^2} = |x-iy|$ the ratio has modulus 1.

Problem 4. (15: 5,10 points)
(a) $\mathrm{e}^{t(1+i)} = \mathrm{e}^t \mathrm{e}^{it}$. So as t grow the magnitude grows exponentially and the argument increases. This gives a spiral!



Note, this is not to scale. The exponential e^t grows too quickly, so we drew $\mathrm{e}^{0.2t}$ instead. Qualitatively this is a good representation of the curve.

(b) The figure shows the image.
In brief we have: $z = 1 \mapsto w = 1$, $z = (1+i)/2 \mapsto w = i/2$, $z = i \mapsto w = -1$.
The sides of the triangle map as follows:
the line segment $[0, 1] \mapsto$ the line segment $[0, 1]$
the line segment $[0, i] \mapsto$ the line segment $[0, -1]$
the line segment $[1, i] \mapsto$ the parabola shown in the figure (proved below).



It takes a small amount of algebra to find the equation for the image of the segment from 1 to i. We parametrize the segment as $z = (1-t) + ti$ with $0 \le t \le 1$. Then $w = z^2 = (1-2t) + 2t(1-t)i$. As a curve in the plane this is

$$w = (u(t), v(t)), \text{ where } u(t) = 1 - 2t \text{ and } v(t) = 2t(1-t).$$

A little more algebra shows that $t = (1-u)/2$ and therefore $v = -u^2 + 1/2$. This is the formula for the parabola shown.

Problem 5. (10 points)
The nth roots of 1 are all the roots of the equation $z^n - 1 = 0$. Since we know that $z = 1$ is a root we can factor the equation as

$$(z-1)(z^{n-1} + z^{n-2} + \ldots + z + 1) = 0$$

Since z_k is a root and $z_k \neq 1$ we must have z_k is a root of the second factor. This is what we were asked to show!

Problem 6. (20: 10,10 points) (Orthogonal lines stay orthogonal!)
(i) As usual, let $z = x + iy$. Vertical line have the equation $x = a$ for some constant a. We can parametrize this line by $z = a + iy$, with $-\infty < y < \infty$. The image of the line under the mapping is

$$w = \mathrm{e}^{a+iy} = \mathrm{e}^a \mathrm{e}^{iy}.$$

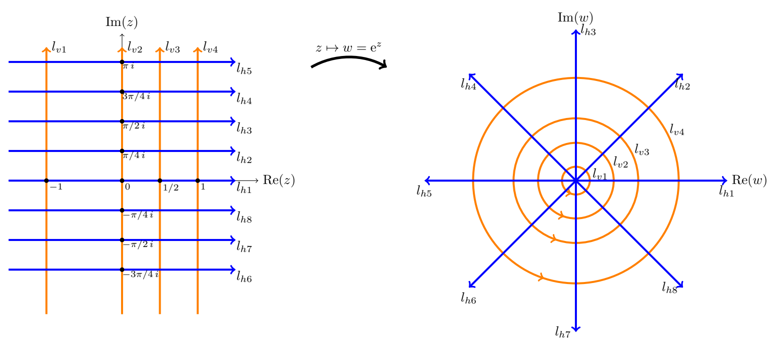
Thus $|w| = \mathrm{e}^a$ is constant, so the image is a circle of radius e^a centered on 0.

(ii) Likewise, horizontal lines have the equation $y = b$ for some constant b. We parametrize this by $z = x + ib$, with $-\infty < x < \infty$. The image of the line under this mapping is

$$w = \mathrm{e}^{x+ib} = \mathrm{e}^x \mathrm{e}^{ib}.$$

Thus $\arg(w) = b$ is constant, so the image is a ray from the origin at angle b.

In the figure below we label the corresponding curves with the same label. So, for example, the image of the vertical line l_{v4} is the circle in the w-plane labeled l_{v4}



(iii) The images found in parts (i) and (ii) are circles and rays. Geometrically we know that radii are perpendicular to circles, so rays from the origin are perpendicular to circles centered on the origin.

(b) This part is similar to part (a), though we will need a little more algebraic manipulation to identify the image curves.

(i) The vertical line $x = a$ is parametrized by $z = a + iy$, with $-\infty < y < \infty$. So the image is $w = a^2 - y^2 + i2ay$. Let's write this as $w = u + iv$. It is easy to see that with $u = a^2 - y^2$ and $v = 2ay$ we have

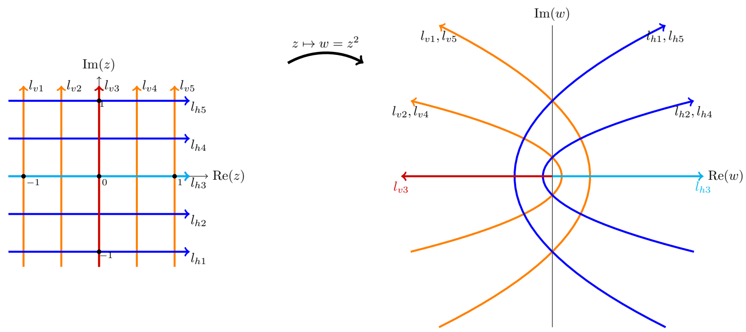
$$u = a^2 - (v/2a)^2.$$

This is a parabola in the uv-plane which opens to the left and has vertex at $(a^2, 0)$.

(ii) Horizontal lines $y = b$ are similar:
Parametrization: $z = x + ib$, with $-\infty < x < \infty$.

Image: $w = x^2 - b^2 + i2bx$.
If $w = u + iv$ then u $= (v/2b)^2 - b^2$. This is a parabola opening to the right with vertex at $(-b^2, 0)$.

The figure below is labeled in the same way as the figure in part (a). Notice the vertical lines $x = a$ and $x = -1$ map to the same parabola as befits the 2-to-1 mapping $w = z^2$. The real z-axis ($x = 0$) maps to the positive real w-axis ($u > 0$). We can view this as a degenerate parabola. Likewise for the imaginary z-axis.



(iii) To show the image curves are orthogonal (meet at right angles) we need to show that their tangent vectors are orthogonal. Happily, we've parametrized the image curves in parts (i) and (ii), so computing tangent vectors is easy. In keeping with the course we will write vectors as $x + iy$ rather than in 18.02 notation as $\langle x, y\rangle$

Horizontal line: $z = a + iy$; image $w = a^2 - y^2 + i2ay$; image tangent vector: $\frac{dw}{dy} = -2y + i2a$.
Vertical line: $z = x + ib$; image $w = x^2 - b^2 + i2xb$; image tangent vector: $\frac{dw}{dx} = 2x + i2b$.

The lines meet at the point $a + ib$, so we need to check the tangents at that point.
Horizontal line image: $\frac{dw}{dy} = -2b + i2a$
Vertical line image: $\frac{dw}{dx} = 2a + i2b$
It is easy to see that these are orthogonal. For 18.04 the nicest way to say this is to note that $-2b + i2a = i(2a + i2b)$. Since multiplication by i is rotation by $90°$ the vectors are at right angles.

Extra problems not for points

Problem 7. (0 points) Answer: the set of points is the circle $|z + 1/2| = 3/2$, excluding the points $z = 1$ and $z = -2$ where $\arg(w)$ is not defined.

Justification. $\mathrm{Arg}(w) = \pm\pi/2$ means that w is on the imaginary axis (and $w \neq 0$). Thus we must have

$$w = \frac{z-1}{z+2} = ib, \text{ where } b \neq 0 \text{ is real.}$$

Solving for z we get $z = \dfrac{2ib+1}{-ib+1}$.

We can view this as a mapping $b \mapsto z$ from the real b-line to the complex z-plane. A tiny amount of algebra gives

$$z = \frac{2ib+1}{-ib+1} = \frac{3}{-ib+1} - 2.$$

Ignoring the scale by 3 and the shift by -2 for the moment we have:

Claim. The image of $w = \frac{1}{-ib+1}$ where b is real is the circle $wz - 1/2| = 1/2$, i.e. the circle with radius 1/2 and center 1/2.

Proof.

$$|w - 1/2| = \left|\frac{1}{-ib+1} - 1/2\right| = \left|\frac{1+bi}{2(1-bi)}\right| = \frac{1}{2}. \qquad \blacksquare$$

(The last equality follow from problem 3.)

Since $z = 3w - 2$ we have the z-image is the circle of radius 3/2 with center $3/2 - 2 = -1/2$.

End of pset 1 solutions