# 1 American options

Most traded stock options and futures options are of American-type while most index options are of European-type.

The central issue is when to exercise? From the holder point of view, the goal is to maximize holder's profit (Note that here the writer has no choice!)

## 1.1 Non arbitrage conditions for no dividend paying stock

### 1.1.1 The Call Option:

1. $C_0^a \geq (S_0 - K)^+$

Proof:

i) $C_0^a \geq 0$ (optionality);

ii) If $C_0^a < S_0 - K$ (assuming $S_0 > K$): buy the option at $C_0^a$ then, exercise immediately. This leads to profit: $S_0 - K$ and the net profit: $S_0 - K - C_0^a > 0$ which gives rise to an arbitrage opportunity. Hence, the no–arbitrage argument yields $C_0^a \geq (S_0 - K)$

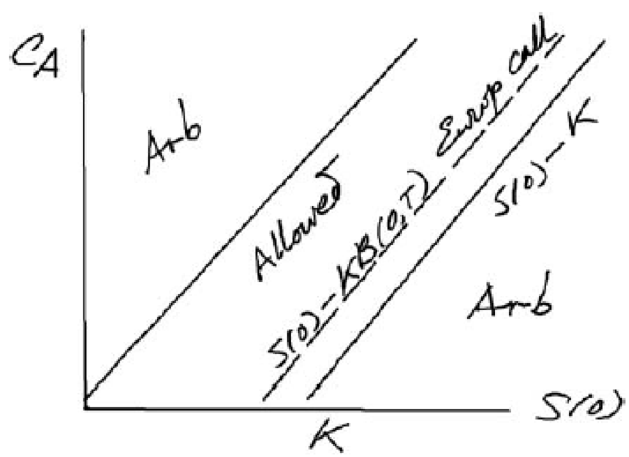
2. $S_0 \geq C_0^a$.

If this is not the case $S_0 < C_0^a$, buy $S_0$ and sell $C_0^a$ yielding a net profit $> 0$ at $t = 0$. Because the possession of the stock can always allow the deliverance of the stock to cover the exercise if exercised, then we are guaranteed to have a positive future profit. Hence, an arbitrage opportunity.

3. $C_0^a \geq C_0^e$ with the same maturity $T$ and strike $K$.

Compared with the European non–arbitrage condition we have $C_0^e \geq (S_0 - KB(0,T))^+$, where $B(t,T)$ is the value at time $t$ of zero coupon bond such that $B(T,T) = 1$, therefore

$$C_0^a \geq (S_0 - KB(0,T))^+$$




4. If the stock has no dividend payment, and the risk–free interest rate is positive, i.e., $B(0,T) < 1$, $\forall\ T > 0$, then one should never prematurely exercise the American call, i.e., $C_0^a = C_0^e$

Indeed

(1) $C_0^a \geq C_0^e \geq (S_0 - KB(0,T))^+$ i.e., the call is "alive"

(2) If exercised now hence the profit $S_0 - K$ i.e., the call is "dead"

Remark that

$$\underbrace{S_0 - KB(0,T)}_{alive} > \underbrace{S_0 - K}_{dead}$$

therefore, it is worth more "alive" than "dead"

(a) Question: Should one exercise the call if $S_0 > K$ and if he believes the stock will go down below $K$?

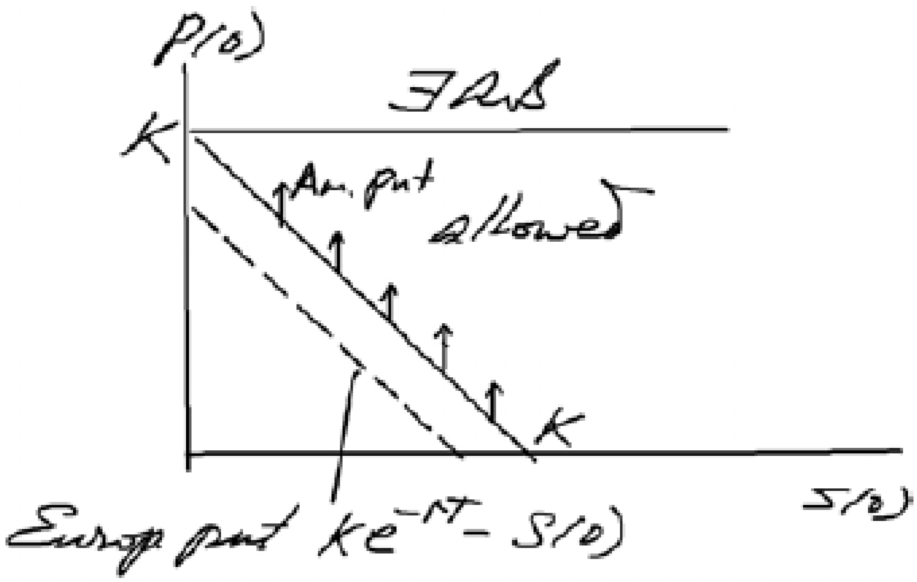
No! If exercise, $(\text{profit})_1 = S_0 - K$
If sell the option, $(\text{profit})_2 = C_0^a$
Since $C_0^a \geq (S_0 - K)^+$ one should sell the option rather than exercise it
(b) With dividend, early exercise may be optimal
(c) Intuition: consider paying $K$ to get a stock now versus paying $K$ to get a stock later, one gets the interest on $K$, therefore, the difference is $Ke^{rT} - K$ if wait
5. For two American call options, $C_t^a(K, T_1)$ and $C_t^a(K, T_2)$ ,with the same strike $K$ on the same stock but with different maturities $T_1$ and $T_2$, then we have $C_0^a(K, T_2) \leq C_0^a(K, T_1)$ if $T_1 \leq T_2$.

### 1.1.2 The Put Option:

1. $P_0^a \geq (K - S_0)^+$
For the European put we have $P_0^e \geq (KB(0,T) - S_0)^+$
Indeed, if $P_0^a < K - S_0$ , buy a put at the price $P_0$ and exercise it immediately, yielding, then, the total cash flow:

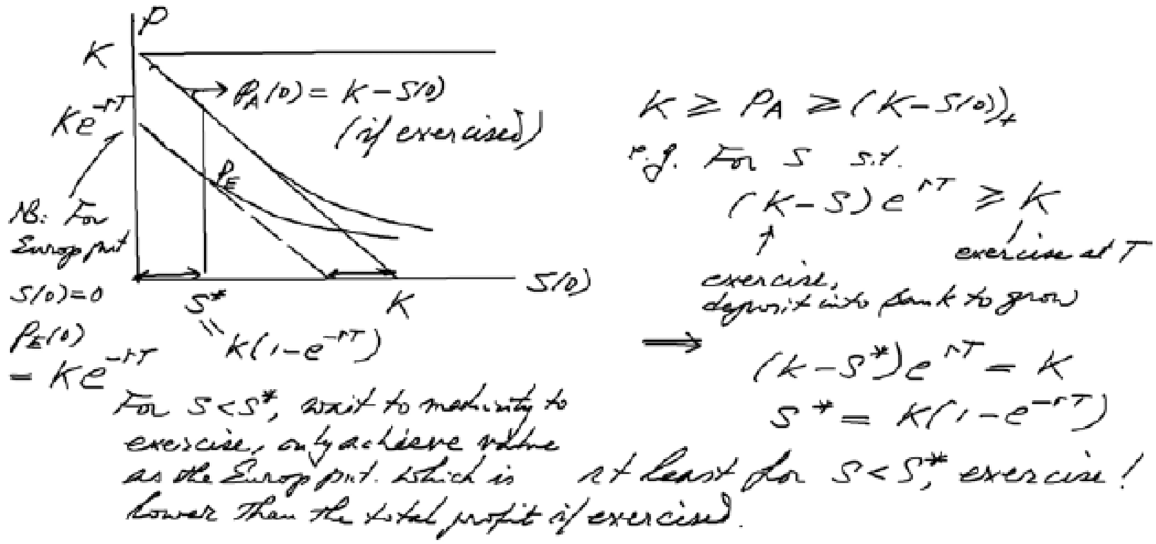
$$\underbrace{-P_0^a}_{\text{buy a put}} + \underbrace{S_0 - K}_{\text{exercise}} > 0.$$

giving rise to an arbitrage opportunity.
2. $P_0^a \leq K$



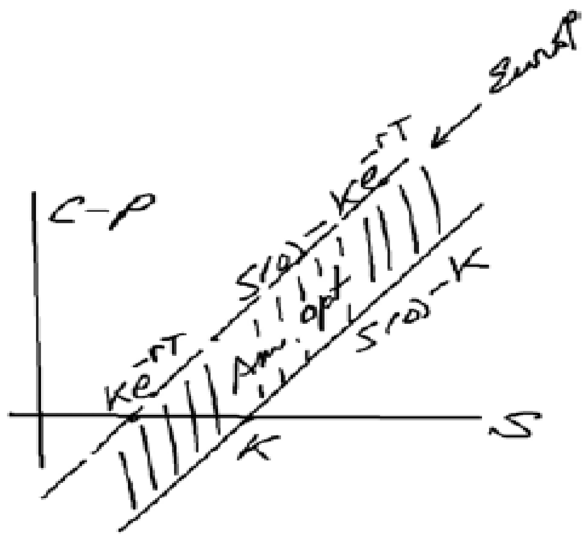

3. $P_0^a \geq P_0^e$
Remark that for a put, the profit is bounded by $K$. This fact limits the benefit from waiting to

exercise and its financial consequence is that one may exercise early if $S_0$ is very small.




4. Put–call parity for American options:

$$S_0 - K \leq C_0^a - P_0^a \leq S_0 - Ke^{-rT}$$




Put–call parity for American options on an non–dividend–paying stock:
(a) $P_0^a + S_0 - KB(0,T) \geq C_0^a$;
(b) $C_0^a \geq P_0^a + S_0 - K$
That is

$$S_0 - K \leq C_0^a - P_0^a \leq S_0 - KB(0,T)$$

**Proof:**
(1) $P_0^a \geq P_0^e = C_0^e - S_0 + KB(0,T)$ then $C_0^e = C_0^a$ implies that

$$P_0^a \geq C_0^a - S_0 + KB(0,T)$$

(2) Consider portfolio: long one call, short one put, short the stock hold $K$ dollars in cash that is:

| | | | |
|---|---|---|---|
| $C_0^a$ | $-P_0^a$ | $K - S_0$ | $> 0$ |
| Never exercised early | Can be exercised early | Exercise | |

If the put is exercised early at $t^o$, our position is

$$C^a_{t^o} - [K - S_{t^o}] - S_{t^o} + KB(0,t^o)^{-1} = C^a_{t^o} + K(B(0,t^o)^{-1} - 1) \geq 0$$

implies liquidated with net positive profit (note that the above inequality holds ">" strictly if $S_{t^o} > 0$ and $t^o = 0$)

If not exercised earlier, at maturity $t = T$, we have

(i) If $S_T \leq K$,

$$\text{The profit} = 0 - [K - S_T] - S_T + KB(0,T)^{-1} = K\left(B(0,T)^{-1} - 1\right) > 0$$

(ii) If $S_T > K$,

$$\text{The profit } = (S_T - K) - 0 - S_T + KB(0,T)^{-1} = K\left(B(0,T)^{-1} - 1\right) > 0$$

therefore, the payoff of the portfolio is positive or zero, implies the present value of the portfolio $\geq 0$, i.e.,

$$C^a_0 - P^a_0 - S_0 + K \geq 0$$

Combining (1) and (2) implies

$$S_0 - K \leq C^a_0 - P^a_0 \leq S_0 - KB(0,T).$$

Which end the proof.

Notice that: If the stock is dividend–paying, for European options, we have

$$C^e_0 - P^e_0 = P.V.[S_T] - KB(0,T)$$

where $P.V.[S_T]$ is the present value of the stock whose price at $T$ is $S_T$, e.g. If there is a dividend $D_{t_1}$ at $t_1$, then

$$P.V.[S_T] = S_0 - D_{t_1}B(0,t_1)$$

for American options, we have

$$C^a_0 - P^a_0 \leq S_0 - KB(0,T)$$

which is unchanged by dividend, however, in general

$$P.V.[S_T] - K \leq C^a_0 - P^a_0 \leq S_0 - KB(0,T).$$

## 1.2 American Calls

### 1.2.1 Time Value

Consider American calls on no–dividend–paying stocks:

Consider the following strategy: Exercise it at maturity no matter what (obviously, suboptimal if $K > S_T$), the present value of the American call under this strategy is:

$$P.V.[S_T - K] = S_0 - KB(0,T)$$

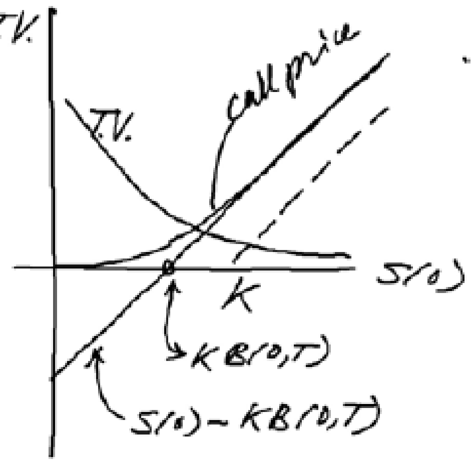
which is equivalent to a forward.

The time value of an American call on a stock without dividends is

$$T.V.(0) = C^a_0 - [S_0 - KB(0,T)]$$

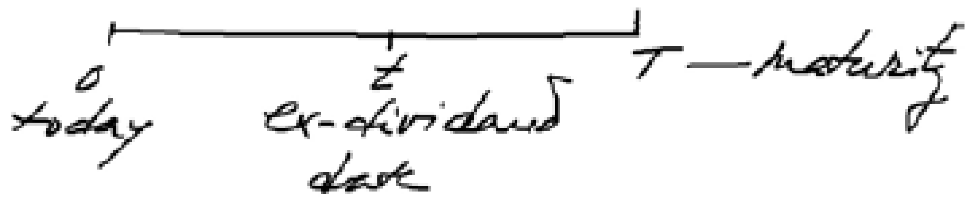
Note that $T.V.(0) \geq 0$ this is because

$$C^a_0 \geq C^e_0 \geq (S_0 - KB(0,T))^+ \quad \text{hence} \quad T.V.(0) \geq 0$$

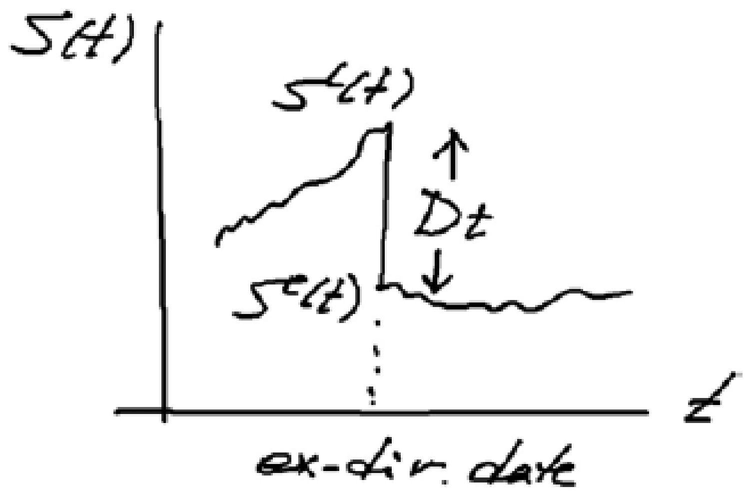



If $S_0 \ll K$, then $T.V.$ is high
If $S_0 \gg K$, then there is a high probability of expiring in–the–money, therefore, $C_0^a \gtrsim S_0 - KB(0,T)$ that is $T.V. \approx 0$.

### 1.2.2 Dividends




**Result**: Given interest rate $r > 0$, it is never optimal to exercise an American call between ex–dividends dates or prior to maturity.




Proof:
**Strategy 1**: Exercise immediately, (the value)$_1 = S_0 - K$
**Strategy 2**: Wait till just before the ex–dividend date, and exercise for sure (even if out–of–money) (the value)$_2 = S_t^c - K$ where $S_t^c$ is the cum stock price just before going ex–dividend. Therefore, the present value is $S_0 - KB(0,t)$. Since $B(0,t) < 1$, the value of Strategy 2 **is greather than** the value of Strategy 1 therefore, it is best to wait.

**Next question: to exercise at anytime after the ex–dividend date and prior to maturity?** The same argument leads to the same conclusion: best to wait.
**Question**: To exercise or not to exercise?
If exercised just prior to the ex–dividend date,

$$\textit{the value} = S_t - K = S_t^c + D_t - K$$

If not exercised, the value $= C_t$ (based on the ex–dividend stock price)

$$C_t = S_t^c - KB(t,T) + T.V.(t)$$

where $T.V.(t)$ is the time value at time $t$.

Since it should be exercised if and only if the exercised value $>$ the value not exercised, that is

$$S_t^c + D_t - K > S_t^c - KB(t,T) + T.V.(t)$$

implies that

$$D_t > K(1 - B(t,T)) + T.V.(t) > 0 \tag{1.1}$$

therefore, exercise is optimal at time $t$ if and only if the dividend is greater than the interest lost on the strike price $K(1 - B(t,T))$ plus the time–value of the call evaluated using the ex–dividend stock price.

Notice that:

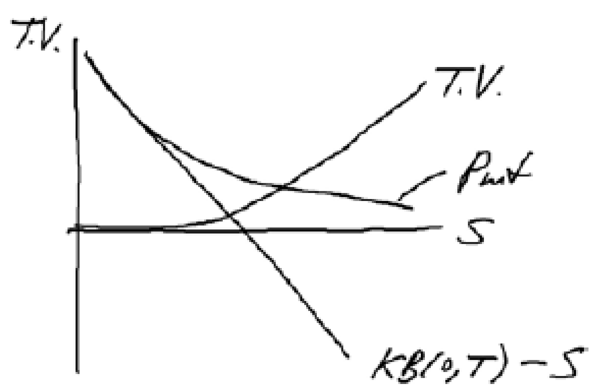
1. If $D_t = 0$ (i.e., no dividend) , the equation (1.1) does not hold. Hence, never exercise early.

2. Exercise is optimal if and only if the dividend is large enough ($>$ interest loss $+T.V.$), therefore, if the dividend is small, time–to–maturity is large, it is unlikely to exercise early.

## 1.3 American Puts

### 1.3.1 Time Value (if no dividend)

| | | | | |
|---|---|---|---|---|
| $T.V.(0)$ | $=$ | $P_0^a$ | $-$ | $\underbrace{KB(0,T) - S_0} \ \geq \ 0$ |
| | | | | the present value of exercising |
| | | | | the American put for sure at maturity |
| $P_0^a \geq P_0^e \geq (KB(0,T) - S_0)^+$. | | | | |




If $S_0 \gg K$, then $T.V.$ is large, best to wait
If $S_0 \ll K$, then $T.V.$ is small

### 1.3.2 Dividend:

Suppose $D_t$ is the dividend per share at time $t$. The present value of exercising the American put for sure at maturity is

$$P.V.[K - S_T] = KB(0,T) - [S_0 - D_t B(0,t)]$$

Note that the dividend leads to a stock price drop, hence, added value for the put. The time–value of the put is

$$T.V.(0) = P_0^a - [KB(0,T) - (S_0 - D_t B(0,t))]$$

To exercise or not to exercise?

1. if exercise: the value is $K - S_0$
2. if not exercise,

$$P_0^a = KB(0,T) - [S_0 - D_t B(0,t)] + T.V.(0)$$

It is optimal to exercise if and only if $K - S_0 > P_0^a$ i.e.

$$K - S_0 > KB(0,T) - [S_0 - D_t B(0,t)] + T.V.(0)$$

or

| $\underbrace{K(1 - B(0,T))}$ | $>$ | $\underbrace{D_t B(0,t)}$ | $+$ | $T.V.(0)$ |
|---|---|---|---|---|
| Interest earned due to early exercise | | Dividend lost due to exercise | | |

(1.2)

**Results**:

1. It may be optimal to exercise prematurely even if the stock pays no dividends.

**Proof**: If $D_t = 0$, the equation (1.2) becomes $K(1 - B(0,T)) > T.V.(0)$, if $T.V.(0)$ is small, then, early exercise.

2. Dividends tend to delay early exercise.

Proof: As $D_t$ increases, $K(1 - B(0,T)) > D_t B(0,t) + T.V.(0)$ may not hold. Hence, to wait.

3. It never pays to exercise just prior to an ex-dividend date.

**Proof**: Consider the following two strategies:

a. Strategy 1: Exercise just before the ex–dividend date,

$$(value)_1 = K - [S_t^c + D_t]$$

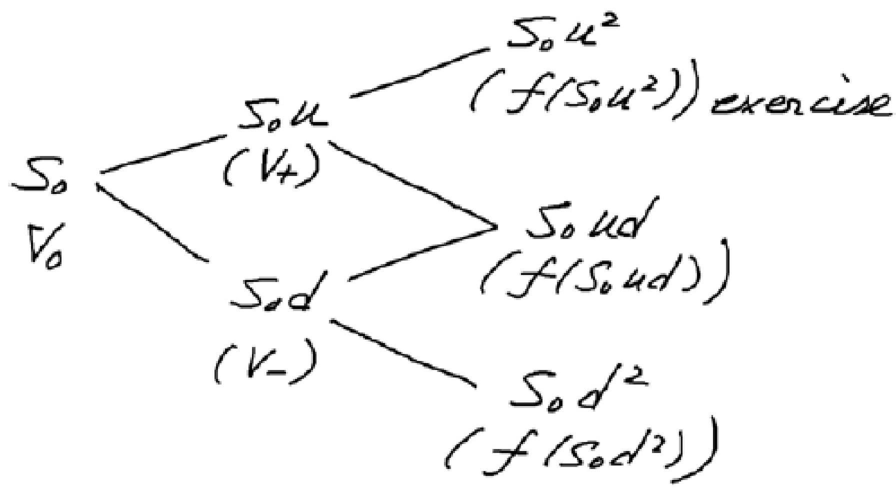
b. Strategy 2: Exercise just after the ex–dividend date

$$(value)_2 = K - S_t^c$$

Since $(value)_2 > (value)_1$, one should exercise after the ex–dividend date.

## 1.4 Valuation Using a Binomial Tree

Consider an American option with payoff $f(S_T)$. in the framework of a binomial model with growth parameters $u$ and $d$ chosen as follows $u = e^{(r-\frac{\sigma^2}{2})h+\sigma\sqrt{h}}$ and $d = e^{(r-\frac{\sigma^2}{2})h-\sigma\sqrt{h}}$. **(This choice will be explained later)**




At the $S_0u$–node,

$$\text{The option is worth} \begin{cases} \text{exercised at } t = h & f(S_0u) & \text{dead} \\ \text{not exercised at } t = h & e^{-rh}(qf(S_0u^2) + (1-q)f(S_0ud)), & \text{alive} \end{cases}$$

Compare these two values, choose the larger one, that is the value is

$$V_1^u = \max\left(f(S_0u); e^{-rh}(qf(S_0u^2) + (1-q)f(S_0ud))\right)$$

Similarly, at the $S_0d$–node,

$$V_1^d = \max\left(f(S_0d); e^{-rh}(qf(S_0du) + (1-q)f(S_0d^2))\right)$$

At $t = 0$,

$$\text{The option is worth} \begin{cases} \text{exercised at } t = 0 & f(S_0) & \text{dead} \\ \text{not exercised at } t = 0 & e^{-rh}\left(qV_1^u + (1-q)V_1^d\right), & \text{alive} \end{cases}$$

that is

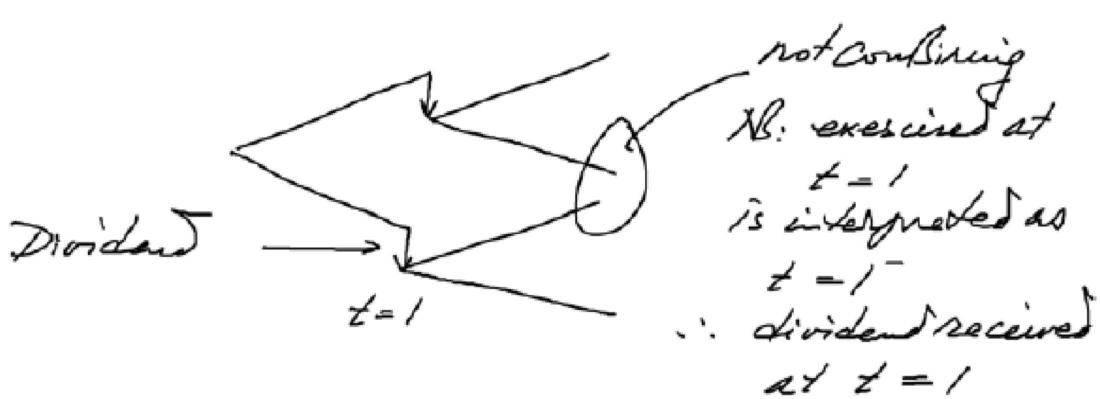
$$V_0 = \max\left(f(S_0); e^{-rh}\left(qV_1^u + (1-q)V_1^d\right)\right)$$

Note that, for an American call,

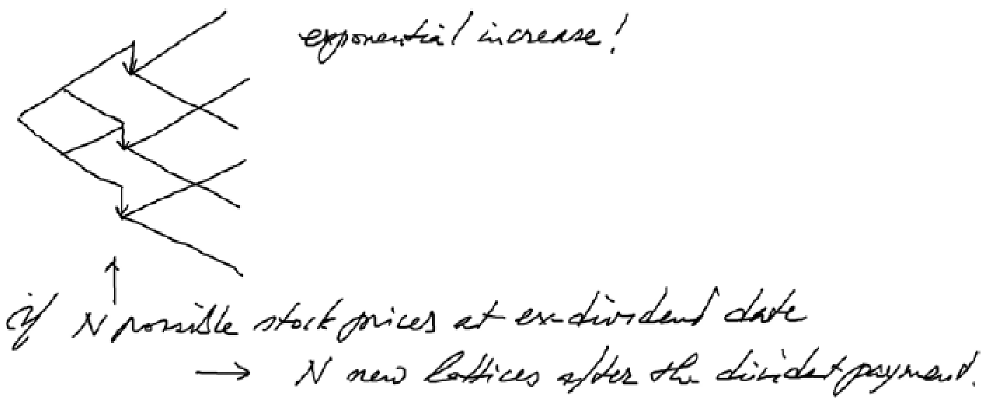
1. If no dividend,

$C_0^a = C_0^e$

2. If there are dividends,



3. Computational complexity:




The adaptive mesh methods: a high resolution (small $\Delta t = h$) tree is grafted onto a low resolution (large $\Delta t$) tree. This yields numerical efficiency over regular binomial or trinomial trees. In particular, for American options, there is a need for high resolution close to strike price and to maturity.