Contents

1 Lebesgue Integral

1 Simple Functions

Definition 1.1
Let $(X, \mathscr{A})$ be a measurable space. A function $f\colon X \longrightarrow \mathbb{R}$ (or $(\overline{\mathbb{R}})$) is called a **simple function** *if it is measurable and takes a finite number of values.*

Let $f\colon X \longrightarrow \bar{\mathbb{R}}$ be a simple function. If $\{c_1, \ldots, c_m\}$ is the set of values of f; $c_j \neq c_k$ for $j \neq k$, and $A_j = \{x \in X \text{ such that } f(x) = c_j\}$, then $X = \bigcup_j A_j$, $A_j \cap A_k = \emptyset$ if $j \neq k$ and $f = \sum_{j=1}^{m} c_j \chi_{A_j}$.
We remark that f is measurable if and only if A_j is measurable for all $j = 1, \ldots, m$.

Theorem 1.2
Let $(X, \mathscr{A})$ be a measurable space and $f\colon X \longrightarrow \overline{\mathbb{R}}$:

1. *If f is a measurable and bounded, there exists a sequence $(f_n)_{n\in\mathbb{N}^*}$ of simple functions which converges uniformly on X to f.*

2. *If f is a non-negative measurable function. Then there exists a sequence of non-negative measurable simple functions which increases to f.*

Proof .

1. Let $M > 0$ such that $\forall x \in X$, $|f(x)| < M$. We denote by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $(n, k) \in \mathbb{N}_0 \times \mathbb{Z}$ and $-2^n \leq k \leq 2^n - 1$, we set

$$A_{n,k} = \{x \in X;\ \frac{kM}{2^n} \leq f(x) < \frac{(k+1)M}{2^n}\}$$

   and we define f_n by:

$$f_n = \sum_{k=-2^n}^{2^n-1} \frac{kM}{2^n} \chi_{A_{n,k}}.$$

   The sets $A_{n,k}$ are measurables and f_n is measurable, for all $n \in \mathbb{N}$.

For $x_0 \in X$, there exists k_0 such that $x_0 \in X_{n,k_0}$. Then $f_n(x_0) = \dfrac{Mk_0}{2^n}$ and $|f(x_0) - f_n(x_0)| < \dfrac{M}{2^n}$. Then the sequence $(f_n)_n$ converges uniformly on X to f.

2. For all $n \in \mathbb{N}$, let $g_n = \inf(f, n) - \frac{1}{n}$. The function g_n is bounded measurable, then from the first case there exists a sequence of simple measurable functions $(f_m)_m$ such that $||f_n - g_n||_\infty < \dfrac{1}{2^n}$. We conclude that:

$$\lim_{n \longrightarrow +\infty} f_n = \lim_{n \longrightarrow +\infty} g_n = \lim_{n \longrightarrow +\infty} \inf(f, n) = f.$$

$f_n \leq g_n + \dfrac{1}{2^n} = \inf(f, n) - \dfrac{1}{n} + \dfrac{1}{2^n} \leq \inf(f, n+1) - \dfrac{1}{n+1} + \dfrac{1}{2^{n+1}} \leq f_{n+1}$. (It suffices to prove that for n big enough that $-\dfrac{1}{n} + \dfrac{1}{2^n} < -\dfrac{1}{n+1} + \dfrac{1}{2^{n+1}}$.)

□

2 Integration

For constructing the integral of real measurable functions on a measure space $(X, \mathscr{B}, \mu)$, we proceed by steps. We begin by the case of the integral of simple functions, then we define the integral of non-negative measurable functions by the increasing limit and we show that the monotone limit allows to define the integral of the measurable non-negative functions, and finally the decomposition of a measurable arbitrary functions $f = f^+ - f^-$ as the difference of two measurable non-negative functions extends the definition of the integral to the measurable functions.

Definition 2.1
If $f = \sum_{k=1}^{N} \lambda_k \chi_{\{f=\lambda_k\}}$ is a non-negative measurable simple function, we define the integral of f by:

$$\int_X f(x) d\,\mu(x) = \sum_{k=1}^{N} \lambda_k \mu(\{f = \lambda_k\})$$

In particular if $f = \chi_A$, A is a measurable subset, then $\int_X \chi_A(x) d\,\mu(x) = \mu(A)$, with the convention that $0 \times (+\infty) = 0$.

Proposition 2.2
Let $\mathscr{E}^+$ be the cone of non-negative simple functions on the measure space $(X, \mathscr{B}, \mu)$. The integral defined on $\mathscr{E}^+$ have the following properties:

1. $\forall\ \alpha\ \in \mathbb{R}^+, \quad \forall f \in\ \mathscr{E}^+; \qquad \int_X \alpha\ f(x) d\,\mu(x) = \alpha \int_X f(x) d\,\mu(x).$

2. $\forall f, g \in \mathscr{E}^+$; $\displaystyle\int_X (f+g)(x)d\,\mu(x) = \int_X f(x)d\,\mu(x) + \int_X g(x)d\,\mu(x)$.

3. $\forall f, g \in \mathscr{E}^+$ such that $f \leq g$; $\displaystyle\int_X f(x)d\,\mu(x) \leq \int_X g(x)d\,\mu(x)$.

4. If $(f_n)_n$ is an increasing sequence in $\mathscr{E}^+$ and if f is the limit of the sequence $(f_n)_n$ belongs to $\mathscr{E}^+$, then $\displaystyle\int_X f(x)d\,\mu(x) = \lim_{n\to+\infty}\int_X f_n(x)d\,\mu(x)$.

Proof .
It is evident that if $\alpha \geq 0$ and f and g of $\mathscr{E}^+$ then αf and $f+g \in \mathscr{E}^+$. ($\mathscr{E}^+$ is a convex cone).

1. The first property is evident.
2. Let f and g be two elements of $\mathscr{E}^+$. We denote by F (resp G) the set of values of f (resp of g).

$$f = \sum_{a\in F} a\chi_{\{f=a\}}, \quad g = \sum_{b\in G} b\chi_{\{g=b\}}.$$

We have

$$\forall\; a \in F;\; \{f=a\} = \bigcup_{b\in G}\{f=a, g=b\}.$$

$$\forall\; b \in G;\; \{g=b\} = \bigcup_{a\in F}\{f=a, g=b\}.$$

$$\int_X f(x)d\,\mu(x) = \sum_{a\in F} a\mu\{f=a\} = \sum_{(a,b)\in F\times G} a\mu\{f=a, g=b\}$$

$$\int_X g(x)d\,\mu(x) = \sum_{b\in G} a\mu\{g=b\} = \sum_{(a,b)\in F\times G} b\mu\{f=a, g=b\}$$

$$\int_X f(x)d\,\mu(x) + \int_X g(x)d\,\mu(x) = \sum_{(a,b)\in F\times G}(a+b)\mu\{f=a, g=b\}$$

$\{f+g=u\} = \bigcup_{(a,b)\in F\times G, a+b=u}\{f=a, g=b\}$. It results that

$$\mu\{f+g=u\} = \sum_{(a,b)\in F\times G, a+b=u}\mu\{f=a, g=b\}$$

Then

$$\int_X f(x)d\,\mu(x) + \int_X g(x)d\,\mu(x) = \sum_u u\mu\{f+g=u\} = \int_X (f+g)(x)d\,\mu(x).$$

3. If $\int_X f(x)d\,\mu(x) = +\infty$, then $\int_X g(x)d\,\mu(x) = +\infty$. The result is evident if $\int_X f(x)d\,\mu(x) < +\infty$ and $\int_X g(x)d\,\mu(x) = +\infty$. Assume now that $\int_X f(x)d\,\mu(x) < +\infty$ and $\int_X g(x)d\,\mu(x) < +\infty$, then the subsets $\{x \in X;\ f(x) = +\infty\}$ and $\{x \in X;\ g(x) = +\infty\}$ are a null sets.
   Let $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ the sets of finite values of f respectively of g. Set $\tilde{f} = \sum_{j=1}^{n} a_j \chi_{\{x\in X;\ f(x)=a_j\}}$ and $\tilde{g} = \sum_{j=1}^{m} b_j \chi_{\{x\in X;\ g(x)=b_j\}}$, then $\int_X f(x)d\,\mu(x) = \int_X \tilde{f}(x)d\,\mu(x)$ and $\int_X g(x)d\,\mu(x) = \int_X \tilde{g}(x)d\,\mu(x)$ and $h = \tilde{g} - \tilde{f} \in \mathcal{E}^+$.
   We deduce from 2) that
$$\int_X g(x)d\,\mu(x) = \int_X f(x)d\,\mu(x) + \int_X h(x)d\,\mu(x) \geq \int_X f(x)d\,\mu(x).$$

□

Lemma 2.3
Let $(f_n)_n$ be an increasing sequence in $\mathscr{E}^+$, and if $g \in \mathscr{E}^+$ such that $g \leq \lim_{n\longrightarrow+\infty} f_n$, then
$$\int_X g(x)d\,\mu(x) \leq \lim_{n\longrightarrow+\infty} \int_X f_n(x)d\,\mu(x).$$

Proof .
For $y \in g(X)$, let $E_y = \{x \in X;\ g(x) = y\}$. To prove the lemma it suffices to prove that for all $y \in g(X)$
$$\int_X g(x)\chi_{E_y}(x)d\,\mu(x) = y\mu(E_y) \leq \lim_{n\longrightarrow+\infty} \int_X f_n(x)\chi_{E_y}(x)d\,\mu(x).$$
The result is trivial if $y = 0$. For $0 < t < y$, we set $A_n = E_y \cap \{x \in X;\ f_n(x) \geq t\}$. $(A_n)_n$ is an increasing sequence of measurable sets and $E_y = \lim_{n\to+\infty} A_n$, because for all $x \in E_y$, $f_n(x) > t$ for n large.
$$t\mu\{E_y\cap\{x \in X;\ f_n(x) > t\}\} = \int_X t\chi_{E_y\cap\{x\in X;\ f_n(x)>t\}}(x)d\,\mu(x) \leq \int_X f_n(x)\chi_{E_y}(x)d\,\mu(x).$$
So $t\mu(E_y) \leq \lim_{n\to+\infty} \int_X f_n(x)\chi_{E_y}(x)d\,\mu(x)$. This is for any $0 < t < y$, then
$$y\mu(E_y) \leq \lim_{n\to+\infty} \int_X f_n(x)\chi_{E_y}(x)d\,\mu(x).$$

To prove 4) of the proposition 2.2, we denote $g = \lim\limits_{n\to+\infty} f_n$. Then $f_n \leq g$, $\forall n \in \mathbb{N}$ and the increasing sequence $\left(\int_X f_n(x)d\,\mu(x)\right)_n$ is bounded above by $\int_X g(x)\ d\,\mu(x)$. For the other sense we applied the lemma 2.3. $\square$

Definition 2.4
Let f be a non-negative measurable function on a measure space $(X, \mathscr{B}, \mu)$, we define

$$\int_X f(x)d\,\mu(x) = \text{Sup}\{\int_X g(x)d\,\mu(x); g \leq f \text{ and } g \in \mathscr{E}^+\}$$

this is a non-negative number finite or infinite.

Remark .
If f is a non-negative measurable function on a measure space $(X, \mathcal{B}, \mu)$, the theorem 1.2 yields the existence of an increasing sequence $(f_n)_n$ of $\mathscr{E}^+$ which converges to f. Then we have $\lim\limits_{n\to+\infty}\int_X f_n(x)d\,\mu(x) \leq \int_X f(x)d\,\mu(x)$. In the other hand for every function $g \in \mathscr{E}^+$ such that $g \leq f = \lim_{n\to+\infty} f_n$, we have from lemma 2.3 that $\int_X g(x)d\,\mu(x) \leq \lim\limits_{n\to+\infty}\int_X f_n(x)d\,\mu(x)$. So from the definition 2.4; $\int_X f(x)d\,\mu(x) \leq \lim\limits_{n\to+\infty}\int_X f_n(x)d\mu(x)$ and then $\int_X f(x)d\,\mu(x) = \lim\limits_{n\to+\infty}\int_X f_n(x)d\,\mu(x)$. This result is independent of the increasing sequence $(f_n)_n$ which converges to f. Then we have now the following theorem:

Theorem 2.5
Let f and g be two non-negative measurable functions on a measure space $(X, \mathscr{B}, \mu)$, and let λ be a non-negative real number, then we have:

1. $\int_X \lambda f(x)d\,\mu(x) = \lambda\int_X f(x)d\,\mu(x)$
2. $\int_X (f+g)(x)d\,\mu(x) = \int_X f(x)d\,\mu(x) + \int_X g(x)d\,\mu(x)$
3. *If $f \leq g$ then* $\int_X f(x)d\,\mu(x) \leq \int_X g(x)d\,\mu(x)$.

Proof .
For the proof it is enough to consider two increasing sequences $(\varphi_n)_n$ and $(\psi_n)_n$ of $\mathscr{E}^+$ which converge respectively to f and g, and then we apply the proposition 2.2.

3 Convergence Theorems

3.1 Monotone Convergence Theorem

Theorem 3.1 *(Monotone Convergence Theorem or Beppo-Levi's Theorem)*
Let $(f_n)_n$ be an increasing sequence of measurable non-negative functions on a measure space $(X, \mathcal{B}, \mu)$, then

$$\int_X \lim_{n\to+\infty} f_n(x)d\,\mu(x) = \lim_{n\to+\infty} \int_X f_n(x)d\,\mu(x).$$

Proof .
For all integer n, there exists an increasing non-negative sequence $(\varphi_{n,j})_j$ of $\mathscr{E}^+$ which converges to f_n. For any j, set $\psi_j = \operatorname{Sup}_{1\le n\le j} \varphi_{n,j}$. Then the sequence $(\psi_j)_j \in \mathscr{E}^+$ is increasing because $\psi_j = \operatorname{Sup}_{1\le n\le j} \varphi_{n,j} \le \operatorname{Sup}_{1\le n\le j} \varphi_{n,j+1} \le \operatorname{Sup}_{1\le n\le j+1} \varphi_{n,j+1} = \psi_{j+1}$.
We want to prove now that the sequence $(\psi_j)_j$ converges to f. We have for all $j \ge n$, $\varphi_{n,j} \le \psi_j$, then $f_n = \lim_{j\longrightarrow+\infty} \varphi_{n,j} \le \lim_{j\longrightarrow+\infty} \psi_j$, and then $f = \lim_{n\longrightarrow+\infty} f_n \le \lim_{j\longrightarrow+\infty} \psi_j$. In the other hand, the inequalities $\varphi_{n,j} \le f_n \le f$ shows that $\psi_j \le f$ and $\lim_{j\longrightarrow+\infty} \psi_j \le f$. The sequence $(\psi_j)_j$ is an increasing sequence of $\mathscr{E}^+$ and converges to f. Then $\displaystyle\int_X f(x)d\,\mu(x) = \lim_{j\longrightarrow+\infty} \int_X \psi_j(x)d\,\mu(x)$. Moreover we have

$$\psi_j \le f_j \Rightarrow \lim_{j\longrightarrow+\infty} \int_X \psi_j(x)d\,\mu(x) \le \lim_{j\longrightarrow+\infty} \int_X f_j(x)\ d\,\mu(x) \le \int_X f(x)d\,\mu(x),$$

which ends the proof of the theorem.

3.2 Fatou's Lemma

Lemma 3.2 *(Fatou's Lemma)*
Let $(f_n)_n$ be a sequence of non-negative measurable functions on a measure space $(X, \mathscr{B}, \mu)$, then:

$$\int_X \underline{\lim}_{n\longrightarrow+\infty} f_n(x)d\,\mu(x) \le \underline{\lim}_{n\longrightarrow+\infty} \int_X f_n(x)d\,\mu(x).$$

Proof .
$\underline{\lim}_{n\longrightarrow+\infty} f_n = \lim_{n\longrightarrow+\infty}(\inf_{j\ge n} f_j)$. We have $\displaystyle\int_X \inf_{j\ge n} f_j(x)\,d\,\mu(x) \le \inf_{j\ge n} \int_X f_j(x)d\,\mu(x)$.
The result follows from the Monotone Convergence Theorem.

Corollary 3.3
Let $(f_n)_n$ be a sequence of measurable non-negative functions on a measure space $(X, \mathscr{B}, \mu)$, then:

$$\int_X \sum_{n=1}^{+\infty} f_n(x)d\,\mu(x) = \sum_{n=1}^{+\infty} \int_X f_n(x)d\,\mu(x)$$

Example .

Let $f_n = n^2\chi_{[0,\frac{1}{n}]}$, $\int_{\mathbb{R}} \underline{\lim}_{n\longrightarrow+\infty} f_n(x)d\lambda(x) = 0$ and $\underline{\lim}_{n\longrightarrow+\infty}\int_{\mathbb{R}} f_n(x)d\lambda(x) = +\infty$.

Corollary 3.4

Let $(X,\mathscr{B},\mu)$ be a measure space and let f be a measurable non-negative function. For all $A \in \mathscr{B}$, let $\tau(A) = \int_X f(x)\chi_A(x)d\,\mu(x)$. Then τ is a non-negative measure on $(X,\mathscr{B})$ called measure of density f by respect to the measure μ. The integral of a measurable non-negative function g by this measure is given by:

$$\int_X g(x)\ d\,\tau(x) = \int_X f(x)g(x)d\,\mu(x).$$

Proof .

Let $(A_n)_n$ be a finite or infinite sequence of measurable pairwise disjoints sets. We have: $f\chi_{\cup_n A_n} = \sum_{n=1}^{+\infty} f\chi_{A_n}$. This which yields that

$$\tau\,(\bigcup_n A_n) = \int_X f(x)\chi_{\cup_n A_n}(x)d\,\mu(x) = \int_X \sum_{n=1}^{+\infty} f(x)\chi_{A_n}(x)d\,\mu(x) = \sum_{n=1}^{+\infty}\int_X f(x)\chi_{A_n}(x)d\,\mu(x).$$

The second part of the corollary is verified by any characteristic function χ_A of a measurable set A. Then it is valid for any simple non-negative function. By using the increasing continuity of the integrals, the result will be valid for non-negative measurable functions.

Definition 3.5

Let f, g be two functions defined on $(X,\mathscr{B},\mu)$. We say that $f = g$ almost everywhere, written $f = g$ a.e., if $\{x \in X;\ f(x) \neq g(x)\}$ is of measure zero. In particular if A is a measurable subset, then $\chi_A = 0$ a.e. if and only if $\mu(A) = 0$.

Definition 3.6

Let f be a function defined on $(X,\mathscr{B},\mu)$. We say that f is defined almost everywhere on X if there exist a null subset N such that f is defined on the complementary of N.

Definition 3.7

A sequence $(f_n)_n$ of functions defined on $(X,\mathscr{B},\mu)$ is said that converges almost everywhere to a function f if the set of x where this fails has measure zero.

Proposition 3.8

Let f and g be two non-negative measurable functions defined on a measure space $(X,\mathscr{B},\mu)$.

1. $\int_X f(x)d\,\mu(x) = 0$ *if and only if* $f = 0$ *almost everywhere.*

2. *If $f = g$ almost everywhere then* $\displaystyle\int_X f(x)d\,\mu(x) = \int_X g(x)\ d\,\mu(x)$.

Proof .

1. We suppose that $\displaystyle\int_X f(x)d\,\mu(x) = 0$. If $A_n = \{x \in X/f(x) \geq 1/n\}$, then $\chi_{A_n} \leq nf$ and $\displaystyle\int_X \chi_{A_n}(x)d\,\mu(x) = \mu(A_n) \leq n\int_X f(x)d\,\mu(x) = 0$. Then for all $n \in \mathbb{N}$; $\mu(A_n) = 0$. It results that $\{x/f(x) \neq 0\} = \bigcup_n A_n$ is a null set.
   If $f = 0$ almost everywhere then for all $n \in \mathbb{N}$, we define $f_n = \inf(f, n)$. The sequence $(f_n)_n$ is increasing and $\displaystyle\int_X f_n(x)d\,\mu(x) = 0$, then it follows from the monotone convergence theorem $\displaystyle\int_X f(x)d\,\mu(x) = 0$.

2. We suppose that $f \leq g$. Then the function $h = g - f$ is defined almost everywhere and equal to 0 almost everywhere.
   If $\displaystyle\int_X f(x)d\,\mu(x) = \int_X g(x)d\,\mu(x) = +\infty$, then we have the desired result.
   If $\displaystyle\int_X f(x)d\,\mu(x) = \int_X g(x)d\,\mu(x) < +\infty$, we have
   $\displaystyle 0 = \int_X h(x)d\,\mu(x) = \int_X g(x)d\,\mu(x) - \int_X f(x)d\,\mu(x)$.
   Let new define the function $h = \inf(f, g)$. h is a non-negative measurable function and we have: $h = f = g$ almost everywhere. Since $h \leq f$ then $\displaystyle\int_X h(x)d\,\mu(x) = \int_X f(x)d\,\mu(x)$, and since $h \leq g$ then $\displaystyle\int_X h(x)d\,\mu(x) = \int_X g(x)d\,\mu(x)$.
   It results that $\displaystyle\int_X f(x)d\,\mu(x) = \int_X g(x)d\,\mu(x)$.

□

Definition 3.9
Let $f\colon X \longrightarrow \bar{\mathbb{R}}$ be a measurable function. If $f^+ = \mathrm{Sup}(f, 0)$ and $f^- = \mathrm{Sup}(-f, 0)$, then $f = f^+ - f^-$. The function f is called integrable by respect to the measure μ if and only if $\displaystyle\int_X f^+(x)d\,\mu(x)$ and $\displaystyle\int_X f^-(x)d\,\mu(x)$ are finite.

The integral of f will be denoted $\displaystyle\int_X f(x)\ d\,\mu(x) = \int_X f^+(x)d\,\mu(x) - \int_X f^-(x)d\,\mu(x)$, and if f is measurable and $\displaystyle\int_X f^+(x)d\,\mu(x) < +\infty$ or $\displaystyle\int_X f^-(x)d\,\mu(x) < +\infty$ we will denote of the same way $\displaystyle\int_X f(x)d\,\mu(x) = \int_X f^+(x)d\,\mu(x) - \int_X f^-(x)d\,\mu(x)$.
We define $L^1(X)$ the space of integrable functions on X.

Proposition 3.10
The set $L^1(X)$ is a vector space on $\mathbb{R}$ and the map $f \longmapsto \int_X f(x)d\,\mu(x)$ is a linear form on $L^1(X)$ and we have $\Big|\int_X f(x)d\,\mu(x)\Big| \leq \int_X |f(x)|d\,\mu(x)$.

Proof .
Let f and g be two integrable functions.
Since $|f+g| \leq |f|+|g|$, then $\int_X |f(x)+g(x)|d\,\mu(x)| \leq \int_X |f(x)|d\,\mu(x)+\int_X |g(x)|\, d\,\mu(x)$, and then $f+g \in L^1(X)$.
We have $f+g = (f+g)^+ - (f+g)^- = f^+ - f^- + g^+ - g^-$, then $(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$. It follows that

$$\begin{aligned}\int_X (f+g)^+(x)d\,\mu(x) &+ \int_X f^-(x)d\,\mu(x) + \int_X g^-(x)\; d\,\mu(x)\\ &= \int_X (f+g)^-(x)d\,\mu(x) + \int_X f^+(x)\; d\,\mu(x) + \int_X g^+(x)d\,\mu(x)\end{aligned}$$

and

$$\begin{aligned}\int_X (f+g)(x)d\,\mu(x) &= \int_X (f+g)^+(x)d\,\mu(x) - \int_X (f+g)^-(x)\; d\,\mu(x)\\ &= \int_X f^+(x)d\,\mu(x) - \int_X f^-(x)\; d\,\mu(x) + \int_X g^+(x)d\,\mu(x) - \int_X g^-(x)d\,\mu(x)\\ &= \int_X f(x)d\,\mu(x) + \int_X g(x)d\,\mu(x).\end{aligned}$$

The other properties are evidents.

Corollary 3.11

1. *If f is measurable and $a \leq f \leq b$ and $\mu(X) < +\infty$, then $f \in L^1(X)$ and we have: $a\mu(X) \leq \int_X f(x)\; d\,\mu(x) \leq b\mu(X)$.*
2. *If f is measurable and $g \in L^1(X)$ and $f \leq g$, then $\int_X f(x)d\,\mu(x) \leq \int_X g(x)d\,\mu(x)$.*
3. *If E is a measurable null set, then $\int_E f(x)d\,\mu(x) = 0$ for any measurable function f.*
4. *Any bounded measurable function and equal to zero in the complementary of a subset of finite measure is integrable.*

Remarks .

1. Let f be an integrable function with respect to a measure μ. Then $\{x \in X/f(x) = \pm\infty\}$ is a null set.

2. On a measure space $(X, \mathscr{B}, \mu)$, the set of functions that are $f = 0$ a.e. is a vector space of $L^1(X, \mathscr{B})$ closed under countable $(\mathrm{Sup}, \inf)$. We denote $L^1(X, \mathscr{B})$ or $L^1(\mu)$ the quotient space $L^1(X, \mathscr{B})$ by the space of null a.e functions. We call that $f = g$ in $L^1(X)$ if $f = g$ μ-almost everywhere.

Definition 3.12
A sequence $(f_n)_n$ of measurable functions on a measure space $(X, \mathscr{B}, \mu)$ converges almost everywhere if the set of divergence of the sequence is a null set. We will denote by $\lim f_n$ *any arbitrary measurable function f such that $(f_n)_n \longrightarrow f$ almost everywhere on X.*

3.3 Dominate Convergence Theorem

Theorem 3.13 *(Dominate Convergence Theorem (or Lebesgue theorem)*
Let $(f_n)_n$ be a sequence of measurable functions on a measure space $(X, \mathscr{B}, \mu)$. We assume that:

i) the sequence $(f_n)_n$ converges almost everywhere on X to a measurable function f definite almost everywhere.

ii) There exist a non-negative integrable function g such that: $|f_n| \leq g$ almost everywhere for all n. Then the sequence $(f_n)_n$ and the function f are integrable and we have:

$$\int_X f(x)\ d\,\mu(x) \;=\; \lim_{n \longrightarrow +\infty} \int_X f_n(x) d\,\mu(x).$$

The interest of the Dominated Convergence Theorem is that it does not require uniform convergence to permute the limit and the integral.

Theorem 3.14
Let $(f_n)_n$ be a sequence of measurable functions on a measure space $(X, \mathscr{B}, \mu)$. We assume that there exist a non-negative integrable function g such that for all n, $|f_n| \leq g$ almost everywhere. Then:

$$\int_X \underline{\lim} f_n(x) d\,\mu(x) \leq \underline{\lim} \int_X f_n(x) d\,\mu(x) \tag{3.1}$$

$$\int_X \overline{\lim} f_n d\,\mu(x) \geq \overline{\lim} \int_X f_n(x) d\,\mu(x) \tag{3.2}$$

and if the sequence $(f_n)_n$ converges almost everywhere on X to a measurable function f defined almost everywhere, then $f \in L^1(X)$ and we have:

$$\int_X f(x) d\,\mu(x) \;=\; \lim_{n \to +\infty} \int_X f_n(x) d\,\mu(x) \tag{3.3}$$

Proof .
The function g is finite almost everywhere on X because it is integrable. If we replace g by the function $g\chi_{\{x/g(x)<+\infty\}}$ this which not change the inequalities $|f_n| \leq g$ almost everywhere. Thus we can suppose that g is finite on X. We replace the sequence $(f_n)_n$ by the functions $f_n\chi_{\{|f_n|\leq g\}}$, this which not modified the integrals $\displaystyle\int_X f_n(x)d\,\mu(x)$ neither the equivalence classes $\lim_{n\longrightarrow+\infty} f_n$ almost everywhere. Then we can suppose that $|f_n| \leq g$ on X. From these modifications, the functions $(f_n)_n$, $\overline{\lim} f_n$ and $\underline{\lim} f_n$ are finite and integrable on X. We apply the Fatou's lemma to the sequence $f_n + g$ we shall have

$$\int_X \underline{\lim}(f_n + g)(x)d\,\mu(x) \leq \underline{\lim} \int_X (f_n + g)(x)d\,\mu(x)$$

Since $\underline{\lim}_{n\longrightarrow+\infty}(f_n + g) = (\underline{\lim}_{n\longrightarrow+\infty} f_n\ ) + g$ on X, we shall have

$$\int_X \underline{\lim}_{n\longrightarrow+\infty} f_n(x)d\,\mu(x) \leq \underline{\lim}_{n\longrightarrow+\infty} \int_X f_n(x)d\,\mu(x)$$

And from Fatou's lemma applied to the sequence $(-f_n + g)_n$ we shall have

$$\int_X \underline{\lim}_{n\longrightarrow+\infty}(-f_n)(x)d\,\mu(x) \leq \underline{\lim}_{n\longrightarrow+\infty} \int_X -f_n(x)d\,\mu(x)$$

Then

$$\int_X \overline{\lim}_{n\longrightarrow+\infty} f_n(x)d\,\mu(x) \geq \overline{\lim}_{n\longrightarrow+\infty} \int_X f_n(x)d\,\mu(x)$$

The result follows easily.

3.4 Applications

3.4.1 Double Series

If we apply the Dominate Convergence Theorem on the measure space $(\mathbb{N}, \mathscr{P}(\mathbb{N}), \mu)$ with the measure μ defined by: $\mu(n) = 1$ for all n of $\mathbb{N}$, we have the following result:

Theorem 3.15
Let $(a_{m,n})_{m,n}$ be a double sequence of complex numbers such that:
i) $\lim\limits_{n\longrightarrow+\infty} a_{m,n} = a_m$ *for all* $m \in \mathbb{N}$.

ii) There exist a sequence $(b_m)_n$ of non-negative real numbers such that $\displaystyle\sum_{m=1}^{+\infty} b_m < +\infty$ *and* $|a_{m,n}| \leq b_m$ *for all* $n \in \mathbb{N}$.
Then we have: $\displaystyle\lim_{n\longrightarrow+\infty} \sum_{m=1}^{+\infty} a_{m,n} = \sum_{m=1}^{+\infty} a_m.$

3.4.2 Integral Depending on Parameter

Let $(X, \mathscr{B}, \mu)$ be a measure space, and let E be a metric space.

Proposition 3.16
Let $f\colon\ E \times X \longrightarrow \mathbb{C}$ such that for all $t \in E$; the mapping $x \longrightarrow f(t,x)$ is integrable. We define

$$F(t) = \int_X f(t,x)d\,\mu(x)$$

Let $a \in E$, we assume that:
For almost any $x \in X$; the mapping $t \longmapsto f(t,x)$ is continuous in a.
There exist a neighborhood $V(a)$ of a and an integrable function g such that $\forall\ t \in V(a)$, $|f(t,.)| \leq g(.)$. Then F is continuous in a.

Proof .
It suffices to apply the Dominate Convergence Theorem to the sequence $(f(a_n,.))_n$ for $n \in \mathbb{N}$; where $(a_n)_n$ is a sequence in $V(a)$ which converges to a.

Proposition 3.17
Let Ω be an open set of $\mathbb{R}$ (resp $\mathbb{C}$). Let $f\colon\ \Omega \times X \longrightarrow \mathbb{C}$; such that for all $t \in \Omega$; the mapping $x \longmapsto f(t,x)$ is integrable. We define

$$F(t) = \int_X f(t,x)d\,\mu(x)$$

We assume that:
For almost all $x \in X$; the mapping $t \longmapsto f(t,x)$ is derivable on Ω (resp holomorphic on Ω). We denote $|\dfrac{\partial}{\partial t}f(t,x)|$ its derivative.
The function $f(t,.)$ is integrable on X and there exist a non-negative integrable function g such that for almost all $x \in X$, $|\dfrac{\partial}{\partial t}f(t,x)| \leq g(x)$ for any $t \in \Omega$. Then F is derivable on Ω (resp holomorphic) and for any t in Ω:

$$\frac{d}{dt}\int_X f(t,x)d\,\mu(x) = \int_X \frac{\partial}{\partial t}f(t,x)d\,\mu(x).$$

Proof .
Let $a \in \Omega$ and $(h_n)_n$ be a sequence of real numbers converging to 0 and such that $a + h_n \in \Omega$. ($h_n \neq 0$, for all n). We define the sequence $(\varphi_n)_n$ by:

$$\varphi_n(x) = \frac{f(a+h_n,x) - f(a,x)}{h_n}$$

For almost all $x \in X$, $\lim_{n\longrightarrow\infty}\varphi_n(x) = \frac{\partial}{\partial t}f(a,x)$ and for such x we have $|\varphi_n(x)| \leq g(x)$. Then according to the mean value theorem. The Dominate Convergence Theorem shows that the function $\frac{d}{dt}f(t,x)$ is integrable and:

$$\int_X \frac{\partial}{\partial t} f(a,x)\, d\,\mu(x) = \lim_{n \longrightarrow +\infty} \int_X \varphi_n(x) d\,\mu(x) \ = \lim_{n \longrightarrow +\infty} \frac{F(a+h_n) - F(a)}{h_n}$$

Remarks .

1. If for each $a \in \Omega$ there exists a neighborhood $V(a)$ and an integrable function g such that for almost all $x \in X$, $|\frac{\partial}{\partial t} f(t,x)| \leq g(x)$ for all $t \in V(a)$. Then F is derivable on Ω and for all $t \in \Omega$:

$$\frac{d}{dt} \int_X f(t,x) d\,\mu(x) = \int_X \frac{\partial}{\partial t} f(t,x) d\,\mu(x).$$

2. If in addition $\frac{\partial}{\partial t} f(t,x)$ is continuous, then F is $\mathcal{C}^1$.

Exercises .

1. Let f be an integrable function on $[0,1]$, then $\lim_{n \longrightarrow +\infty} \int_0^1 x^n f(x) dx = 0$, in fact $|x^n \ f(x)| \leq |f(x)|$ which is integrable, and $\lim_{n \to +\infty} x^n \ f(x) = 0$ a.e. Then the result follows from the Dominate Convergence Theorem.

2. Let $(f_n)_n$ be the sequence defined in $[0,1]$ by: $f_n(x) = \frac{nx}{1+n^4x^4}$.
   It is easy to prove that the sequence $(f_n)_n$ is uniformly bounded on $[0,1]$ and $\lim_{n \to +\infty} f_n(x) = 0$. Then from the Dominate Convergence Theorem $\lim_{n \longrightarrow +\infty} \int_0^1 \frac{nx}{1+n^4x^4} dx =$ 0.

3. Let $(f_n)_n$ the sequence defined in $[0,1]$ by: $f_n(x) = \frac{nx}{1+n^2x^4}$. $\lim_{n \to +\infty} f_n(x) = 0$ but $\int_0^1 \frac{nx}{1+n^2x^4} dx = \frac{1}{2} \int_0^n \frac{dt}{1+t^2}$, then $\lim_{n \longrightarrow +\infty} \int_0^1 \frac{nx}{1+n^2x^4} dx = \frac{\pi}{4}$.

4 Comparison of Riemann and Lebesgue integrals

4.1 Riemann and Lebesgue Integrals

Let a and b two reals numbers, $a < b$. We consider the measure space $([a,b], \mathscr{B}^*, \lambda)$, where λ is the Lebesgue measure on $\mathbb{R}$ and $\mathscr{B}^*$ is the Lebesgue σ-algebra of $[a,b]$. For a bounded function f on $[a,b]$, we denote $\int_a^b f(x)dx$ the Riemann integral for f on $[a,b]$ and $\int_a^b f(x) d\,\lambda(x)$ the Lebesgue integral, if they exist.

Let f be a bounded function on $[a,b]$. Then from the definition of the Riemann integral and the proprieties of the lower and upper Darboux sum of f, there exists an increasing sequence of partitions $(\sigma_n)_n$ of $[a,b]$ such that if $\sigma_n = \{x_0 = a, \dots x_{p_n} = b\}$ the sequence $(\delta_n)_n$ defined by: $\delta_n = \mathrm{Sup}_{0\leq k\leq p_n-1}\ |x_{k+1} - x_k|$ converges to 0. (δ_n is called the norm of the partition). We denote

$$(U)\int_a^b f(x)dx = \lim_{n\longrightarrow+\infty} S(\sigma_n, f)$$

$$(L)\int_a^b f(x)dx = \lim_{n\longrightarrow+\infty} s(\sigma_n, f)$$

Let $(g_n)_n$ and $(h_n)_n$ the sequences of simple functions defined by:

$$g_n(x) = \begin{cases} m_k = \inf_{t\in[x_k,x_{k+1}[} f(t) \quad if \quad x_k \leq x < x_{k+1} \\ g_n(b) = f(b) \end{cases}$$

$$h_n(x) = \begin{cases} M_k = \mathrm{Sup}_{t\in[x_k,x_{k+1}[} f(t) \quad if \quad x_k \leq x < x_{k+1} \\ h_n(b) = f(b) \end{cases}$$

The sequence $(g_n)_n$ is increasing and the sequence $(h_n)_n$ is decreasing. For $x \in [a,b]$, the sequence $(g_n)_n$ converges to a function g and the sequence $(h_n)_n$ converges to a function h. We remark that

$$S(\sigma_n, f) = \int_a^b h_n(x)dx = \int_a^b h_n(x)d\,\lambda(x)$$

$$s(\sigma_n, f) = \int_a^b g_n(x)dx = \int_a^b g_n(x)d\,\lambda(x)$$

Since g and h are measurables, it follows from the monotone convergence theorem,

$$\lim_{n\to+\infty}\int_a^b g_n(x)dx = (L)\int_a^b f(x)dx = \int_a^b g(x)d\,\lambda(x) \tag{4.4}$$

$$\lim_{n\to+\infty}\int_a^b h_n(x)dx = (U)\int_a^b h(x)dx = \int_a^b f(x)d\,\lambda(x) \tag{4.5}$$

In the other hand $g(x) \leq f(x) \leq h(x)\ \forall x \in [a,b]$.

Theorem 4.1
Let f be a bounded function on $[a,b]$.

a) If f is Riemann-integrable on $[a,b]$, then f is Lebesgue integrable on $[a,b]$ and:

$$\int_a^b f(x)d\,\lambda(x) = \int_a^b f(x)dx$$

b) f is Riemann-integral on $[a,b]$ if and only if, the set of discontinuity of f is a null set.

c) If the set of discontinuity of f is a null set, then f is Lebesgue integrable and

$$\int_a^b f(x)d\,\lambda(x) = \int_a^b f(x)dx$$

For the proof we need the following lemma:

Lemma 4.2
Let f, g and h as above. For $x \in [a,b] \setminus \left(\bigcup_{n=1}^{+\infty} \sigma_n\right)$, $g(x) = h(x)$ if and only if f is continuous in the point x.

Proof of the lemma .
Let $x \in [a,b] \setminus \left(\bigcup_{n=1}^{+\infty} \sigma_n\right)$ and $\delta_n = ||\sigma_n||$. The sequence $(\delta_n)_n$ converges to 0.
If f is continuous in x, then for $\varepsilon > 0, \exists \eta > 0$ such that $\forall t \in [a,b]$ and $|t-x| < \eta$, then $|f(x) - f(t)| < \varepsilon$.
Let n_0 such that $\forall n \geq n_0$, $\delta_{n_0} < \eta$.
For $n > n_0$, σ_n is a partition of $[a,b]$, then there exist $k \in \{0, \ldots, p_n - 1\}$ such that $x_k < x < x_{k+1}$. It results that $\forall t \in]x_k, x_{k+1}[$, $|f(x) - f(t)| < \varepsilon$, then $h_n(x) = M_k \leq f(x) + \varepsilon$ and $g_n(x) = m_k \geq f(x) - \varepsilon$ and $h_n(x) - g_n(x) \leq \varepsilon$. This is for all $n \geq n_0$. Then $h(x) - g(x) \leq \varepsilon$ and this for all $\varepsilon > 0$ and then $g(x) = h(x)$.
Conversely if $g(x) = h(x)$ and $x \notin (\bigcup_n \sigma_n)$. Since $g(x) \leq f(x) \leq h(x)$, then $f(x) = g(x) = h(x)$, $(g_n(x))_n$ and $(h_n(x))_n$ converges to $f(x)$.
Let $\varepsilon > 0$, it follows from the above result that there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$: $0 \leq f(x) - g_n(x) < \varepsilon$ and $0 \leq h_n(x) - f(x) < \varepsilon$.
σ_{n_0} is a partition of $[a,b]$, then there exist $k \in \{0, \ldots, p_{n_0} - 1\}$ such that $x \in [x_k, x_{k+1}[= I$. We have:

$$h_{n_0}(x) - \varepsilon < f(x) < g_{n_0}(x) + \varepsilon$$

Moreover $h_{n_0}(x) = \underset{t \in]x_k, x_{k+1}[}{\text{Sup}} f(t)$ and $g_{n_0}(x) = \inf_{t \in]x_k, x_{k+1}[} f(t)$.
Then $\forall t \in I,\ f(t) - \varepsilon < f(x) < f(t) + \varepsilon$ this which yields that f is continuous in the point x.

Proof of the theorem .
a) If f is Riemann-integrable on $[a,b]$, we have

$$(L)\int_a^b f(x)dx = (U)\int_a^b f(x)dx = \int_a^b f(x)dx$$

and from (4.4) and (4.5) we shall have: $\displaystyle\int_a^b h(x)d\lambda(x) = \int_a^b g(x)d\lambda(x)$. Thus $\displaystyle\int_a^b (h(x) - g(x))d\lambda(x) = 0$. Moreover $h - g$ is a non-negative integrable function, then $h = g$ λ-almost everywhere and then $f = g$ a.e on $[a,b]$. Thus f is measurable and $\displaystyle\int_a^b f(x)dx = \int_a^b f(x)d\,\lambda(x)$.

b) f is Riemann-integrable $\Longleftrightarrow$ $(U)\displaystyle\int_a^b f(x)dx = (L)\int_a^b f(x)dx \Longleftrightarrow h = g$ a.e and the result is deduced from the previous lemma; in fact:
f Riemann-integrable $\Longleftrightarrow$ $h = g$ a.e which is equivalent to $\{x/h(x) \neq g(x)\} \cup (\bigcup_n \sigma_n)$ is a null set with respect to the Lebesgue measure, and this is equivalent to the fact that f is continuous a.e on $[a, b]$.

c) If the set of discontinuity of f is a null set. Then $\lim\limits_{n\to+\infty} g_n(x) = \lim\limits_{n\to+\infty} h_n(x) = f(x)$ at each point of continuity of f, then f is measurable and Dominate convergence theorem yields

$$\lim_{n\to+\infty}\int_{[a,b]} g_n(x)dx = \int_{[a,b]} f(x)dx$$

$$\lim_{n\to+\infty}\int_{[a,b]} h_n(x)dx = \int_{[a,b]} f(x)dx$$

Thus f is Riemann integrable and

$$\int_a^b f(x)d\,\lambda(x) = \int_a^b f(x)dx.$$

□

We give now a new proof of the theorem (4.1)

Proposition 4.3
Let $f\colon [a, b] \to \mathbb{R}$ is a bounded function. f is Riemann integrable iff it is continuous almost everywhere on $[a, b]$.

Proof .
a) Suppose that f is Riemann integrable. For each $x \in [a, b]$, set

$$g(x) = \operatorname*{Sup}_{\delta>0}\ \inf_{y\in[a,b],|y-x|\leq\delta} f(y),$$

$$h(x) = \inf_{\delta>0}\ \operatorname*{Sup}_{y\in[a,b],|y-x|\leq\delta} f(y),$$

so that f is continuous at x iff $g(x) = h(x)$. We have $g \leq f \leq h$, so if σ is any partition of $[a, b]$ then $S(g, \sigma) \leq S(f, \sigma) \leq S(h, \sigma)$ and $s(g, \sigma) \leq s(f, \sigma) \leq s(h, \sigma)$. But $S(f, \sigma) = S(h, \sigma)$ and $s(g, \sigma) = s(f, \sigma)$, because on any open interval $]c, d[\subset [a, b]$ we must have

$$\inf_{x\in]c,d[} g(x) = \inf_{x\in]c,d[} f(x), \quad \operatorname*{Sup}_{x\in]c,d[} f(x) = \operatorname*{Sup}_{x\in]c,d[} h(x).$$

It follows that

$$s(f) = s(g) \leq S(g) \leq S(f), \quad s(f) \leq s(h) \leq S(h) = S(f).$$

Because f is Riemann integrable, both g and h must be Riemann integrable, with integrals equal to $\displaystyle\int_a^b f(x)dx$. Then, they are both Lebesgue integrable, with the same integral. But $g \leq h$, so $g = h$ a.e. Now f is continuous at any point where g and h agree, so f is continuous a.e.

(b) Now suppose that f is continuous a.e. For each $n \in \mathbb{N}$, let σ_n be the partition of $[a,b]$ into 2^n equal intervals. Set

$$h_n(x) = \operatorname*{Sup}_{y\in]c,d[} f(y), \quad g_n(x) = \inf_{y\in]c,d[} f(y)$$

if $]c,d[$ is an open interval of σ_n containing x; for definiteness, say $h_n(x) = g_n(x) = f(x)$ if x is one of the points of the list σ_n. Then $(g_n)_n$, $(h_n)_n$ are, respectively, increasing and decreasing sequences of functions, each function constant on a each of a finite family of intervals covering $[a,b]$; and $s_(f,\sigma_n) = \int g_n d\,\mu$, $S(f,\sigma_n) = \int h_n d\,\mu$. Next,

$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} h_n(x) = f(x)$ at any point x at which f is continuous; so $f = \lim_{n\to\infty} g_n = \lim_{n\to\infty} h_n$ a.e. By Lebesgue's Dominated Convergence Theorem, $\lim_{n\to\infty} \int g_n d\,\mu = \int f d\,\mu = \lim_{n\to\infty} \int h_n d\,\mu$; but this means that

$s(f) \geq \int f d\,\mu \geq S(f)$,

so these are all equal and f is Riemann integrable.

□

4.2 Generalized Integral and Lebesgue Integral

Let $]a,b[$ be an open interval of $\mathbb{R}$ and let f be a locally Riemann-integrable function on $]a,b[$ (i.e. f is Riemann-integrable on $[\alpha,\beta]$ for all α,β such that $a < \alpha < \beta < b$.) We say that the generalized Riemann integral of f exists on $]a,b[$ if and only if $\lim\limits_{\beta\to b} \int_{x_0}^{\beta} f(x)dx$ exists and $\lim\limits_{\alpha\to a} \int_{\alpha}^{x_0} f(x)dx$ exists. (x_0 fixed in $]a,b[$). This limit when it exists does not depend on x_0 and is denoted by: $\int_a^b f(x)dx$.

Proposition 4.4
Let f be a locally Riemann-integrable function defined on $]a,b[$. Then f is Lebesgue-integrable on $]a,b[$ if and only if the improper integral $\int_a^b f(x)dx$ is absolutely convergent and in this case the Riemann integral and the Lebesgue integral coincide (i.e. $\int_a^b f(x)dx = \int_a^b f(x)d\,\lambda(x)$.)

Proof .
We consider two sequences $(a_n)_n$ and $(b_n)_n$ of $]a,b[$ such that the sequence $(a_n)_n$ decreases to a and the sequence $(b_n)_n$ increases to b. Let $\varphi_n(x) = |f(x)|\chi_{[a_n,b_n]}$. The sequence $(\varphi_n)_n$ increases to $|f|\chi_{]a,b[}$. The functions φ_n are measurable then f is measurable. It follows from monotone convergence theorem that

$$\lim_{n\longrightarrow+\infty} \int_{\mathbb{R}} \varphi_n(x)d\,\lambda(x) = \int_a^b |f(x)|d\,\lambda(x)$$

Moreover it follows from the previous theorem that $\int_{\mathbb{R}} \varphi_n(x)d\,\lambda(x) = \int_{a_n}^{b_n} |f(x)|dx$.

Then from the previous definition $\lim_{n\longrightarrow+\infty} \int_{\mathbb{R}} \varphi_n(x)d\,\lambda(x) = \int_a^b |f(x)|dx$. Then it follows that f is Lebesgue integrable. To show that the two integrals coincide we set $g_n = f\chi_{[a_n,b_n]}$. Then $(g_n)_n$ converges to $f\chi_{]a,b[}$. The functions g_n are integrable and $|g_n| \leq |f|\chi_{[a,b]}$. It follows from the Dominate Convergence Theorem that

$$\lim_{n\longrightarrow+\infty} \int_a^b g_n(x)d\,\lambda(x) = \int_a^b f(x)\ d\,\lambda(x)$$

The result follows from previous result.

Conversely If f is Lebesgue-integrable on $]a,b[$, then $|f|$ is Lebesgue-integrable on $]a,b[$. Let $(a_n)_n$ and $(b_n)_n$ two sequences in $]a,b[$, such that the sequence $(a_n)_n$ decreases to a and $(b_n)_n$ increases to b. The sequence $f_n = |f|\chi_{[a_n,b_n]}$ fulfill the hypotheses of the monotone convergence theorem, then

$$\lim_{n\longrightarrow+\infty} \int_a^b f_n(x)d\,\lambda(x) = \int_a^b f(x)\ d\,\lambda(x) < +\infty$$

Moreover $\int_a^b f_n(x)d\,\lambda(x) = \int_{a_n}^{b_n} |f(x)|dx$ which follows from the previous theorem. Then

$\lim_{n\longrightarrow+\infty} \int_{a_n}^{b_n} |f(x)|dx$ exists in $\mathbb{R}$. Then $\int_a^b |f(x)|dx < +\infty$.

5 Fubini's Theorem

5.1 Product Measure Spaces

Let $(X_1, \mathscr{A}_1, \mu_1)$ and $(X_2, \mathscr{A}_2, \mu_2)$ two measure spaces. We intend to construct the product measure on a suitable σ-algebra contained in the power set of the Cartesian product $X = X_1 \times X_2$. By a rectangular set R in X we mean any set of the form $R = A \times B$ where $A \in \mathscr{A}_1$ and $B \in \mathscr{A}_2$. We will take as the family of elementary sets for the product measure

$$\mathscr{C} = \{E = \bigcup_{j=1}^{n} R_j;\ R_j = A_j \times B_j, A_j \in \mathscr{A}_1,\ B_j \in \mathscr{A}_2\}, \tag{5.6}$$

where R_j are disjoint rectangles and n is an arbitrary natural number. $\mathscr{C}$ is an algebra.

Definition 5.1
We define the product measure

$$\mu_1 \otimes \mu_2(E) = \sum_{j=1}^{n} \mu_1(A)\mu_2(B)$$

for each elementary set $E \in \mathscr{C}$ as defined by the equation (5.6).

This definition requires justification, because the decomposition given in equation (5.6) is not unique. Suppose

$$E = \bigcup_{j=1}^{n} A_j \times B_j = \bigcup_{j=1}^{m} C_j \times D_j$$

It follows from the finite additivity of each of the measures μ_1 and μ_2 that

$$\mu_1(A_j)\mu_2(B_j) = \sum_{k=1}^{m} \mu_1(A_j \cap C_k)\mu_2(B_j \cap D_k),$$

and

$$\mu_1(C_k)\mu_2(D_k) = \sum_{j=1}^{n} \mu_1(A_j \cap C_k)\mu_2(B_j \cap D_k).$$

and then

$$\sum_{k=1}^{m} \mu_1(C_k)\mu_2(D_k) = \sum_{j=1}^{n} \mu_1(A_j)\mu_2(B_j).$$

Lemma 5.2
Suppose $A \times B = \bigcup_{j=1}^{+\infty} A_j \times B_j$, *where* $A, A_j \in \mathscr{A}_1$ *and* $B, B_j \in \mathscr{A}_2$ *and the* $A_j \times B_j$ *are disjoint. Then*

$$\mu_1 \otimes \mu_2(A \times B) = \sum_{j=1}^{+\infty} \mu_1 \otimes \mu_2(A_j \times B_j).$$

Proof .
We have

$$\chi_A(x)\chi_B(y) = \chi_{A\times B}(x, y) = \sum_{j=1}^{+\infty} \chi_{A_j}(x)\chi_{B_j}(y)$$

By the Monotone Convergence Theorem

$$\chi_A(x)\mu_2(B) = \sum_{j=1}^{+\infty} \chi_{A_j}(x)\mu_2(B_j),$$

and also by the Monotone Convergence Theorem

$$\mu_1(A)\mu_2(B) = \sum_{j=1}^{+\infty} \mu_1(A_j)\mu_2(B_j).$$

Definition 5.3
If $E \subset X_1 \times X_2$; we define the $x-$section of E by

$$E_x = \{y \in X_2;\ (x,y) \in E\},\ y \in X_2$$

and the $y-$section by

$$E^y = \{x \in X_1;\ (x,y) \in E\},\ y \in X_2.$$

Similarly, if $f\colon X \longrightarrow \bar{\mathbb{R}}$, then the x and y-sections of f are the mappings $f_x\colon X_2 \longrightarrow \bar{\mathbb{R}}$ and $f^y\colon X_1 \longrightarrow \bar{\mathbb{R}}$ defined by $f_x(y) = f(x,y)$ and $f^y(x) = f(x,y)$.

5.2 Fubini-Tonelli's Theorem

Theorem 5.4 *(Fubini Tonelli)*
Let $(X_1, \mathscr{A}_1, \mu_1)$ and $(X_2, \mathscr{A}_2, \mu_2)$ be two $\sigma-$ finite measure spaces, and let the product measure space be denoted by $(X, \mathscr{A}, \mu)$. Let f be a non negative measurable function on X. Then the functions $g(x) = \int_{X_2} f(x,y)d\,\mu_2(y)$ and $h(y) = \int_{X_1} f(x,y)d\,\mu_1(x)$ are measurable on X_1 and X_2 respectively; and

$$\begin{aligned} \int\int_X f(x,y)d\,\mu(x,y) &= \int_{X_2}\left(\int_{X_1} f(x,y)d\,\mu_1(x)\right)d\,\mu_2(y) \qquad (5.7)\\ &= \int_{X_1}\left(\int_{X_2} f(x,y)d\,\mu_2(y)\right)d\,\mu_1(x) \end{aligned}$$

These three integrals may be $+\infty$.

5.3 Fubini's Theorem

Theorem 5.5 *(Fubini)*
Let $(X_1, \mathscr{A}_1, \mu_1)$ and $(X_2, \mathscr{A}_2, \mu_2)$ be two $\sigma-$ finite measure spaces, and let the product measure space be denoted by $(X, \mathscr{A}, \mu)$. Let $f \in L^1(X, d\,\mu)$. Then the functions $\int_{X_2} f(x,y)d\,\mu_2(y) \in L^1(X_1, \mu_1)$ and $\int_{X_1} f(x,y)d\,\mu_1(x) \in L^1(X_2, \mu_2)$ and (5.7) holds

The strategy of the proof is to begin by proving the result for characteristic functions of rectangles, then simple functions, and extend the result to general measurable functions on X.

Proposition 5.6
If $E \in \mathscr{A}$, then the sections E_x and E^y respectively belong to $\mathscr{A}_2$ for each $x \in X_1$, and to $\mathscr{A}_1$ for each $y \in X_2$. If f is measurable with respect to the product algebra $\mathscr{A}_1 \otimes \mathscr{A}_2$, then its sections f_x and f^y are measurable with respect to the factors $\mathscr{A}_2$ and $\mathscr{A}_1$ respectively.

Proof .
Let $\mathscr{C}$ be the collection of all subsets $E \subset X$ such that $E_x \in \mathscr{A}_2$ for all $x \in X_1$ and $E^y \in \mathscr{A}_1$ for all $y \in X_2$. Then $(A \times B)_x = B$ if $x \in A$ and $(A \times B)_x = \emptyset$ if $x \in A^c$. and similarly for the section $(A \times B)^y$. Hence $\mathscr{C}$ contains all rectangles. Moreover, $\mathscr{C}$ is a $\sigma-$ algebra, since $\left(\bigcup_{j=1}^{+\infty} E_j\right)_x = \bigcup_{j=1}^{+\infty} (E_j)_x$ and $(E_x)^c = (E^c)_x$, and similarly for $y-$sections. Therefore $\mathscr{C} \subset \mathscr{A}_1 \otimes \mathscr{A}_2$. The measurability of f_x and f^y follows from the first statement and the relationships

$$(f_x)^{-1}(B) = (f^{-1}(B))_x; (f^y)^{-1}(B) = (f^{-1}(B))^y.$$

□

Lemma 5.7
Let $(X_1, \mathscr{A}_1, \mu_1)$ and $(X_2, \mathscr{A}_2, \mu_2)$ be two $\sigma-$ finite measure spaces and $(X, \mathscr{A}, \mu)$ be the product measure space. Given $E \in \mathscr{A}$, the sections $(\chi_E)_x$ and $(\chi_E)^y$ are measurable in $(X_1, \mathscr{A}_1, \mu_1)$ and $(X_2, \mathscr{A}_2, \mu_2)$ respectively; and

$$\begin{aligned} \mu(E) = \int\int_X \chi_E(x,y) d\mu(x,y) &= \int_{X_2} \left(\int_{X_1} (\chi_E)^y(x) d\mu_1(x) \right) d\mu_2(y) \qquad (5.8) \\ &= \int_{X_1} \left(\int_{X_2} (\chi_E)_x(y) d\mu_2(y) \right) d\mu_1(x) \end{aligned}$$

Proof .
We shall establish the lemma for the case in which μ_1 and μ_2 are finite measures. Let $\mathscr{C}$ be the class of sets in $\mathscr{A}$ for which the lemma holds. When E is a rectangle, $E = A \times B$, $(\chi_E)^y(x) = (\chi_E)_x(y) = \chi_A(x)\chi_B(y)$ and (5.8) is equal to $\mu_1(A)\mu_2(B)$. Then $E \in \mathscr{C}$.
$\mathscr{C}$ contain finite disjoint rectangles. It suffices to prove that $\mathscr{C}$ is a monotone class (cf prop ???).
If $E = \bigcup_{j=1}^{+\infty} E_j$ with $(E_j)_j$ is an increasing sequence of sets of $\mathscr{C}$. Then by Monotone Convergence Theorem $E \in \mathscr{C}$.
If $E = \bigcap_{j=1}^{+\infty} E_j$ with $(E_j)_j$ is a decreasing sequence of sets of $\mathscr{C}$. Then since μ_1 and μ_2 are finite measures, then by the Monotone Convergence Theorem $E \in \mathscr{C}$. □

Proof (Tonelli Theorem) .
Lemme 5.7 proves that theorem 5.4 is valid for characteristic functions for measurable subsets and by additivity the theorem is valid for simples functions. If f is a non negative measurable on $(X, \mathscr{A}, \mu)$, there exists an increasing sequence of simples functions and the result is deduced from the Monotone Convergence Theorem. □

Proof (Fubini Theorem) .
If f is integrable on X, we decompose $f = f^+ - f^-$ and apply Tonelli Theorem for f^+ and f^-.

□

A Regularity of Measures on Metric Spaces

1 Regularity of Measures on Metric Spaces

Theorem 1.1
Let X be a metric space. We denote by $\mathscr{B}_X$ the Borel $\sigma-$algebra on X. Then any finite measure on X is regular. (i.e. $\forall A \in \mathscr{B}_X$ and $\forall \varepsilon > 0$, there exist an open set U and a closed set F of X such that $F \subset A \subset U$ and $\mu(U \setminus F) \leq \varepsilon$.)

Proof .
If A is a closed set we take $F = A$ and we consider the sequence $(U_n)_n$ defined by: $U_n = \{x; d(x, A) < \frac{1}{n}\}$. The sequence of open subsets $(U_n)_n$ decreases to the set A and since the measure μ is finite, then $\lim_{n\to+\infty} \mu(U_n) = \mu(A)$.
We consider $\mathscr{C}$ the class of subsets which fulfill the desired property:
$\mathscr{C}$ contains the closed sets. Let prove that $\mathscr{C}$ is a σ-algebra.
Let $(A_n)_n$ be a sequence of $\mathscr{C}$, $(F_n)_n$ a sequence of closed subsets and $(U_n)_n$ a sequence of open subsets such that $F_n \subset A_n \subset U_n$ and $\mu(U_n \setminus F_n) < \dfrac{\varepsilon}{2^{n+1}}$. We define $U = \bigcup_{n=1}^{+\infty} U_n$, $\tilde{F} = \displaystyle\bigcup_{n=1}^{+\infty} F_n$ and $F = \displaystyle\bigcup_{n=1}^{n_0} F_n$. The integer n_0 is selected so that $\mu(\displaystyle\bigcup_{n=n_0}^{+\infty} F_n) \leq \varepsilon/2$.

$$F \subset \bigcup_{n=1}^{+\infty} A_n \subset U \text{ and } \mu(U \setminus F) < \varepsilon$$

$\mathscr{C}$ is closed under complementarity, then $\mathscr{C}$ is a σ-algebra.
If there exists a sequence $(K_m)_m$ of compacts such that $X = \bigcup_m K_m$ and μ is finite on any compact, then μ is regular in the sense that for all $A \in \mathscr{B}_X$ and $\forall \varepsilon > 0$, there exist an open set U and a compact set K of X such that $K \subset A \subset U$ and $\mu(U \setminus K) \leq \varepsilon$.

Lemma 1.2
Let X be a metric space and let F be a closed subset of X and $\varepsilon > 0$. There exists a continuous function f on X such that $f = 1$ on F and $f(x) = 0$ if $d(x, F) > \varepsilon$ and $0 \leq f(x) \leq 1$ on X.

Proof .
We take $\varphi(t)) = \begin{cases} 1 & \text{if } t \leq 0 \\ 1-t & \text{if } 0 \leq t \leq 1 \\ 0 & \text{if } t \geq 1 \end{cases}$ and $f(x) = \varphi(\frac{d(x,F)}{\varepsilon})$.

2 Application

Theorem 2.1
Let X be a metric space and $\mathscr{B}_X$ the Borel σ-algebra of X. Let μ and ν two finite measures on $(X, \mathscr{B}_X)$. We assume that

$$\int_X f(x)d\,\mu(x) = \int_X f(x)d\,\nu(x)$$

for any continuous function f on X, then $\mu = \nu$.

Proof .
Let F be a closed set of X. We take the same notations of the previous lemma and we define for all $n \in \mathbb{N}$

$$\varphi_n(t) = \varphi(nt), \quad f_n(x) = \varphi_n(d(x,F))$$

$(f_n)_n$ is a decreasing sequence of continuous functions which pointwise converges to χ_F. We use the Dominate Convergence Theorem.

$$\mu(F) = \lim_{n \longrightarrow +\infty} \int_X f_n(x)d\,\mu(x) = \int_X f_n(x)d\,\nu(x)$$

Then the two measures coincide.

B Riesz Theorem and Lebesgue-Stieljes Measure

1 Riesz Theorem

Theorem 1.1
Let f be a measurable function almost everywhere finite on $[0,1]$. Then for all $\varepsilon > 0$ there exist a closed set $F \subset [0,1]$ such that the restriction of f on F is continuous and $\lambda(F) > 1-\varepsilon$.

Proof .
Since $\{x; f(x) = \pm\infty\}$ is a null set, there exist a closed set $B \subset [0,1]$, such that $\lambda(B) > 1-\varepsilon/2$ and the restriction of f on B is bounded. Thus without loss of generalities, we can suppose that f is bounded on $[0,1]$.
We assume in the first time that $f = \chi_A$, for A a measurable subset in $[0,1]$. Since the measure λ is regular there exist a closed set $G \subset A$ in $[0,1]$ such that $\lambda(A \setminus G) < \varepsilon/2$ and there exist an open set $H \supset A$ in $[0,1]$ such that $\lambda(H \setminus A) < \varepsilon/2$. It suffices to take $F = G \cup H^c$.
If f is a simple function, $f = \sum_{j=1}^{n} \lambda_j \chi_{A_j}$, with A_j measurable subsets. We can always assume that the numbers λ_j are real. For the function $f_j = \lambda_j \chi_{A_j}$, there exist a closed set F_j such that the restriction of f_j on F_j is continuous and $\lambda(F_j) > 1-\varepsilon/2^n$. The function f is continuous on the closed set $F = \bigcap_{j=1}^{n} F_j$ and $\lambda(F) > 1-\varepsilon$.
If f is a measurable function bounded on $[0,1]$. There exist a sequence of simple functions $(f_n)_n$ which converges uniformly to f. For any function f_n there exists a closed set F_n such that f_n is continuous on F_n and $\lambda(F_n) > 1-\varepsilon/2^n$. We consider the compact set $F = \bigcap_n F_n$. The sequence $(f_n)_n$ converges uniformly on F to f. Then the restriction of f on F is continuous and $\lambda(F) > 1-\varepsilon$.

2 Lebesgue-Stieljes Measure

We recall that if we take the Borel σ-algebra $\mathscr{B}_{\mathbb{R}}$ on $\mathbb{R}$ and if μ is a non-negative measure and finite on the compacts, we associate the increasing function $\varphi\colon \mathbb{R} \longrightarrow \mathbb{R}$ defined by:

$$\varphi(x) = \begin{cases} \mu([0, x[) \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -\mu([x, 0[) \text{ if } x < 0 \end{cases}$$

φ is continuous at left.
Inversely, any increasing and continuous at left function $\varphi: \mathbb{R} \longrightarrow \mathbb{R}$, we can associate a unique non-negative measure μ such that $\mu([a, b[) = \varphi(b) - \varphi(a)$, for all a and b in $\mathbb{R}$. We recall that if two non-negative measures μ and ν on $\mathscr{B}_{\mathbb{R}}$ are such that $\mu([a, b[) = \nu([a, b[)$, for all a and b in $\mathbb{R}$, then $\mu = \nu$.
Let $\mathcal{C}_c(\mathbb{R})$ the space of continuous functions on $\mathbb{R}$ with compact support. A linear form L on $\mathcal{C}_c(\mathbb{R})$ is called non-negative if $L(f) \geq 0$, for all non-negative function $f \in \mathcal{C}_c(\mathbb{R})$. Then we have the following theorem.

Theorem 2.1 *(F.Riesz's theorem)*
For any non-negative linear form L on $\mathcal{C}_c(\mathbb{R})$, we can associate a unique measure μ on
$B_{\mathbb{R}}$ such that

$$\int_{\mathbb{R}} f(x) d\,\mu(x) = L(f) \quad \forall f \in \mathcal{C}_c(\mathbb{R})$$

Proof .
Uniqueness: Let μ and ν two measures on $\mathscr{B}_{\mathbb{R}}$ such that

$$\int_{\mathbb{R}} f(x) d\,\mu(x) = \int_{\mathbb{R}} f(x) d\,\nu = L(f) \quad \forall f \in \mathcal{C}_c(\mathbb{R})$$

For any a, b of $\mathbb{R}$, we take the sequence of functions $(f_n)_n \in \mathcal{C}_c(\mathbb{R})$ defined by:

$$f_n(x) = \begin{cases} 0 \text{ if } x < a - \frac{1}{n} \text{ or } x > b \\ 1 \text{ if } a \leq x \leq b - \frac{1}{n} \\ nx + (1 - na) \text{ if } a - \frac{1}{n} \leq x \leq a \text{ and } -nx + nb \text{ if } b - \frac{1}{n} \leq x \leq b \end{cases}$$

The sequence $(f_n)_n$ is dominated by any function $\varphi \in \mathcal{C}_c(\mathbb{R})$ equal to 1 on $[a - 1, b]$. Then it follows from the Dominate Convergence Theorem that: $\mu([a, b[) = \nu([a, b[)$, and then $\mu = \nu$.
Existence:
Let $\mathcal{E}_1$ the set of functions $f: \mathbb{R} \longrightarrow \mathbb{R}$ such that there exist an increasing sequence of $\mathcal{C}_c(\mathbb{R})$ converging to f and dominated by a function $h \in \mathcal{C}_c(\mathbb{R})$.
For any interval $]a, b[\in \mathbb{R}$, the characteristic function of the interval $]a, b[$ is in $\mathcal{E}_1$. We extend L on $\mathcal{E}_1$ by:

$$L(f) = \lim_{n \longrightarrow +\infty} L(f_n)$$

for all $f \in \mathcal{E}_1$. It suffices to prove that $L(f)$ does not depends on the sequence $(f_n)_n$. Let $(f_n)_n$ and $(g_n)_n$ be two increasing sequences of $\mathcal{E}_1$ dominated by a function $h \in \mathcal{C}_c(\mathbb{R})$ and $\lim\limits_{n \longrightarrow +\infty} f_n = \lim\limits_{n \longrightarrow +\infty} g_n$.

We want to show that $\lim_{n\longrightarrow+\infty} L(f_n) = \lim_{n\longrightarrow+\infty} L(g_n)$. Let $(\varphi_k)_k$, the sequence defined by $\varphi_k(x) = (g_n(x) - f_k(x))^+$. $\varphi_k \in \mathcal{C}_c(\mathbb{R})$ and $\varphi_k \geq 0$. The sequence $(\varphi_k)_k$ is decreasing to 0. From the Dini theorem The convergence is uniform. Let $\psi \in \mathcal{C}_c(\mathbb{R})$ such that $\psi = 1$ on the support of h. Then for all $\varepsilon > 0$, there exist k_0 such that for all $k \geq k_0$, $0 \leq \varphi_k \leq \varepsilon\psi$. Then $0 \leq L(\varphi_k) \leq \varepsilon L(\psi)$ and then $\lim_{k\longrightarrow+\infty} L(\varphi_k) = 0$. We have: $g_n - f_k \leq \varphi_k$, then $L(g_n) \leq L(f_k) + \varepsilon L(\psi)$. It follows that $L(g_n) \leq \lim_{k\longrightarrow+\infty} L(f_k)$ and then $\lim_{k\longrightarrow+\infty} L(g_k) = \lim_{k\longrightarrow+\infty} L(f_k)$.
L fulfill $L(f+g) = L(f) + L(g)$ and $L(\lambda f) = \lambda L(f)$, for all $\lambda \geq 0$ and all $f, g \in \mathcal{E}_1$. We put

$$\mathcal{E} = \mathcal{E}_1 - \mathcal{E}_1 = \{f - g, f \in \mathcal{E}_1, g \in \mathcal{E}_1\}$$

$\mathcal{E}$ is a vector space and contains the characteristic functions of bounded intervals. For $h \in \mathcal{E}$ and $h = f - g$ with f and g in $\mathcal{E}_1$, we still put $L(h) = L(f) - L(g)$. This definition of L on $\mathcal{E}$ does not depend of the representing. L in this way is defined on $\mathcal{E}$ and non-negative. Let

$$\varphi(x) = \begin{cases} L(\chi_{[0,x[}) & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -L(\chi_{[x,0[}) & \text{if} \quad x < 0 \end{cases}$$

We have $L(\chi_{[a,b[}) = \varphi(b) - \varphi(a)$. φ is increasing and continuous at the left. Let μ the measure associated to φ. We have: $\mu([a,b[) = \varphi(b) - \varphi(a) = L(\chi_{[a,b[})$, then for all bounded interval I, $\mu(I) = L(\chi_I)$, then for any simple function f, $\int f(x)d\,\mu(x) = L(f)$. Let f be a continuous function with compact support, there exist a sequence of simple functions $(f_n)_n$ which converges uniformly to f and the support of the functions f_n are in a compact K fixed. Let $\psi \in \mathcal{C}_c(\mathbb{R})$ such that $\psi = 1$ on K. Let $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that $|f - f_n| \leq \varepsilon\psi$ for $n \geq N$. Then $|L(f) - L(f_n)| \leq \varepsilon L(\psi)$ and $\lim_{n\longrightarrow+\infty} L(f_n) = L(f)$. The Dominate Convergence Theorem gives that $\lim_{n\longrightarrow+\infty} \int_{\mathbb{R}} f_n(x)d\,\mu(x) = \int_{\mathbb{R}} f(x)d\,\mu(x)$.

C Measure Image

0.1 Measures Image

Proposition 0.2
Let $(X, \mathscr{A}, \mu)$ be a measure space, Y any non empty set, and $\phi\colon X \to Y$ a mapping. Set

$$\mathscr{B} = \{F \subset Y,\, \phi^{-1}\{F\} \in \mathscr{A}\}, \quad \nu(F) = \mu(\phi^{-1}\{F\}) \text{ for every } F \in \mathscr{B}.$$

Then $(Y, \mathscr{B}, \nu)$ is a measure space.

Proof .
- $\emptyset = \phi^{-1}\{\emptyset\} \in \mathscr{A}$ so $\emptyset \in \mathscr{B}$.
- If $F \in \mathscr{B}$, then $\phi^{-1}\{F\} \in \mathscr{A}$, so $X \setminus \phi^{-1}\{F\} \in \mathscr{A}$; but $X \setminus \phi^{-1}\{F\} = \phi^{-1}\{Y \setminus F\}$, so $Y \setminus F \in \mathscr{B}$.
- If $(F_n)_n$ is a sequence in $\mathscr{B}$, then $\phi^{-1}\{F_n\} \in \mathscr{A}$ for every n, so $\bigcup_{n=1}^{+\infty} \phi^{-1}\{F_n\} \in \mathscr{A}$; but $\phi^{-1}\left(\bigcup_{n=1}^{+\infty} F_n\right) = \bigcup_{n=1}^{+\infty} \phi^{-1}\{F_n\}$, so $\bigcup_{n=1}^{+\infty} F_n \in \mathscr{B}$.

Thus $\mathscr{B}$ is a σ-algebra.
- $\nu(\emptyset) = \mu(\phi^{-1}\{\emptyset\}) = \mu(\emptyset) = 0$.

If $(F_n)_n$ is a disjoint sequence in $\mathscr{B}$, then $(\phi^{-1}\{F_n\})_n$ is a disjoint sequence in $\mathscr{A}$, so

$$\nu(\bigcup_{n=1}^{+\infty} F_n) = \mu(\phi^{-1}\left(\bigcup_{n=1}^{+\infty} F_n\right)) = \mu(\bigcup_{n=1}^{+\infty} \phi^{-1}\{F_n\}) = \sum_{n=0}^{+\infty} \mu(\phi^{-1}\{F_n\}) = \sum_{n=0}^{+\infty} \nu(F_n).$$

So ν is a measure.

Definition 0.3
The measure ν is called the image measure of μ by ϕ.

0.2 Examples