Some Simple Black Hole Thermodynamics

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In his recent popular book *The Universe in a Nutshell*, Steven Hawking gives expressions for the entropy[1] and temperature (often referred to as the "Hawking temperature"[2]) of a black hole:[3]

$$S = \frac{kc^3}{4\hbar G} A \qquad (1)$$

$$T = \frac{\hbar c^3}{8\pi kGM}, \qquad (2)$$

where A is the area of the event horizon, M is the mass, k is Boltzmann's constant, $\hbar = \frac{h}{2\pi}$ (h being Planck's constant), c is the speed of light, and G is the universal gravitational constant. These expressions can be used as starting points for some interesting approximations on the thermodynamics of a Schwarzschild black hole, of mass M, which by definition is nonrotating and spherical with an event horizon of radius $R = \frac{2GM}{c^2}$.[4,5]

Hawking has theorized that during pair production occurring just outside the event horizon, a black hole slowly loses mass or evaporates as particles are radiated away. This, now known as "Hawking radiation," was initially described to many in his first popular book *A Brief History of Time*. In this process, anti-particles with negative energy fall into the black hole actually causing the mass to *decrease*.[6]

Using the expression for the Schwarzschild radius, the entropy of a black hole of event-horizon area $A = \pi R^2$ can be written in terms of its mass using Eq. (1) as $S = \frac{4\pi kG}{\hbar c} M^2$. As mass is lost, the change in entropy will be $dS = \frac{8\pi kG}{\hbar c} mdm$. Multiplying by both sides of Eq. (2) gives $Tds = c^2 dm$. Integrating from the greater initial mass M to a lesser final mass M' yields $Q = (M' - M)c^2 = \Delta Mc^2$, which is the heat energy *lost* when the black hole radiates away a mass $\Delta M = M' - M$. Note that ΔM is intrinsically a negative quantity.

The heat lost during a temperature change can also be written as $dQ = mCdT$, which allows the specific heat (at constant pressure) of a black hole C to be determined. Substituting $dQ = c^2 dm$ and from Eq. (2) $dT = -\frac{\hbar c^3}{8\pi kG}\frac{dm}{m^2}$ yields $C = -\frac{8\pi kG}{\hbar c} M$. The negative sign is not a surprise, since it can be clearly seen from Eq. (2) and the above temperature change that as a black hole loses mass and therefore energy, its temperature increases.

Note that integrating Tds over the *entire* mass M of a black hole would give $Q = -Mc^2$, which suggests that its *total* energy would eventually be radiated away as heat. Therefore, by the first law of thermodynamics, $\Delta E = Q - W$, there is no work being done on the event horizon as the volume decreases. This suggests that the pressure on the event horizon is negligible and verifies that the specific heat is indeed at constant pressure.

The temperature increase with loss of mass shown in Eq. (2) suggests that over time, the rate at which the energy is radiated from the black hole should also increase. The proportionality between specific

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heat and mass seen above supports the assertion as well.

The rate of energy loss can be approximated with the Stephan-Boltzmann radiation law, $J = \frac{dU/dt}{A} = \sigma T^4$, where $\sigma = \frac{\pi^2 k^4}{60\hbar^3 c^4}$, the Stephan-Boltzmann constant.[7] Rearranging the radiation equation and substituting the area of the event horizon gives $\frac{dU}{dt} = 4\pi R^2\ \sigma T^4$. Substituting for the constant σ, the Schwarzschild radius, and the Hawking temperature gives, after further rearranging, a rate of mass loss $\frac{dm}{dt} = \frac{\hbar c^4}{15360\pi G^2}\frac{1}{M^2}$, which will indeed increase as mass and energy are lost.[8,9]

Solving for dt in the last expression and integrating yields $t = \frac{5120\pi G^2}{\hbar c^4} M^3 = 10^{-16} M^3$, the lifetime of a black hole once it begins to evaporate.[10,11] The Hawking temperature of a black hole can be approximated from the values of the constants as $T \cong \frac{10^{23}}{M}$; this is only about 10^{-7} K[12] above absolute zero even for the smallest stellar black holes (approximately 3 solar masses).[13] Since the average temperature of the universe is about 2.7 K, most black holes are absorbing more energy than they emit and will not begin to evaporate for some time, until the universe has expanded and cooled below their temperature.[14] Even once evaporation begins, by the above equation, a 3-solar-mass black hole would last 10^{75} s or 10^{68} yr! However, primordial, mini-black holes, theorized by Hawking to have been created during the big bang,[15] with masses of about 10^{12} kg would have been much hotter and would evaporate in about 10^{20} s or 10^{13} yr. These may not have evaporated yet either, but they should much sooner than their stellar cousins.

What actually happens when a black hole finally radiates away the last of its mass is not clear, but at such a high temperature a huge burst of x-rays or gamma rays is likely. Nothing of this nature of the expected magnitude has even been observed.[16]

The above expression for the change in the entropy of a black hole shows that as a black hole loses mass through evaporation its entropy will decrease. However, the second law of thermodynamics states that the entropy of a closed system must increase.[17] If a black hole is in a reservoir of volume V and temperature T, and total energy $E = aVT^4$, where $a = \frac{4\sigma}{c}$ in an alternative form of the Stephan-Boltzmann Law,[18] it can be shown with the first law of thermodynamics that the entropy of the reservoir is $S = \frac{4}{3}\ aVT^3$.[19] The total energy of the black hole and reservoir system $E = mc^2 + aVT^4$ remains constant and total entropy of the system $S = \frac{4\pi kG}{\hbar c}\ m^2 + \frac{4}{3}\ aVT^3$ has to increase as the black hole radiates its energy into the reservoir.

By conservation of energy, $dE = 0 = c^2 dm + aT^4\ dV + 4aVT^3\ dT$. The entropy change of the system would be $dS = \frac{8\pi kG}{\hbar c}\ mdm + \frac{4}{3} aT^3\ dV + 4aVT^2\, dT$. Multiplying the entropy change by the temperature T and subtracting the energy equation, then dividing the temperature back out gives $dS = \frac{8\pi kG}{\hbar c}\ mdm - \frac{c^2}{T}\ dm + \frac{1}{3}\ aT^3\ dV$. Substituting the Hawking temperature in the second term (physically appropriate as it was from the energy of the black hole) shows that it is equivalent to the first term. The first two terms therefore cancel and leave $dS = \frac{1}{3} aT^3\ dV$. Since the volume of the reservoir increases by the same amount that the event horizon's volume decreases, this surprisingly simple expression for dS is indeed a positive quantity and therefore shows an overall increase in the entropy of the system.

Substituting the value $a = \frac{4\sigma}{c}$ and the increase in the volume of the reservoir, which is the *opposite* of the decrease in the event horizon's volume, $dV = 4\pi R^2\ dR = \frac{32\pi G^3}{c^6}\ m^2 dm$, shows, after quite a few cancellations, that once the system reaches equilibrium (when the reservoir reaches the Hawking temperature), the expression for the entropy change simplifies to $dS = -\frac{k}{720}\frac{dm}{m}$. Integrating from the original to the final mass of the black hole gives $\Delta S = -\frac{k}{720}\ln\frac{M'}{M}$ or finally, $\Delta S = \frac{k}{720}\ln\frac{M}{M'}$. Again, since the initial mass M is greater than the final mass M', the result is a positive quantity showing an increase in entropy.

Note that the entropy depends only on the mass of the black hole. Entropy is defined as the logarithm of the number of states accessible to a system[20] and in the case of a Schwarzschild black hole, mass is the only "state variable."[21,22]

Despite the fact that these approximations apply only to the simplest of black holes (the Schwarzschild black hole), the results are still informative and intriguing, perhaps especially for those who wish to begin investigating some of the fascinating properties of black holes[23] in a bit more detail than Hawking and others can go into in popular books, but without necessarily having to delve into the details of general relativity and quantum mechanics.[24,25]

Appendix

If the energy of the reservoir is $E = aVT^4$, a change in the energy $dE = dQ - dW$ can also be written as $dE = \left(\frac{\partial E}{\partial S}\right)_{\mathrm{v}} dS + \left(\frac{\partial E}{\partial V}\right)_{\mathrm{s}} dV$. This is equivalent to $aT^4\, dV + 4aT^3\, dT = TdS + aT^4\, dV$, which reduces to $dS = 4aVT^2 dT$, or integrating, $S = \frac{4}{3} aVT^3$.

References

1. S. Hawking, *The Universe in a Nutshell* (Bantam Books, New York, 2001), p. 63.
2. R.M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (University of Chicago Press, 1994), p. 124.
3. Ref. 1, p. 118.
4. Ref. 1, p. 111.
5. C.A. Pickover, *Black Holes: A Traveler's Guide* (Wiley, New York, 1996), p. 52.
6. S. Hawking, *The Illustrated Brief History of Time*, updated and expanded ed. (Bantam Books, New York, 1996), pp. 136–137.
7. C. Kittell and H. Kroemer, *Thermal Physics*, 2nd ed. (Freeman, New York, 1980), p. 96.
8. Ref. 5, p. 116, gives an expression of similar form, but does not go into as much detail with the constants.
9. B.W. Carroll and D.A. Ostlie, *An Introduction to Modern Astrophysics* (Addison Wesley Longman, New York, 1996), p. 673, does not give an expression but mentions the M^{-2} dependence.
10. Ref. 5, pp. 112 and 114, again gives an expression of similar form, without as much detail with the constants, but does not give a derivation.
11. Ref. 9, p. 674, gives a similar expression without derivation.
12. Agrees with value given in Ref. 6, p. 137.
13. Ref. 9, p. 662.
14. Ref. 6, p. 137.
15. Ref. 6, pp. 127, 138–139.
16. Ref. 6, p. 139.
17. Ref. 16, Hawking states that the increase in entropy from the radiation more than makes up for the entropy decrease of the black hole.
18. Ref. 7.
19. See the Appendix for a proof of this.
20. Ref. 7, p. 42.
21. Ref. 9, p. 668, states that a black hole can be defined in terms of its mass, angular momentum, and charge.
22. Ref. 5 identifies various types of black holes (Kerr, etc.) that do rotate (therefore having angular momentum) and have charge, but defines a Schwarzschild black hole as nonrotating and noncharged.
23. D.V. Schroeder, *An Introduction to Thermal Physics* (Addison Wesley Longman, New York, 2000), pp. 83, 84, 92, and 304, has derivations of the entropy, temperature, and rate of mass loss of a black hole as end-of-chapter problems.
24. Ref. 2 is an excellent resource if one wishes to explore these details at a higher level.
25. E.F. Taylor and J.A. Wheeler, *Exploring Black Holes: Introduction to General Relativity* (Addison Wesley Longman, New York, 2000) is also excellent and at a more intermediate level.

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