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Introduction to Quantum Entanglement

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Abstract: This paper contains a characterization of certain aspects of bipartite quantum entanglement. We discuss relationship between entropy and entanglement, as well as qualitative and quantitative aspects of entanglement. Qualitative characterization of entanglement concerns the criteria: reduction criterion, positive partial transpose, positive maps, entanglement witness and majorization criterion. Measures of entanglement have been discussed as the quantitative aspects of entanglement.

Keywords: Quantum information theory, states entanglement, reduction criterion, positive partial transpose, positive maps, entanglement witness, majorization, measures of entanglement.

1. Introduction

In 1935 Einstein, Podolsky and Rosen designed a thought experiment to demonstrate the incompleteness of quantum mechanics [1]. The presented thought experiment assumed the principle of locality and reality of quantum mechanics. The logic of the experiment of Einstein, Podolsky and Rosen was as follows. If we consider a system of two particles in state $|\phi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, then the measurement made on the first particle has an impact to a outcome on the second particle. After measurement of the first subsystem, the first particle is in state $|0\rangle$ or $|1\rangle$ with probability $\frac{1}{2}$. The same results are obtained for the second particle. Suppose that the particles are separated from each other in millions of light years. By measuring the first particle if it is *obtained* state $|0\rangle$, then it is known that second one is in state $|1\rangle$. It looks like, the knowledge on the state of second particle came to the observer the first particle faster than the speed of light. It follows that it is not satisfied at least principle of quantum mechanics. Einstein, Podolsky and Rosen came to the conclusion that some quantum effects travel faster than light, which is contradiction to the theory of relativity. The presented experiment is called the EPR paradox. In response to the EPR paradox, Irish physicist John Stewart Bell performed a thought experiment showing that at least one of the quantum mechanics assumptions must be false [12]. Bell introduced inequalities that satisfy the assumptions of local realism, and then showed that for certain quantum states they are violated. Experimental violation of Bell's inequalities was confirmed repeatedly by some quantum systems [18, 24]. EPR paradox has become the basis to define the term *entanglement* of states, which is a type of correlation between particles that have no classical counterpart [7].

The recent development of quantum information theory showed wide practical applications of entanglement. Entanglement became the basis for quantum information processing and is used in quantum cryptography, quantum teleportation, quantum error correction codes and quantum computation.

This paper is organized as follows. Section 2 contains definition of pure states and entanglement of pure states. Similarly, section 3 presents the definition of entanglement of mixed states. Section 4 shows relationship between entropy and entanglement. Section 5 presents some methods of identifying entanglement for bipartite pure and mixed states. Finally, section 6 describes the axioms of measure of entanglement and presents chosen measure of entanglement.

2. Entanglement of pure states

A *pure quantum state* can be represented by a vector in a complex Hilbert space with unit length. Thus for each pure state $|\psi\rangle$ and any basis $\{|u_1\rangle, ..., |u_n\rangle\}$ the state $|\psi\rangle$ can be extended to

$$|\psi\rangle = \alpha_1|u_1\rangle + \cdots + \alpha_n|u_n\rangle, \tag{1}$$

where $\sum_{i=1}^{n} |\alpha_i|^2 = 1$. Now we consider the entanglement of pure states. Let $\{|u_i\rangle, i = 1, 2, ..., n\}$ and $\{|v_i\rangle, j = 1, 2, ..., m\}$ be orthonormal bases of n – dimensional Hilbert space $\mathcal{H}_n$ and m – dimensional Hilbert space $\mathcal{H}_m$ respectively. Denote by $\mathcal{H}_{nm}$ a Kronecker product of spaces $\mathcal{H}_n$ and $\mathcal{H}_m$. Thus $\mathcal{H}_{nm} = \mathcal{H}_n \otimes \mathcal{H}_m$ is a nm – dimensional Hilbert space with orthonormal basis $\{|u_i\rangle \otimes |v_j\rangle, i = 1, 2, ..., n, j = 1, 2, ..., m\}$, where $|u_i\rangle \otimes |v_j\rangle = |u_i\rangle|v_j\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \gamma_{ij}|u_i\rangle|v_j\rangle$ and $\sum_{i=1}^{n} \sum_{j=1}^{m} |\gamma_{ij}|^2 = 1$.

Definition 1 *A pure state $|\chi\rangle \in \mathcal{H}_{nm}$ is called separable if and only if it can be written as Kronecker product of states $|\psi\rangle = \sum_{i=1}^{n} \alpha_i|u_i\rangle \in \mathcal{H}_n$ and $|\phi\rangle = \sum_{j=1}^{n} \beta_i|v_i\rangle \in \mathcal{H}_m$*

$$|\chi\rangle = |\psi\rangle \otimes |\phi\rangle. \tag{2}$$

Otherwise $|\chi\rangle$ is entangled.

Consider a pure state $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ from $\mathcal{H}_2 \otimes \mathcal{H}_2$. Let $|\phi_1\rangle = \alpha|0\rangle + \beta|1\rangle$, $|\phi_2\rangle = \gamma|0\rangle + \delta|1\rangle$ be states from Hilbert space $\mathcal{H}_2$, where $|\alpha|^2 + |\beta|^2 = 1$ and $|\gamma|^2 + |\delta|^2 = 1$. Suppose that $|\psi^+\rangle$ is separable, thus it can be written in form

$$|\psi^+\rangle = |\phi_1\rangle \otimes |\phi_2\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle). \tag{3}$$

Hence

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle. \tag{4}$$

There are no values $\alpha, \beta, \gamma, \delta$ that $\alpha\gamma = \beta\delta = \frac{1}{\sqrt{2}}$ and $\alpha\delta = \beta\gamma = 0$ thus we obtain

$$|\psi^+\rangle \neq |\phi_1\rangle \otimes |\phi_2\rangle \tag{5}$$

so the state $|\psi^+\rangle$ is entangled.

3. Entanglement of mixed states

Any pure state $|\psi\rangle$ can be identified with the density operator expressed as $\rho = |\psi\rangle\langle\psi|$. Mixed states are statistical mixture of density operators of pure states. Each density operator for pure state ρ is a projection operator into one-dimensional space thus satisfies the property $\rho^2 = \rho$.

Consider statistical mixture of pure states $\{\rho_i = |\psi_i\rangle\langle\psi_i|, i = 1, 2, ..., n\}$ with probabilities $\{p_i, i = 1, 2, ..., n\}$. Then the density operator for the system is expressed as

$$\rho = \sum_{i=1}^{n} p_i \rho_i. \tag{6}$$

Density operator has characteristic properties and they are follows: ρ is Hermitian, positive and has a trace equal to 1.

Definition 2 *Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be Hilbert spaces. Denote by ρ a density operator of state from $\mathcal{H}_A \otimes \mathcal{H}_B$. Operator ρ is called separable if there exist a sequence $\{p_i\}_{i=1}^n$ of positive real numbers summing to 1, a sequence density operators $\{\rho_i^A\}_{i=1}^n$ corresponding with states from $\mathcal{H}_A$ and a sequence density operators $\{\rho_i^B\}_{i=1}^n$ corresponding with states from $\mathcal{H}_B$ such that*

$$\rho = \sum_{i=1}^{n} p_i \rho_i^A \otimes \rho_i^B. \tag{7}$$

In other words, if the mixed state can be written as a convex combination of Kronecker product of density operators then the state is separable. For the first separability of mixed states was raised by Werner [10]. Equality (6) is more restrictive than Bell's inequalities, thus each separable state satisfies Bell's inequalities [5]. The condition (6) is also true for pure states. The decomposition state ρ into (6) is not unique. Entanglement detection is a difficult problem except in the case of pure states. In special cases it is possible to detect entanglement and it is described in section (5).

4. Entropy of entangled states

In information theory, Shannon entropy is a measure of expected value of the information contained in a message [25]. The quantum counterpart of Shannon entropy is the von Neumann entropy. In classical information theory entropy of a single random variable is never greater than the entropy of joint random variables. In the case of quantum systems, von Neumann entropy for the joint system can be smaller than the entropy of its subsystems [3]. This fact is useful for detection of entanglement of quantum states.

Definition 3 ([11]) *Let ρ be a density operator of a quantum system, then von Neumann entropy is defined as*

$$S(\rho) = -\mathrm{tr}(\rho \log \rho). \tag{8}$$

Using the spectral decomposition, function $\log$ can be extended on operators. Thus the von Neumann entropy can be written as

$$S(\rho) = -\sum_{i=1}^{n} \lambda_i \log \lambda_i. \tag{9}$$

Here after we assume that default logarithm base is equal to 2.

Any pure state has a spectrum of the form $\lambda_1 = 1, \lambda_2 = 0, ..., \lambda_n = 0$. Thus, the von Neumann entropy of the system is equal to

$$S(\rho) = -1 \log 1 = 0. \tag{10}$$

Mixed state is called *maximally mixed* if it is represented by the density operator $\rho = \frac{1}{N}\mathbb{1} \in \mathcal{H}$, where N is a dimension of space $\mathcal{H}$. Von Neumann entropy for this states takes the highest value equal to

$$S(\rho) = -\sum_{i=1}^{n} \frac{1}{n} \log \frac{1}{n} = \log n. \tag{11}$$

Von Neumann entropy can be interpreted as a measure of the unpredictability of measurement of a quantum state.

Conditional and joint quantum entropies are defined similarly as in the classical information theory. It is necessary to introduce the reduced operators and partial trace. For any separable state $\rho^{AB} = \rho^A \otimes \rho^B$ the partial traces are defined as

$$\mathrm{tr}_A(\rho^{AB}) = \mathrm{tr}(\rho^A)\rho^B \quad \text{and} \quad \mathrm{tr}_B(\rho^{AB}) = \mathrm{tr}(\rho^B)\rho^A. \tag{12}$$

Since ρ^A and ρ^B are density operators, thus $\mathrm{tr}(\rho^A) = 1$ and $\mathrm{tr}(\rho^B) = 1$. Hence, the reduced density operators can be expressed as

$$\rho^A = \mathrm{tr}_B(\rho^{AB}) \quad \text{and} \quad \rho^B = \mathrm{tr}_A(\rho^{AB}). \tag{13}$$

We extend the definition of partial trace for all state by trace linearity.

Definition 4 ([11]) *Let ρ^A, ρ^B and ρ^{AB} be the density operators of quantum systems $\mathcal{A}$, $\mathcal{B}$ and composite system $\mathcal{AB}$ respectively. The joint von Neumann entropy of the quantum systems $\mathcal{A}$ and $\mathcal{B}$ is defined as*

$$S(\rho^A, \rho^B) = S(\rho^{AB}). \tag{14}$$

Definition 5 ([11]) *Let ρ^A and ρ^B be the density operators of quantum systems $\mathcal{A}$ and $\mathcal{B}$. The conditional von Neumann entropy is given by*

$$S(\rho^A|\rho^B) = S(\rho^A, \rho^B) - S(\rho^B). \tag{15}$$

Conditional quantum entropy can be negative which is not possible in the classical theory of information [2].

Consider the entangled state $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ of the system $\mathcal{AB}$. The matrix form of the density operator ρ^{AB} for the state $|\psi^+\rangle$ is given by

$$\rho^{AB} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}. \tag{16}$$

The spectrum of the operator ρ is the sequence $\{1, 0, 0, 0\}$, thus conditional entropy of the state ρ^{AB} is

$$S(\rho^{AB}) = -\log 1 = 0. \tag{17}$$

By equations (9) shows that $\rho^A = \mathrm{tr}_B(\rho^{AB})$ and $\rho^B = \mathrm{tr}_A(\rho^{AB})$, thus

$$\rho^A = \rho^B = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|). \tag{18}$$

Sequence $\{\frac{1}{2}, \frac{1}{2}\}$ is a spectrum of the operators ρ^A and ρ^B. Hence the entropy of operators ρ^A and ρ^B is equal to

$$S(\rho^A) = S(\rho^B) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = 1. \tag{19}$$

Thus, we obtain the inequality $S(\rho^A) \geq S(\rho^{AB})$, which shows that the entropy of the subsystem $\mathcal{A}$ is greater than the entropy of the joint systems $\mathcal{AB}$. Hence, knowledge of the joint system is greater than knowledge of the subsystem. This property occurs only in the case of entangled states [26,27]. Thus the von Neumann entropy can be a tool used for the detection of entanglement.

Von Neumann entropy is applied in the research of mixed states in many aspects, such as quantitative measure of entanglement [9] or a necessary condition for the non-violation of Bell's Inequalities [13].

5. Separability criteria

Generally the problem of deciding whether the state is separable or entangled is difficult. In the case of bipartite pure states, there exists an effective method for detecting entanglement. The criteria for checking whether the state is entangled exists for mixed states, but they are effective only for low dimensional cases.

5.1. Separability criterion for pure states

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a Hilbert space defined as a Kronecker product of two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. For any pure state $|\psi\rangle$ from $\mathcal{H}_{AB}$ there exist orthonormal bases $\{|a_i\rangle, i = 1, ..., n\}$ and $\{|b_i\rangle, i = 1, ..., m\}$ respectively in spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ such that

$$|\psi\rangle = \sum_{i=1}^{k} \sqrt{\lambda_i} |a_i\rangle \otimes |b_i\rangle, \tag{20}$$

where $k \leq \min(n, m)$, $\sum_{i=1}^{k} \lambda_i = 1$ and λ_i are positive coefficients called Schmidt coefficients [14]. For a pure state $|\psi\rangle$ the Schmidt coefficients are equal to the eigenvalues of $\rho^A = \mathrm{tr}_B(|\psi\rangle\langle\psi|)$ [15]. The state $|\psi\rangle$ is separable if and only if it has unique Schmidt coefficient that λ_i is equal to 1 [4, 11].

5.2. Peres-Horodecki criterion

The Peres-Horodecki criterion is also called *positive partial transpose* (**PPT**). The Peres-Horodecki criterion is a necessary condition for separability of mixed states. Let $M \times N$ dimensional state be a state from Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\dim \mathcal{H}_A = M$ and $\dim \mathcal{H}_B = N$. For 2×2 and 2×3 dimensional states, the **PPT** criterion is also sufficient condition for separability [6].

Let ρ be a density operator of composite system $\mathcal{AB}$. Matrix element of an operator ρ is given by

$$\rho_{m\mu,n\nu} = \langle m|\langle\mu|\rho|n\rangle|\nu\rangle, \tag{21}$$

where Latin letters describe first subsystem and Greek letters describe second subsystem. The partial transposition of operator ρ is defined in [4, 6] as

$$\rho^{T_B}_{m\mu,n\nu} = \rho_{m\nu,n\mu} \quad \text{and} \quad \rho^{T_A}_{m\mu,n\nu} = \rho_{n\mu,m\nu}. \tag{22}$$

Hence the operation (2) for any separable state $\rho = \rho^A \otimes \rho^B$ can be expressed as

$$\rho^{T_B} = \rho^A \otimes (\rho^B)^T \quad \text{and} \quad \rho^{T_A} = (\rho^A)^T \otimes \rho^B. \tag{23}$$

Theorem 1 (Peres [5]) *If a state ρ is separable, then ρ^{T_A} and ρ^{T_B} are positive operators.*

Theorem 2 (Horodecki [6]) *Composite states ρ of dimension 2×2 and 2×3 is separable if and only if ρ^{T_A} is positive operator.*

A $M \times N$ dimensional Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, where $M = 2, N > 3$ or $M \geq 3$ contains entangled states [16], so partial transpose criterion is effective only for low dimensional states.

Consider an entangled state $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. The matrix form of the density operator ρ^{AB} for the state $|\phi^+\rangle$ is given by

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{24}$$

Hence, the density matrix for operator ρ^{T_B} is in the form

$$\rho^{T_B} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}. \tag{25}$$

The spectrum of the operator ρ^{T_B} is the sequence $\{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$. One eigenvalue is smaller than 0, thus ρ^{T_B} is not positive and the state ρ is entangled.

5.3. Entanglement and positive maps

Let $\mathcal{B}(\mathcal{H}_A)$ and $\mathcal{B}(\mathcal{H}_B)$ be spaces of bounded operators on Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. The space of the linear maps from $\mathcal{B}(\mathcal{H}_A)$ to $\mathcal{B}(\mathcal{H}_B)$ is denoted by $\mathcal{L}(\mathcal{B}(\mathcal{H}_A), \mathcal{B}(\mathcal{H}_B))$. A linear map $\Lambda \in \mathcal{L}(\mathcal{B}(\mathcal{H}_A), \mathcal{B}(\mathcal{H}_B))$ is called positive if $\rho \geq 0$ implies $\Lambda(\rho) \geq 0$. Map Λ is completely positive (CP), if extended map $\mathbb{1} \otimes \Lambda : \mathcal{B}(\mathbb{M} \otimes \mathcal{H}_A) \to \mathcal{B}(\mathbb{M} \otimes \mathcal{H}_B)$ is positive for any space $\mathbb{M}$.

Consider completely positive map Λ and state $\rho_A \otimes \rho_B$. The state $\rho_A \otimes \rho_B$ is positive thus $(\mathbb{1} \otimes \Lambda)(\rho_A \otimes \rho_B) = \rho_A \otimes \Lambda(\rho_B)$ is positive. Map Λ is positive for all states also for entangled ones, hence Λ cannot be used for the detection of entanglement. The above property is important for the recognizing entangled states. For any entangled state ρ there exist a positive but not CP map Λ such that $(\mathbb{1} \otimes \Lambda)(\rho)$ is not positive.

Theorem 3 ([4, 6]) *Let ρ be a density operator acting on Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Then ρ is separable if and only if for any positive map $\Lambda \in \mathcal{L}(\mathcal{B}(\mathcal{H}_A), \mathcal{B}(\mathcal{H}_B))$ the operator $(\mathbb{1} \otimes \Lambda)(\rho)$ is positive.*

Suppose that Λ is a transposition operator such that $\Lambda(\sigma) = \sigma^T$. Hence $(\mathbb{1} \otimes \Lambda)(\rho) = \rho^{T_B}$ and ρ is entangled if operator ρ^{T_B} is not positive. Thus, if Λ is a transposition operator, then we obtain **PPT** criterion.

Positive maps are a strong tool to detection of entanglement. Unfortunately it is not easy to find a good mapping for solving problem of entanglement. Proposals of positive maps used for deciding whether state is entangled are contained in [17].

5.4. Reduction criterion for separability

Reduction criterion is based on the theory of positive maps [4]. Theorem (1) shows that for any positive map Λ which is not CP, state ρ is separable if

$$(\mathbb{1} \otimes \Lambda)(\rho) \geq 0. \tag{26}$$

Consider the map Λ of the form $\Lambda(\sigma) = \mathrm{tr}(\sigma)\mathbb{1} - \sigma$. If σ is positive then $\Lambda(\sigma)$ is also positive, hence Λ is a positive map. Thus, condition (4) can be written as

$$\rho_A \otimes \mathbb{1} - \rho \geq 0, \tag{27}$$

where $\rho_A = \mathrm{tr}_B(\rho)$. Dual criterion for (4) is given by

$$\mathbb{1} \otimes \rho_B - \rho \geq 0. \tag{28}$$

In general reduction criterion is weaker than the PPT criterion, but for 2×2 and 2×3 dimensional states, PPT criterion and reduction criterion are equivalent.

5.5. Entanglement witness

Entanglement witness is a strong criterion used for distinguish an entangled state from separable ones. The idea of entanglement witnesses is based on the Hahn-Banach theorem.

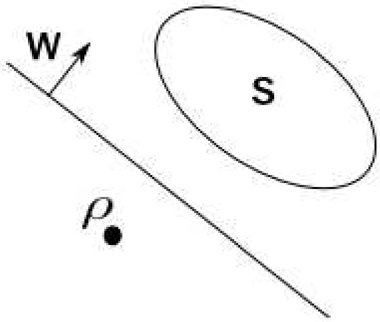

Fig. 1. Graphical interpretation of the Hahn-Banach theorem

Theorem 4 ([20]) *Denote by S a convex, compact set in a finite dimensional Banach space. Let ρ be a point in the space with $\rho \notin S$. Then there exist a hyperplane that separates ρ from S.*

Figure 1 shows the geometric interpretation of the Hahn-Banach theorem. Hyperplane separating the set S of the point ρ is determined by the orthonormal vector W, which is selected from outside the set S. Each point ρ may be characterized by the signum of the scalar product $\mathrm{tr}(W\rho)$. If state ρ is separable then distance is positive otherwise the distance is negative. By using entanglement witnesses it is possible to approximation the set of separable states. Since the separable states form a convex, compact set, hence Hahn-Banach theorem is interesting in terms of entanglement detection

Theorem 5 ([19]) *A density operator ρ is entangled if and only if there exist a Hermitian operator W with $\mathrm{tr}(W\rho) < 0$ and $\mathrm{tr}(W\sigma) \geq 0$ for any separable state σ.*

Operator W is called an entanglement witness (EW). Between positive maps and entangled witness exists a close relationship [19]. For any entanglement operator W exists positive map Λ such that

$$W = (\mathbb{1} \otimes \Lambda)(P_+), \tag{29}$$

where P_+ is the projector operator

$$P_+ = \frac{1}{M} \left(\sum_{i=1}^{M} |ii\rangle \right) \left(\sum_{j=1}^{M} \langle jj| \right). \tag{30}$$

Operator P_+ is a projector operator onto the maximally entangled state. Correspondence between W and Λ is determined by Jamiołkowski isomorphism.

Definition 6 ([21]) *An entanglement witness W is decomposable (DEW) if it can be written as*

$$W = \alpha P + (1-\alpha)Q^{T_A}, \tag{31}$$

where $P, Q \geq 0$, $\alpha \in [0,1]$ and T_A is a partial transpose on subsystem A.

Decomposable entanglement witnesses cannot detect entanglement of **PPT** states. If exists at least one **PPT** entangled state which cannot be detected by entanglement witness W, then W is non-decomposable [21].

Each entangled state can be detected by some witness W, but there is no general method for construction of entanglement witnesses. Thus the entanglement witness is difficult to use. Construction a entanglement witnesses for different classes of entangled states are described in [21, 23, 22].

5.6. Majorization

The majorization criterion proposed by Nielsen and Kempe [3] is necessary but not sufficient condition for separability.

Definition 7 *Denote by $X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_n\}$ non-increasing sequence such that $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_i \geq y_2 \geq \cdots \geq y_n$. Sequence X majorizes Y, if the conditions*

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i \tag{32}$$

is held for any $k = 0, ..., n-1$ and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i. \tag{33}$$

If X majorizes Y, then we write that $X \prec Y$.

Theorem 6 ([3]) *Let ρ^A, ρ^B and ρ^{AB} be the density operators of quantum systems $\mathcal{A}, \mathcal{B}$ and composite systems $\mathcal{AB}$. If state ρ^{AB} is separable, then*

$$\lambda(\rho^{AB}) \prec \lambda(\rho^A), \tag{34}$$

and

$$\lambda(\rho^{AB}) \prec \lambda(\rho^B), \tag{35}$$

where $\lambda(\rho_{AB})$ is a non-increasing sequence of eigenvalues of operator ρ_{AB} and similarly $\lambda(\rho_A)$, $\lambda(\rho_B)$ are non-increasing sequences of eigenvalues of operators ρ_A and ρ_B respectively.

Sequences $\lambda(\rho_A)$, $\lambda(\rho_B)$ are shorter than $\lambda(\rho_{AB})$, thus they are enlarged by appending zeros to equalize their dimensions with sequence $\lambda(\rho_{AB})$.

Conditions (7) and (8) imply that

$$S(\rho^{AB}) \geq S(\rho^A) \text{ and } S(\rho^{AB}) \geq S(\rho^B), \tag{36}$$

so this criterion is stronger than entropic criterion [3].

6. Measures of entanglement

6.1. Local Operations and Classical Communication

Local Operations and Classical Communication $LOCC$ is an important class of maps which are necessary to define measures of entanglement. The class of $LOCC$ operations includes all quantum operations, also measurements, but only performed locally in each subsystems. $LOCC$ operations are characterized by two properties:

– *Local Operations:* all operations are performed locally on the respective subsystems. Simple example of local operation is trace, which can be performed locally on the subsystems: $T = T^A \otimes T^B$.

– *Classical Communication:* this properties means that information between subsystems is exchanged by means of classical communication channels.

6.2. Requirements for entanglement measures

Entanglement criteria are helpful in detecting entanglement, but they do not give quantitative information on how much the state is entangled. Despite the fact that it is still being discussed what conditions should be fulfilled by a measure of entanglement, it is widely assumed that the measure of entanglement should satisfy the following requirements [8].

1. A measure of entanglement E is a function which assigns non-negative value for each state ρ.
2. *LOCC Monotonicity:* Measure E cannot increase under $LOCC$ operations. Alternative weaker condition is *invariant under local unitary operations*. Measure E is invariant under local unitary operations, because they are reversible thus $E(\rho) = E(U \otimes V \rho U^\dagger \otimes V^\dagger)$, where U, V are unitary.
3. *Normalization:* For any state σ holds $E(\sigma) \leq E(\rho) = S(\rho)$, where ρ is maximally entangled.
4. *Convexity:* $E(\lambda\rho + (1-\lambda)\sigma) \leq \lambda E(\rho) + (1-\lambda)E(\sigma)$, with $\lambda \in [0, 1]$.

5. *Continuity:* Let ρ_n and σ_n be a sequences of states acting with composite Hilbert spaces $(\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n}$. If $\lim_{n\to\infty} ||\rho_n - \sigma_n||_1 = 0$ then

$$\lim_{n\to\infty} \frac{E(\rho_n) - E(\sigma_n)}{n \ln(\dim(\mathcal{H}_A \otimes \mathcal{H}_B))} = 0. \tag{37}$$

6. Depending on the requirements of measurement functions, additivity condition can be formulated in several ways

   - *Additivity:* $E(\rho \otimes \sigma) = E(\rho) + E(\sigma)$ for any states ρ and σ.
   - *Subadditivity:* $E(\rho \otimes \sigma) \leq E(\rho) + E(\sigma)$ for any states ρ and σ.
   - *Weak additivity:* $E(\rho) = \frac{1}{N} E(\rho^{\otimes N})$ for any state ρ.
   - *Existence of regularization:* $E(\rho) = \lim_{N\to\infty} = \frac{E(\rho^{\otimes N})}{N}$.

6.3. Important entanglement measures

For bipartite pure states which satisfies requirements from section (1) there exists only one entanglement measure – von Neumann entropy of reduced density operator [8]. Let ρ be a density operator of a pure state from Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Thus measure of entanglement E for ρ is given by

$$E(\rho) = S(\rho^B) = -\mathrm{tr}(\rho^B \log \rho^B), \tag{38}$$

where $\rho^B = \mathrm{tr}_A(\rho)$.

For mixed states, there exist a lot of measures of entanglement. The most important of these are contained in [9].

1. *Entanglement cost:* Let Λ be a trace preserving $LOCC$ operations and denote by $\Psi(K)$ a density operator corresponding to the maximally entangled state in K dimensions. Thus entanglement cost for operator ρ is given by

$$E_C(\rho) = \inf \left\{ r : \lim_{n\to\infty} \left[\inf_{\Lambda} \mathrm{tr}(|\rho^{\otimes n} - \Lambda(\Psi(2^{rn}))|) \right] = 0 \right\}. \tag{39}$$

   This measure gives information how expensive it is to create an entangled state ρ using $LOCC$ operations in bipartite entangled state.

2. *Distillable entanglement:* Similarly let Λ be a trace preserving $LOCC$ operations and $\Psi(K)$ be a density operator of maximally entangled state in K dimensions. The distillable entanglement for state ρ is defined as

$$E_D(\rho) = \sup \left\{ r : \lim_{n\to\infty} \left[\inf_{\Lambda} \mathrm{tr}(|\Lambda(\rho^{\otimes n}) - \Psi(2^{rn})|) \right] = 0 \right\}. \tag{40}$$

Distillable entanglement is reverse to the entanglement cost and tells how many $LOCC$ operations should be used to extract state ρ into composition of bipartite entangled states.

3. *Entanglement of formation:* For a mixed states ρ the entanglement of formation is given by

$$E_F(\rho) = \inf \left\{ \sum_i p_i E(|\psi_i\rangle\langle\psi_i|) : \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \right\}, \tag{41}$$

   where E is measure of entanglement for pure states defined as (5). This measure is minimized over all possible decompositions state ρ. This measure is close to entanglement cost. Measure E_C is asymptotic version fo E_F and can be expressed as

$$E_C(\rho) = \lim_{n\to\infty} \frac{E_F(\rho^{\otimes n})}{n}. \tag{42}$$

4. *Relative entropy of entanglement:* Let S be a set of separable states. Relative entropy of entanglement for state ρ is defined as

$$E_R(\rho) = \inf_{\sigma\in S} S(\rho||\sigma) = \inf_{\sigma\in S} \mathrm{tr}(\rho\log\rho - \rho\log\sigma). \tag{43}$$

   Quantum relative entropy is a measure of similarity between two quantum states. Relative entropy of entanglement gives distance between ρ and the nearest separable state.

For pure states, the above-mentioned measures coincide with the von Neumann entropy of reduced density operator. The situation is more difficult for mixed states, where many entanglement measures exist. It can be shown that for each measure of entanglement E and density operator ρ it holds $E_D(\rho) \leq E(\rho) \leq E_C(\rho)$ [7]. Thus E_D and E_C are the lower and upper limits of values assumed by any entanglement measures.

7. Conclusion

Entanglement of quantum states is the object of intensive research, due to the wide range of applications in quantum information theory. In this paper we present the basic aspects of quantum entanglement. For bipartite pure state there exist efficient methods for detection entanglement. The situation is more complicated for mixed states. Unfortunately, there is no general, efficient method for identifying entanglement. The theory of positive mappings gives strong tools for the deciding whether state is separable or entangled. In special cases, the positive maps can be reduced to the reduction criterion or **PPT** criterion. These criteria are necessary conditions for separability of mixed states.

For 2×2 and 2×3 dimensional states, PPT criterion and reduction criterion are also sufficient conditions of separability. The most of the entanglement detection methods is strongly associated with positive maps, for example described entanglement witness. Of the presented methods, only majorization criterion entanglement is not related to positive mappings. This paper contains also a description of the measures of entanglement, which give information on how much the state is entangled.

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References

1. A. Aspect, P. Grangier, G. Roger: *Experimental Tests of Realistic Local Theories via Bell's Theorem*, Phys. Rev. Lett. 47, 460 (1981).
2. R. Augusiak, J. Tura, M. Lewenstein: *A note on the optimality of decomposable entanglement witnesses and completely entangled subspaces*, J. Phys. A: Math. Theor. 44, 212001 (2011).
3. J.S. Bell: *On The Einstein Podolsky Rosen Paradox*. Department of Physics, University of Wisconsin. Physics Vol. 1, No. 3. pp. 195-200 (1964).
4. I. Bengtsson, K. Życzkowski: *Geometry of Quantum States: An Introduction to Quantum Entanglement*, 1st Edition, Cambridge University Press (2008).
5. E. Bishop, D. Bridges: *Constructive Analysis*, Springer, Berlin (1985).
6. D. Bruß: *Characterizing Entanglement*, J. Math. Phys. 43, 4237 (2002).
7. D. Chruściński, J. Pytel, G. Sarbicki: *Constructing optimal entanglement witnesses*, Phys. Rev. A 80, 062314 (2009).
8. T. M. Cover, J.A. Thomas: *Elements of Information Theory*, Wiley-Interscience New York, NY, USA (1991).
9. A. Einstein, B. Podolsky, N. Rosen: *Can Quantum Mechanical Description of Physical Reality Be Considered Complete?*, Physical Review 47, 777-780 (1935).
10. M. Horodecki, P. Horodecki, R. Horodecki: *Mixed-state entanglement and quantum communication*. Quantum Information, Springer Tracts in Modern Physics, vol. 173, pp.151-195 (2001).

11. M. Horodecki, P. Horodecki, R. Horodecki: *Separability of mixed states: necessary and sufficient conditions*, Phys. Lett A 233, (1996).

12. R. Horodecki, P. Horodecki, M. Horodecki, K. Horodecki: *Quantum entanglement*, Rev. Mod. Phys. 81, 865-942 (2009).

13. M. Horodecki, J. Oppenheim, A. Winter: *Quantum information can be negative*, Nature 436:673-676 (2005) as “Partial quantum information”.

14. M. Keyl: *Fundamentals of Quantum Information Theory*, Phys. Rep. 369, no. 5, 431-548 (2002).

15. J.K. Korbicz, M.L. Almeida, J. Bae, M. Lewenstein, A. Acín: *Structural approximations to positive maps and entanglement-breaking channels*, Phys. Rev. A 78, 062105 (2008).

16. M. Lewenstein, D. Bruß, J.I. Cirac, B. Kraus, M. Kus, J. Samsonowicz, A. Sanpera, R. Tarrach: *Separability and distillability in composite quantum systems -a primer-*, Journal of Modern Optics, 47, 2841 (2000).

17. M. Lewenstein, B. Kraus, J. I. Cirac, P. Horodecki: *Optimization of entanglement witnesses*, Phys. Rev. A 62, 052310 (2000).

18. M.A. Nielsen, J. Kempe: *Separable states are more disordered globally than locally*, Phys. Rev. Lett., Vol. 86, 5184-7 (2001).

19. A. Peres: *Separability Criterion for Density Matrices*, Phys. Rev. Lett 77. 1413 (1996).

20. D. Petz: *Quantum Information Theory and Quantum Statistics*, 1st Edition, Springer (2008).

21. M.B. Plenio, S. Virmani: *An introduction to entanglement measures*, Quant. Inf. Comput. 7:1-51 (2007).

22. E. Santos: *Entropy inequalities and Bell inequalities for two-qubit systems*, Phys. Rev. A 69, 022305 (2004).

23. E. Schmidt: *Zur Theorie der linearen und nichtlinearen Integralgleichungen.* I. Teil: Entwicklung willkürlicher Funktionen nach Systemen vorgeschriebener, Math. Annalen 63, 433 (1907).

24. V. Vedral: *Introduction to Quantum Information Science*, Oxford University Press, USA (2007).

25. G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, A. Zeilinger: *Violation of Bell’s inequality under strict Einstein locality conditions*, Phys. Rev. Lett. 81, 5039 (1998).

26. R.F. Werner: *Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model*, Phys. Rev. A 40, 4277-4281 (1989).

27. K. Życzkowski: *Ingemar Bengtsson, Relativity of pure states entanglement*, Ann. Phys. 295, 115 (2002).

Wprowadzenie do kwantowego splątania

Streszczenie

W artykule zostały poruszone podstawowe aspekty splątania kwantowego. Przytoczone zostały definicje splątania dla stanów czystych oraz mieszanych. Następnie opisano pojęcie entropii von Neumanna oraz jej związek ze splątaniem stanów kwantowych. Kolejne sekcje zawierają opis kryteriów separowalności. Ostatnia sekcja zawiera aksjomaty miar splątania, a także najważniejsze miary splątania.