Relation Between Exponential and Poisson Distributions

An interesting feature of these two distributions is that, if the Poisson provides an appropriate description of the number of occurrences per interval of time, then the exponential will provide a description of the length of time between occurrences.

Consider the probability function for the Poisson distribution,

$$f(x) = \frac{\boldsymbol{e}^{-(\lambda t)}(\lambda t)^{x}}{\boldsymbol{x}!}; \quad \boldsymbol{x} = \mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots,$$

where, λ is the mean rate of arrivals and t is a period of time.

Defining the r.v. Y as the time of an event, we have (by definition),

$$F(t) = P(T \leq t) = 1 - P(T > t).$$

Now, The 1st arrival occurs after time t iff there are no arrivals in the interval $[0, t]$.

Therefore,

$$P(T > t) = P(zero\ events\ occur\ in\ time\ 0\ to\ t) = P(X = 0) = \frac{\boldsymbol{e}^{-(\lambda t)}(\lambda t)^{0}}{\mathbf{0}!} = \boldsymbol{e}^{-\lambda t}.$$

Hence,

$$F(t) = P(T \leq t) = 1 - P(T > t) = 1 - \boldsymbol{e}^{-\lambda t},$$

which is the distribution function for the exponential distribution with parameter λ.

Briefly,

X (no. of occurrences per interval of time t) $\sim Poissin(\lambda t)$,

Y (time between occurrences)$\sim Exp(\lambda)$.

Example

If we know that there are on average 10 customers visiting a store within 2 hours (120 minutes) interval, then the r.v. that represents the number of costumers is $X \sim Poisson(\lambda t = 10)$. From another point of view, the r.v. that represents the time between costumers arrivals $Y \sim Exp\left(\lambda = \frac{10}{120}\right)$, where the average time between customers' arrival is $\frac{1}{\lambda} = \frac{120}{10} = 12$ minutes.