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Residue theorems and their applications : computing integrals once thought impossible to evaluate analytically

Greg Rainwater
California State University, Monterey Bay

Residue Theorems and their Applications: Computing integrals once thought impossible to evaluate analytically

By

GREG RAINWATER

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Abstract

It is common to encounter an integral that seems impossible to evaluate. The Residue Theorems introduce techniques that may make the impossible, possible. In this capstone I introduce Residue Theorems and apply techniques to integrals of certain forms that cannot be computed using methods in calculus of real variables. Not only do the Residue Theorems make it possible to reach a closed solution, they can make evaluating integrals much easier than before.

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Introduction

0.1. Historical Background

Have you ever wondered if it was possible to evaluate integrals like $\int_0^\infty \sin x/x dx$ with ease? This capstone will give some positive answers to such a question by exploring the various theorems and techniques. To accomplish this, it is necessary to transform these integrals into the complex plane.

A complex number is considered to be made up of a real part and imaginary part, where the imaginary part is a real number multiplied by $\sqrt{-1}$ (denoted by i). This idea of the square root of a negative number dates back to the ancient Egyptians, but this wasn't known until "in 1878, when two brothers stole a mathematical papyrus from the ancient Egyptian burial site in the Valley of Kings" [**7**]. Although mathematicians encountered the square root of a negative number in quadratic and cubic equations, no one was able to solve them until the 16th century. The first to solve a general cubic of the form

$$x^3 + px = q \tag{0.1.1}$$

was Scipione Del Ferro (1465-1526)[**1**] just before he died. Using Del Ferro's idea, Italian mathematicians Niccolo Fontana (1499-1557) and Gerolamo Cardano (1501-1576)[**1**] are considered the first to come across complex numbers by algebraic manipulation such that a complex number can be represented as $a + \sqrt{-1}\sqrt{b}$ (where a and b are real numbers). The manipulations are quite ugly, but the real solution to Equation 0.1.1 is

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, \tag{0.1.2}$$

while there still exists two other solutions, which are complex numbers.

The term *imaginary numbers* wasn't defined until philosopher Rene Descartes (1596-1650) "associated imaginary numbers with geometric impossibility"[**6**]. Taken from [**6**], Descartes defined the term *imaginary* as

> "For any equation one can imagine as many roots (as its degree would suggest), but in many cases no quantity exists which corresponds to what one imagines."

One of the greatest mathematicians, Leonhard Euler (1707-1783)[**6**] "introduced the i symbol for $\sqrt{-1}$ in 1777"[**7**]. Building on previous mathematicians breakthroughs, Euler "visualized complex numbers as points with rectangular coordinates" [**6**] such that a complex number z can be expressed as $z = x + iy$. Further, Euler showed $x + iy = r(\cos\theta + i\sin\theta)$ and proved $e^{i\theta} = \cos\theta + i\sin\theta$. As this is just a sample of Euler's contributions to the field of complex numbers, I can confidently say Euler was a pure genius who loved mathematics. According to [**7**]

> the nineteenth-century French astronomer Dominique Arago said of him "Euler calculated without apparent effort, as men breathe, or as eagles sustain themselves in the wind." When he died he had written more brilliant mathematics than had any other mathematician, and to this day he still holds the record.

Even with such a strong foundation, no one had yet published a "suitable presentation of complex numbers"[**6**]. So, the first to do so was a Norwegian mathematician, Caspar Wessel (1745-1818)[**1**] in 1797. Wessel introduced the idea to represent complex numbers as vectors. "He uses the geometric addition of vectors and defined multiplication of vectors in terms of what we call today adding the polar angles and multiplying the magnitudes"[**6**]. This was a big step for mathematics, and led to William Rowan Hamilton (1805-1865) defining complex numbers as ordered pairs and creating the algebraic definition of operations.

The term *complex number* came from Carl Friedrich Gauss (1777-1855) in 1831. It was thought Gauss had this idea of "geometric representation of complex numbers since 1796"[**6**]. [**6**] also states that in 1811 letter, Gauss mentions the theorem that was known later as Cauchy's theorem. Augustin-Luis Cauchy (1789-1857) discovered and rediscovered countless amazing results in the area of complex analysis along with constructing "the set of complex numbers in 1847" [**6**]. Some of these results that will be emphasized are Cauchy's Integral Theorem and Residue Theorem.

Cauchy was "a revolutionary in mathematics and a highly original founder of modern complex function theory" [**9**] and he is credited for creating and proving the Residue Theorem. Even though Cauchy produced the most important theorem (in Complex Analysis) and contributed more than anyone else, he would not have been able to do this without the work of earlier mathematicians in areas such as complex variables, complex function theory, and differential equations. "The earlier mathematicians relevant to Cauchy's work include Clairaut (1713-1765), Euler (1707-1783), d'Alembert (1717-1783), Lagrange (1736-1813), Laplace (1749-1827), and Poisson (1781-1840)"[**9**]. Cauchy, Bernhard Riemann (1826-1866), and Karl Weierstrass (1815-1897) are known as the "three principal architects of complex function theory"[**9**].

At first, other mathematicians did not appreciate Cauchy's "new" discoveries. According to [**4**], "Poisson saw in the paper merely a means of evaluating integrals...though the new method was worthy of consideration, it ought not to replace the old ones". At the time, some of Cauchy's work was considered nontraditional (his notation, language, and lengthy equations) and showed only simple applications. However, "in the course of composing these papers Cauchy went from exclusively using rectangular coordinates and evaluating integrals around rectangles to realizing the importance of considering polar coordinates and evaluating integrals around closed contours"[**9**]. There are other minor reasons why mathematicians criticized Cauchy but the main reason is that they failed to recognize the essential part, "that the integral of an analytic function of a complex variable taken along a closed path depends entirely upon the behavior of the function at points of discontinuity within the path"[**4**]. The mathematicians at the time may have not appreciated Cauchy's Residue Theorem and residue calculus but further down the road his discoveries are appreciated and recognized to be absolutely remarkable. From [**4**], "nevertheless the proofs are so conceived that they correspond in substance to the standards of rigor of present day" and from [**9**], "the calculus of residues is fundamental to complex function theory, and in these papers of Cauchy we have its origins".

Before Cauchy proved his Integral Theorem, he was already thinking and working on the creation of calculus of residues. Cauchy "discussed the convergence of a series within a circle of convergence and its divergence outside the circle, providing the expression for the radius of convergence" in a paper that was published in 1821 [**9**]. The following year

Cauchy figured out that it was possible to represent a function by its MacLaurin or Taylor series only if the series is convergent and has the function as its sum. It was not until Cauchy published a sequence of papers between 1826 and 1829 that this idea became very significant. He defined what a residue is and how the coefficient a_{-1} of a given function's series expansion describes the behavior of the line integrals around undefined points. Cauchy expanded his Integral Theorem to evaluate integrals of complex valued functions around a closed curve where a finite number of points lie inside the curve for which the function is not differentiable, which is also known as the Residue Theorem. "The residue calculus was an important tool for Cauchy in evaluating definite integrals, summing series, and discovering integral expressions for the roots of equations and the solutions of differential equations" [**9**].

The Residue Theorem relies on what is said to be the most important theorem in Complex Analysis, Cauchy's Integral Theorem. The Integral Theorem states that integrating any complex valued function around a curve equals zero if the function is differentiable everywhere inside the curve. Cauchy was not the only one that had this idea, it was Carl Friedrich Gauss (1777-1855) that already knew about it but never proved the theorem. According to [**3**], "in a letter he had written to his friend Friedrich Wilhelm Bessel (1784-1846) in 1811...it can be read that Gauss already knew about the "integral theorem." At the end of the letter Gauss states "This is a very beautiful theorem, for which I will give a not difficult proof at a suitable opportunity." In 1825, Cauchy presented the definition of integrals between complex limits where he stated his integral theorem and proof in his pamphlet "Mémoire sur les intégrales définies, prises entre des limites imaginaires." This was an amazing breakthrough for mathematics because about a year later in 1826, Cauchy created the calculus of residues.

Now, equipped with historical background of complex analysis, how can it be applied in order for it to have any significance? Well, in mathematics, some of the many applications are in areas such as functional analysis, linear algebra, analytic number theory, quantum field theory, and algebraic geometry. Applications in engineering such as fluid flow, electrical engineering, aeronautics, and many other. Even though it would be interesting for this capstone to cover all of these applications, I think it would be wise to settle with just one. So, the application this capstone will focus on is how the Residue Theorems are powerful tools to not only evaluate line integrals, but also to evaluate integrals in the real plane. The objective is to show the various techniques and residue theorems that can be used to simplify the procedure of evaluating all sorts of integrals in the real plane, and as mentioned before, integrals that once were thought to be difficult to compute.

0.2. Understanding the Residue Theorem

The ultimate goal of the Residue Theorem is to find the area under a curve, so suppose you are given a function $f(x)$ whose curve is represented in Figure 1. The shaded part (in Figure 1) underneath the curve represents the area that we want to find. In calculus, we use something called the integral (denoted as $\int f(x)dx$) to find the area under $f(x)$ but at times it can be to difficult to integrate due to the function and/or the limits of integration. By transforming the function to the complex plane, do a couple "tricks" in order to apply the Residue Theorem because that gives us a way to solve the integral by simple calculations. Basically, the Residue Theorem evaluates integrals by means of simple calculation instead of having to identify what integration technique to use and how to apply it.

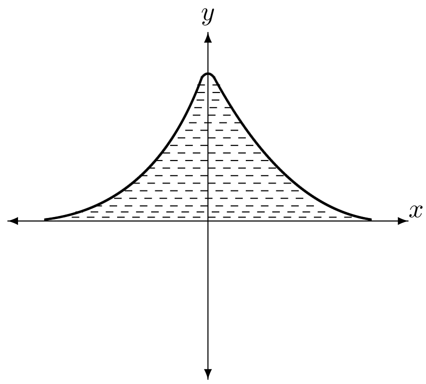
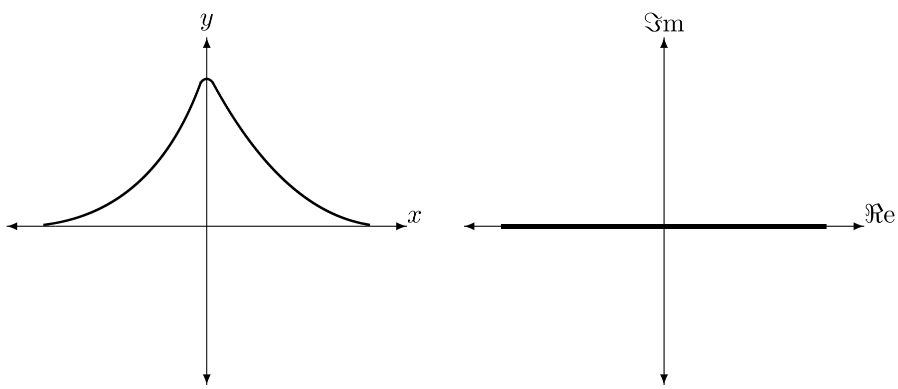


FIGURE 1. Area under curve $f(x)$



$f(x)$ in real plane $f(x)$ in complex plane

FIGURE 2. Visual representation of a real-valued function in the complex plane

For a conceptual understanding of the Residue Theorem, it is important to know the difference between the set of real numbers ($\mathbb{R}$) and the set of complex numbers ($\mathbb{C}$). A similarity between the two sets is that there is a one-to-one correspondence between real-valued functions and the real part of complex valued functions (Figure 2). Since a complex number is made up of two independent variables it would be difficult to graph a function of complex variables and find the area under the curve using the technique used in the real plane where the integration (of a function of a real variable) is over an interval. Unlike the real plane, the technique used in the complex plane is integration (of a function of complex variable) along a contour (known as contour integrals, line integrals, or path integrals). The next key concept to understanding the Residue Theorem is Cauchy's Integral Theorem, which basically states:

If a complex valued function $f(z)$ is analytic (continuous and differential) everywhere in a

(a) (b)

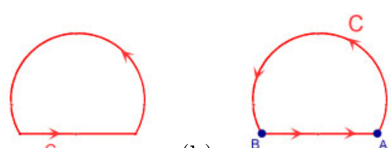


FIGURE 3

simply connected domain and closed contour C (Figure 3a) then

$$\int_C f(z)dz = 0.$$

Since $f(z)$ is analytic everywhere in the domain, integrating along any simply closed contour will equal zero because you always end up back the starting point. For example, the integral along a contour that starts at point z_a then goes to point z_b and back to point z_a will equal zero because integrating any contour from point z_b to point z_a will cancel out integrating along any contour from point z_a to point z_b (Figure 3b) . This is possible because integrating along any two contours from point z_a to z_b will be the same (under the conditions stated above, in the complex plane).

This changes when there exists a non-analytic point inside a closed curve. So, we extend that concept by supposing that $f(z)$ is analytic everwhere except at the point z_0 which lies within C. These points are called *singularities* and below is the idea on how to integrate along a closed contour where singularities exist inside that contour. Using Laurent's Theorem, there exists a disc around z_0 such that $f(z)$ is analytic, let γ denote that curve (Figure 4).

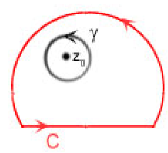


FIGURE 4

Then the contour C is split into two separate contours by cutting a line through that point and the contour C (Figure 5). By doing this, the integral along C becomes

$$\int_C f(z)dz = \int_{C_{\gamma 0}} f(z)dz + \int_{C_{\gamma 1}} f(z)dz + \int_\gamma f(z)dz \tag{0.2.1}$$

Now you have two closed contours ($C_{\gamma 0}$ and $C_{\gamma 1}$) such that $f(z)$ is analytic everywhere

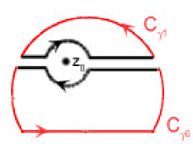


FIGURE 5

inside and therefore $\int_{C_{\gamma 0}} f(z)dz = 0$ and $\int_{C_{\gamma 1}} f(z)dz = 0$. After realizing that

$$\int_C f(z)dz = \int_\gamma f(z)dz$$

and that the integral depends entirely on the behavior at each singular point (also known as the residue), then you have almost met the requirements to conceptually understand the Residue Theorem. The last concept is how the residue gets its value. So suppose you have an isolated singular point z_0 of order 1 (also known as a Simple Pole), then the Laurent Series around that point would be

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k,$$

and since $f(z)$ is analytic everywhere but at the point z_0 then $f(z)$ can be written as

$$f(z) = \frac{a_{-1}}{(z - z_0)} + g(z)$$

where $g(z)$ is analytic at the point z_0. By solving for a_{-1} and then let z approach z_0, $((z - z_0)g(z)) \to 0$ and you are left with

$$a_{-1} = \lim_{z \to z_0} (z - z_0) f(z),$$

where $(z - z_0)f(z)$ will cancel out the singularity of order 1 in $f(z)$. The coefficient a_{-1} is called the Residue of $f(z)$ at z_0 and is denoted as

$$\underset{z=z_0}{Res}\,[\,f(z)\,] = a_{-1} = \lim_{z \to z_0} (z - z_0) f(z). \tag{0.2.2}$$

Similarly for an isolated singular point z_0 of order k (pole of order k), $f(z)$ can be written as

$$f(z) = \frac{a_{-k}}{(z - z_0)^k} + \frac{a_{-k+1}}{(z - z_0)^{k-1}} + ... + \frac{a_{-1}}{(z - z_0)} + g(z)$$

where $g(z)$ is analytic at z_0. By multiplying both sides by $(z - z_0)^k$ yields

$$(z - z_0)^k f(z) = a_{-k} + a_{-k+1}(z - z_0) + ... + (z - z_0)^{k-1} a_{-1} + (z - z_0)^k g(z).$$

Then by taking the $(k-1)^{\text{th}}$ derivative gives

$$\frac{d^{k-1}}{dz^{k-1}} \left[(z - z_0)^k f(z)\right] = a_{-1}\,(\,(k-1)!\,) + \frac{d^{k-1}}{dz^{k-1}} \left((z - z_0)^k\, g(z)\right)$$

As $z \longrightarrow z_0$ implies

$$\left[\frac{d^{k-1}}{dz^{k-1}} \left((z - z_0)^k g(z)\right)\right] \to 0\,,\ \text{so}$$

$$\begin{aligned} \lim_{z \to z_0} \left[\frac{d^{k-1}}{dz^{k-1}} \left((z - z_0)^k f(z)\right)\right] &= \lim_{z \to z_0} [\,a_{-1}\,((k-1)!)\,] + 0 \\ &= a_{-1}\,((k-1)!) \end{aligned}$$

Solving for a_{-1} gives

$$\underset{z=z_0}{Res}\,[\,f(z)\,] = a_{-1} = \frac{1}{(k-1)!} \lim_{z \to z_0} \left[\frac{d^{k-1}}{dz^{k-1}} ((z - z_0)^k f(z))\right]. \tag{0.2.3}$$

Someone lacking the full conceptual understanding of the Residue Theorem would still be able to compute the residue(s) and even apply the Residue Theorem to evaluate integrals. Finding the residue(s) is basically all calculation (at Calculus level), but to use the Residue Theorem, you need to understand the conditions to evaluate integrals.

THEOREM 0.2.1 (Cauchy's Residue Theorem). *Suppose the function f is analytic everywhere in a simply connected domain, except for isolated singularities at* $z_1, z_2, ..., z_n$. *Let C be a simple positively oriented closed contour that does not pass through any singularity. Then*

$$\int_C f(z)dz \quad = \quad 2\pi i \sum_{j=1}^{n} \underset{z=z_j}{Res}\,[\,f(z)\,]\,.$$

PROOF. The contour along C is equivalent to the sum of the contours around each singularity. (Why? Refer to Equation 0.2.1 and Figure 4 & 5)

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + ... + \int_{C_n} f(z)dz \tag{0.2.4}$$

Note that for each C_j represents the contour around the corresponding singularity $(z = z_j)$.

Since $f(z)$ is analytic, there exists a Laurent series expansion at each isolated singular point z_j (for $j = 1, 2, \ldots, n$).
So for each isolated singular point z_j, let

$$f(z) = \sum_{k=-\infty}^{\infty} a_k^{[j]}(z - z_j)^k. \tag{0.2.5}$$

Then by substituting $f(z)$ with Equation 0.2.5, the integral along each C_j becomes

$$\int_{C_j} \sum_{k=-\infty}^{\infty} a_k^{[j]}(z - z_j)^k dz.$$

So,

$$\begin{aligned}
\int_C f(z)dz &= \int_{C_1} \sum_{k=-\infty}^{\infty} a_k^{[1]}(z - z_1)^k dz + \int_{C_2} \sum_{k=-\infty}^{\infty} a_k^{[2]}(z - z_2)^k dz \\
&\qquad + ... + \int_{C_n} \sum_{k=-\infty}^{\infty} a_k^{[n]}(z - z_n)^k dz \\
&= \sum_{k=-\infty}^{\infty} a_k^{[1]} \int_{C_1} (z - z_1)^k dz + \sum_{k=-\infty}^{\infty} a_k^{[2]} \int_{C_2} (z - z_2)^k dz \\
&\qquad + ... + \sum_{k=-\infty}^{\infty} a_k^{[n]} \int_{C_n} (z - z_n)^k dz.
\end{aligned}$$

Therefore

$$\int_C f(z)dz = \sum_{j=1}^{n} \sum_{k=-\infty}^{\infty} a_k^{[j]} \int_{C_j} (z - z_j)^k dz. \tag{0.2.6}$$

Then using the definition of contour integration (refer to [**8**])

$$\int_\gamma \phi(z)dz = \int_a^b \phi(\zeta(t))\zeta'(t)dt$$

where $\phi(\zeta(t))$ is continuous and differentiable on interval $a \le t \le b$,

and polar form of a complex variable $z = R(\cos(\theta) + i\sin(\theta)) = Re^{i\theta}$, where R is the radius and $\theta \in [0, 2\pi]$. We can let $\phi(z) = (z - z_j)^k$ and $\zeta(\theta) = Re^{i\theta} + z_j$ to obtain

$$\begin{aligned}
\int_{C_j} (z - z_j)^k dz &= \int_{C_j} \phi(z)dz \\
&= \int_0^{2\pi} \phi(\zeta(\theta))\zeta'(\theta)d\theta \\
&= \int_0^{2\pi} (Re^{i\theta} + z_j - z_j)^k (Rie^{i\theta})d\theta \\
&= \int_0^{2\pi} (R_j e^{i\theta})^k iR_j e^{i\theta} d\theta.
\end{aligned}$$

Now simplify and pull out the constants,

$$\begin{aligned}\int_{C_j} (z-z_j)^k dz &= \int_0^{2\pi} (R_j)^k (e^{i\theta})^k i R_j e^{i\theta} d\theta \\ &= \int_0^{2\pi} i(R_j)^{k+1} (e^{i\theta})^{(k+1)} d\theta \\ &= i(R_j)^{k+1} \int_0^{2\pi} e^{(k+1)i\theta} d\theta.\end{aligned}$$

The next step is to integrate along each C_j to obtain:

$$i(R_j)^{k+1} \int_0^{2\pi} e^{(k+1)i\theta} d\theta = \left\{ \begin{array}{cc} 2\pi i & \text{if } k=-1 \\ 0 & \text{otherwise} \end{array} \right\}.$$

Hence

$$\int_{C_j} (z-z_j)^k dz = \left\{ \begin{array}{cc} 2\pi i & \text{if } k=-1 \\ 0 & \text{otherwise} \end{array} \right\}.$$

So if $k=-1$ then the following holds:

$$\begin{aligned}\int_C f(z)dz &= \sum_{j=1}^{n} \sum_{k=-1}^{-1} a_k^{[j]} 2\pi i \\ &= 2\pi i \sum_{j=1}^{n} a_{-1}^{[j]}.\end{aligned}$$

Since the coefficient a_{-1} is called the residue of $f(z)$ at z_j (refer to Equations 0.2.2 & 0.2.3)

$$a_{-1} = \underset{z=z_j}{Res}\,[\,f(z)\,]\,.$$

Therefore

$$\int_C f(z)dz = 2\pi i \sum_{j=1}^{n} \underset{z=z_j}{Res}\,[\,f(z)\,]\,.$$

□

If the conditions of the Residue Theorem are met then you would easily be able to evaluate integrals by simply calculating the residue(s), sum them up, and multiply by $2\pi i$.

The Residue Theorem is limited to evaluating certain integrals of rational functions, but extending this theorem produces broader range of applications. The Residue Theorems that are acknowledged in this capstone will merely serve the purpose of showing how powerful and widely useful the Residue Theorems can be when evaluating integrals. Each Residue Theorem has an application to evaluating a specific type of integral that is commonly encountered. The applications are presented through the proof of the corresponding theorems and followed by a concrete example of how it is applied. This capstone focuses on how to use the Residue Theorems to evaluate improper integrals involving trigonometric functions ($\int_{-\infty}^{\infty} f(x)\cos\lambda\, x\, dx$), proper integrals of sine and cosine functions ($\int_0^{2\pi} F(\sin\theta,\cos\theta)\, d\theta$), improper integrals involving exponential functions ($\int_{-\infty}^{\infty} \frac{P(e^x)}{Q(e^x)} e^{ax} dx$), improper integrals involving fractional powers ($\int_0^{\infty} \frac{x^{\alpha}\, P(x)}{Q(x)} dx$) and logarithmic functions ($\int_0^{\infty} \frac{P(x)}{Q(x)} \ln(x) dx$).

The Basics

1.3. Complex Field

Definition 1.3.1 (Complex Field). The set of all complex numbers ($\mathbb{C} = \{a + ib \mid (a, b) \in \mathbb{R} \times \mathbb{R}\}$) with operations defined as: Let $z = (a, b)$ and w $= (c, d)$ be complex numbers

$$\begin{aligned} z \pm w &= (a, b) \pm (c, d) = (a \pm c, b \pm d) \quad \textit{addition/subtraction} \\ z \cdot w &= (a, b) \cdot (c, d) = (ac - bd, ad + bc) \quad \textit{multiplication} \\ \frac{z}{w} &= \frac{(a, b)}{(c, d)} = (\frac{ac + bd}{c^2 + d^2}, \frac{bc - ad}{c^2 + d^2}) \quad \textit{division } (w \neq 0) \end{aligned}$$

Definition 1.3.2. The *modulus* or absolute value of a complex number $z = x + iy$ is a non-negative real number denoted $|z|$ and given by equation

$$|z| = \sqrt{x^2 + y^2}$$

Remark. The modulus is the distance between the origin and point $z = (x, y)$, which can also be considered as the length of a vector.

Definition 1.3.3. The *Conjugate* of $z = x + iy$ is denoted $\overline{z}$ and defined as

$$\overline{z} = x - iy \qquad \text{and} \qquad |z|^2 = \overline{z}\, z$$

Remark. The conjugate can be seen geometrically as the reflection over the real axis.

Definition 1.3.4 (Argument of a Complex Number). The angle between the positive real axis and the vector created by complex number z

Remark. There are infinite number of arguments that differ on intervals of 2π

Definition 1.3.5 (Principal Argument). Unique argument that lies on $(-\pi, \pi]$, denoted as $\arg z$

Note. To compute $\arg z$, consider $z = x + iy = |z|\, e^{i\theta}$ where $\theta \in (-\pi, \pi]$.

$$\arg(x + iy) = \arctan(\frac{y}{x}) \iff \tan(\arg z) = \frac{y}{x}$$

1.4. Complex Functions

1.4.1. Complex Functions.

Definition 1.4.1 (Trigonometric Form). A complex number $z = x + iy$ can be written as polar coordinates where

$$\begin{aligned} x &= Re[z] = r\cos\theta, \\ y &= Im[z] = r\sin\theta, \end{aligned}$$

and

$$z = x + iy = r(\cos\theta + i\sin\theta)$$

1.4.1.1. *Exponential and Logarithmic Functions.*

Note. To derive the relationship of exponential function and the trigonometric functions, use the Taylor series expansion of e^z,$sin(z)$, and $cos(z)$, then replace z with iz.

$$\begin{aligned}
e^z &= \sum_{n=0}^{\infty} \frac{1}{n!} z^n && (1.4.1)\\
&= 1 + z + \frac{z}{2!} + ... + \frac{z^n}{n!} + ...\\
sin(z) &= \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} z^{2n-1} && (1.4.2)\\
&= z - \frac{z^3}{3!} + ... + \frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!} + ...\\
cos(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} && (1.4.3)\\
&= 1 - \frac{z^2}{2!} + ... + \frac{(-1)^n z^{2n}}{(2n)!} + ...
\end{aligned}$$

Rearranging terms and factoring out the i

$$\begin{aligned}
e^{iz} &= 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} + ...\\
&= (1 - \frac{z^2}{2!} + \frac{z^4}{4!} + ...) + i(z - \frac{z^3}{3!} + \frac{z^5}{5!} + ...)
\end{aligned}$$

$$e^{iz} = cos(z) + isin(z) \tag{1.4.4}$$

and

$$e^{-iz} = cos(z) - isin(z) \tag{1.4.5}$$

It follows by adding equations 1.4.4 and 1.4.5 that

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \tag{1.4.6}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{1.4.7}$$

Note. Polar form can be represented as $z = re^{i\theta}$. Also notice the form $e^z = e^x(\cos y + i\sin y)$ has a period of $2\pi i$, so for any integer k,$e^{z+2k\pi i} = e^z$

$$\begin{aligned}
e^{z+2\pi i} &= e^{x+iy+2\pi i} = e^{x+i(2\pi+y)}\\
&= e^x(\cos(y+2\pi) + i\sin(y+2\pi))\\
&= e^x[(\cos y\cos 2\pi - \sin y\sin 2\pi) + i(\sin y\cos 2\pi + \cos y\sin 2\pi)]\\
&= e^x(\cos y + i\sin y)\\
&= e^z
\end{aligned}$$

Furthermore let $w = e^z$ and $z = x + iy$ implies

$$|w| = e^x \qquad \text{or} \qquad x = ln\,|w|$$

and

$$Arg\,w = y + 2k\pi \qquad \text{or} \qquad y = Arg\,w$$

Therefore the inverse images of the point w can only be the points of the form

$$z = ln\,|w| + iArg\,w.\ [\mathbf{8}]$$

REMARK. The principal argument ($\arg w = y$) is usually used, since $Arg\,w$ can take infinite values

PROPERTY 1.4.1.

$$\begin{aligned} e^{2\pi i} &= \cos 2\pi + i \sin 2\pi \\ &= 1 + i0 \\ &= 1 \end{aligned} \tag{1.4.8}$$

$$\begin{aligned} e^{\pi i} &= \cos \pi + i \sin \pi \\ &= -1 + i0 \\ &= -1 \end{aligned} \tag{1.4.9}$$

PROPERTY 1.4.2.

$$\text{Re}\left[e^{iz}\right] = \cos[z] \tag{1.4.10}$$

$$\text{Im}\left[e^{iz}\right] = \sin[z] \tag{1.4.11}$$

DEFINITION 1.4.2 (Logarithm). The logarithm, denoted by Lnz and defined by

$$Lnz = \ln|z| + iArgz$$

REMARK. The value of the logarithm can be evaluated by the principal value,

$$Ln(z) = \ln(|z|) + i\arg(z) = \ln(|z|) + 2k\pi i,$$

then

$$Ln(z) = \ln(\sqrt{x^2 + y^2}) + i\arctan(\frac{y}{x}) \tag{1.4.12}$$

DEFINITION 1.4.3 (Cauchy-Riemann Equations[**8**]). Let $f(z) = u(x, y) + iv(x, y)$ be a function of a complex variable defined on a domain G. Then a necessary and sufficient condition for $f(z)$ to be differentiable (as a function of a complex variable) at the point $z_0 = x_0 + iy_0 \in G$ is that the functions $u(x, y)$ and $v(x, y)$ be differentiable (as functions of two real variables x and y) at the point (x_0, y_0) and satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at (x_0, y_0). If these conditions are satisfied, $f\prime(z_0)$ can be represented in any of the forms

$$f\prime(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$f\prime(z_0) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$
$$f\prime(z_0) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$
$$f\prime(z_0) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

1.4.2. Analytic Functions and Differentiation.

Definition 1.4.4 (Differentiation). A function $f\colon D \to \mathbb{C}$ is said to be differentiable at $z_0 \in D$ if

$$\lim_{z \longrightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}, \text{ exists.}$$

Then

$$f'(z_0) = \lim_{z \longrightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

the function $f(z)$ is said to be differentiable at the point z_0.

Note. $f'(z_0)$ is also called the derivative of the function $f(z)$ at the point z_0.

Definition 1.4.5. A function f is said to be *entire* if f is analytic at all points in $\mathbb{C}$

Definition 1.4.6 (Analyticity). A function f is said to be analytic at z if f is differentiable in a neighborhood of z. f is analytic on set D if f is differentiable at all points of some open set containing D

1.5. Complex Integrals

Definition 1.5.1 (Simple Closed Curve). A simple closed curve is a curve whose only double points are its initial and terminal points. [**2**]

Definition 1.5.2 (Jordan Curve). Let C be a simple closed curve in $\mathbb{C}$. Then $\mathbb{C}$ C has exactly two connected components, one bounded and the other unbounded. [**2**]

Definition 1.5.3 (Definite Integral). Let $f(t) = u(t) + iv(t)$ where $u(t)$ and $v(t)$ are real-valued functions of real variable t for $a \leq t \leq b$. Then

$$\int_a^b f(t)dt = \int_a^b (u(t) + iv(t))dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

Definition 1.5.4 (Contour Integrals). Suppose that C is an arc given by a function ζ : $[a, b] \longrightarrow \mathbb{C}$. A complex valued function f is said to be continous on the arc C if the function $\Psi(t) = f(\zeta(t))$ is continous in $[a, b]$, then

(1) The integral of f on C is defined to be

$$\int_C f(z)dz = \int_a^b f(\zeta(t))\zeta'(t)dt$$

(2) If describe C in opposite direction from $t = b$ to $t = a$, then
$$\int_{-C} f(z)dz = -\int_C f(z)dz$$
(3) Supppose F is analytic in a domain G and has a continous derivative $f = F'$ in G. Also suppose ζ is an arc lying in G with initial point z_1 and terminal point z_2
$$\int_C f(z)dz = F(\zeta(b)) - F(\zeta(a)) = F(z_2) - F(z_1)$$

FIGURE 6. Augustin-Louis Cauchy (1789-1857)

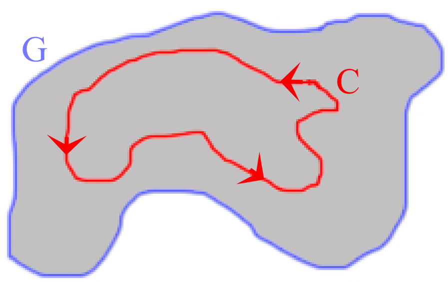


FIGURE 7. Cauchy's Integral Theorem

1.5.1. Cauchy's Integral Theorem.

NOTE. One of the greatest minds in history was Augustin-Louis Cauchy (Figure 6), a French mathematician in the 1800s. Cauchy's understanding of mathematics was far and beyond most mathematicians of his time. His creativity and understanding of mathematics lead to 789 publications that opened the doors to the next mathematicians. Cauchy's integral theorem is one of the most important theorems in complex analysis.

THEOREM 1.5.5 (Cauchy-Goursat Theorem –Cauchy Integral Theorem). *Let $f(z)$ be analytic in a simply connected domain G (Figure 7), then*
$$\int_C f(z)dz = 0$$
for every piecewise smooth closed path $C \subset G$

THEOREM 1.5.6 (Cauchy's Theorem for Multiply Connected Domains). *If $f(z)$ is analytic in a domain D containing the simple closed path C and the simple closed paths C_1, C_2, …, C_n all interior to C, then the integral along C is equal to the sum of all the integrals*

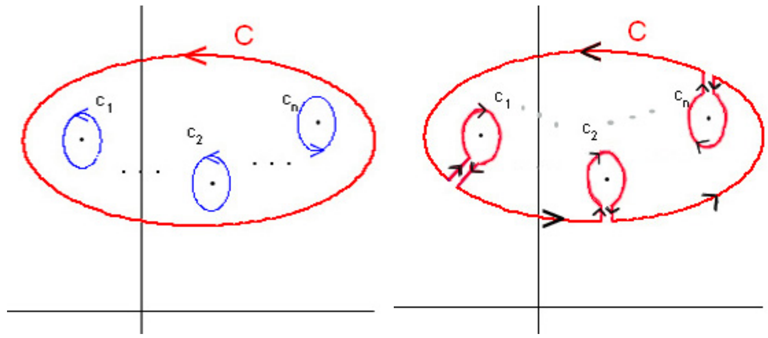


FIGURE 8. Sum of all contours is equivalent to contour along C

along all the C_k, provided all the paths are traversed either counterclockwise or clockwise.[**2**] *(Figure 8)*

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_n} f(z)dz + \cdots + \int_{C_n} f(z)dz \tag{1.5.1}$$

1.5.2. Cauchy's Integral Formula.

THEOREM 1.5.7 (Cauchy's Integral Formula). *Suppose that f is entire, for some complex number a, and for a curve C defined by*

$$C : a + Re^{i\theta}, \text{ where } 0 \leq \theta \leq 2\pi \text{ and } R \geq |a|$$

then

$$f(a) = \frac{1}{2\pi} \int_C \frac{f(z)}{z-a} dz$$

THEOREM 1.5.8 (M L Theorem). *Suppose that a function f is continuous on a contour C. Then*

$$\left| \int_C f(z)dz \right| \leq M\, L$$

where L is the length of C and M is real constant such that $|f(z)| \leq M$

Concepts Of The Residue Theorems

2.1. Laurent Series

Theorem 2.1.1 (Laurent's Theorem). *Let $f(z)$ be an analytic function on an annulus D : $r < |z - z_0| < R$. Then there exists a Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{2.1.1}$$

converging to $f(z)$ on D[**8**, 229]

Remark. When a function is not analytic at some point(s), a Laurent series can be used to represent the function such that it is analytic in and on some annulus. Figure 9 shows where $f(z)$ is analytic (on and between both contours). The coefficient of the term $1/(z-z_0)$ happens to play a significant role in the Residue Theorems.

Theorem 2.1.2. *The coefficients of the Laurent series are given by the formula*

$$a_k = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(z)}{(z - z_0)^{k+1}} dz \qquad (k = 0, \pm 1, \pm 2, ...)$$

where γ_ρ is any circle $|z - z_0| = \rho$, $r < \rho < R$. [**8**, 228]

Remark. The negative powers of the Laurent series are referred to as the principal part of $f(z)$ [**2**]

2.2. Poles, Singular Points, Residues

Note. A point $z = z_0$ is called a singular point or singularity of complex function $f(z)$ if f is not defined (not analytic) at $z = z_0$, but for every neighborhood of z_0 contains at least one point at which $f(z)$ is analytic.

Definition 2.2.1 (Isolated Singular Point). A singular point z_0 of $f(z)$ is called an isolated singular point if there exists $\delta > 0$ such that $f(z)$ is analytic in the punctured disk $0 < |z - z_0| < \delta$

Remark. Getting as arbitrarily close to z_0 as one wants, there exists an arbitrarily small disk around z_0 such that $f(z)$ is analytic.

Remark. A Laurent series exists for isolated singularities because the annulus (Theorem 2.1.1) is the same as the punctured disk.

Definition 2.2.2 (Removable Singularities). Let the function $f(z)$ be analytic in a punctured disk $0 < |z - z_0| < \delta$. If its Laurent series around $z = z_0$ has no principal part for $0 < |z - z_0| < \delta$, then z_0 is said to be a removable singularity of $f(z)$.[**2**]

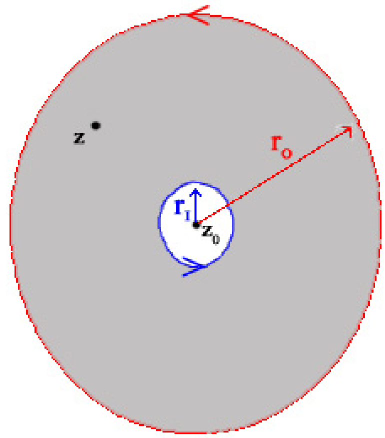


FIGURE 9. $r_I \leq |z - z_0| \leq r_O$

If z_0 is a removable singularity then $f(z)$ has a finite limit as $z \to z_0$

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^n \\ &= a_0 + a_1(z - z_0) + ... + a_n(z - z_0)^n + ... \end{aligned}$$

So, as z approaches z_0, $f(z)$ approaches a_0.

$$\lim_{z \to z_0} f(z) = a_0 \tag{2.2.1}$$

DEFINITION 2.2.3 (Essential Singularity). If $f(z)$ is analytic in a punctured disk $0 < |z - z_0| < \delta$ and the principal part of the Laurent series of $f(z)$ around $z = z_0$ contains an infinite number of terms then the point z_0 is called an essential singularity of $f(z)$

NOTE. If z_0 is an essential singularity then $f(z)$ has no finite or infinite limit as $z \to z_0$. For example, the function $f(z) = Ln(z)$ has a singularity at $z = 0$ but is not isolated. Why? Because there does not exist a neighborhood around z_0 such that $f(z)$ is analytic (Lnz is not analytic on the negative real axis)

DEFINITION 2.2.4 (Poles). Suppose $f(z) = g(z)/(z - z_0)^n$ where f is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$ and where g is analytic in some neighborhood of z_0. If $n > 1$ then f is said to have a pole of order n at z_0. If $n = 1$ then f is said to have a simple pole at z_0.

THEOREM 2.2.5 (Simple Pole). *The point z_0 is a simple pole of the function $f(z)$. Then $f(z)$ can be written as*

$$f(z) = \frac{a_{-1}}{z - z_0} + g(z)$$

where $g(z)$ is analytic at z_0. So as $z \to z_0$, $(z - z_0) \to 0$. Then

$$a_{-1} = \lim_{z \to z_0} (z - z_0) f(z) \tag{2.2.2}$$

THEOREM 2.2.6 (Pole of order k). *If $f(z)$ has a pole of order k at z_0 then $f(z)$ can be written as*

$$f(z) = \frac{a_{-k}}{(z - z_0)^k} + \frac{a_{-k+1}}{(z - z_0)^{k-1}} + ... + \frac{a_{-1}}{(z - z_0)} + g(z)$$

where $g(z)$ is analytic at z_0, so

$$(z - z_0)^k f(z) = a_{-k} + a_{-k+1}(z - z_0) + ... + (z - z_0)^k g(z)$$

is analytic at z_0. Then by taking the $(k-1)^{th}$ derivative gives

$$\frac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k f(z)\right] = a_{-1}\,(\,(k-1)!\,) + \frac{d^{k-1}}{dz^{k-1}}\left((z-z_0)^k\, g(z)\right)$$

As $z \to z_0$ implies

$$\left[\frac{d^{k-1}}{dz^{k-1}}\left((z-z_0)^k g(z)\right)\right] \to 0\,,\ so$$

$$\begin{aligned}\lim_{z\to z_0}\left[\frac{d^{k-1}}{dz^{k-1}}\left((z-z_0)^k f(z)\right)\right] &= \lim_{z\to z_0}\left[\,a_{-1}\,((k-1)!)\,\right] + 0\\ &= a_{-1}\,((k-1)!)\end{aligned}$$

Solving for a_{-1} gives

$$a_{-1} = \frac{1}{(k-1)!}\lim_{z\to z_0}\left[\frac{d^{k-1}}{dz^{k-1}}((z-z_0)^k f(z))\right] \tag{2.2.3}$$

Remark. Since the calculation of a Laurent series expansion can often take a quite a bit of time, Equation 2.2.3 can be used to find the a_{-1} coefficient.

Definition 2.2.7 (Residue). For an isolated singular point z_0 of f, there is a Laurent series

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n,$$

which is valid for $0 < |z-z_0| < R$. The coefficient a_{-1} of $(z-z_0)^{-1}$ is called the residue of f at z_0 and denoted as

$$a_{-1} = \underset{z=z_0}{Res}\,[\,f\,]$$

Residue Theorems

The Residue Theorem is fairly straightforward and easy to understand but there is a difficult part and that is knowing how to apply the theorem to evaluate definite integrals. Since the purpose of this capstone is to show how the Residue Theorems can make evaluating integrals easier, this section will help identify what approach should be taken to evaluate certain types of integrals. With practice this ability becomes routine, which shows the power of the Residue Theorems and how they can make life so much easier when evaluating integrals.

In order to use the Residue Theorem to evaluate real integrals, the integral needs to be extended to the complex plane. Actually, there is a one-to-one correspondence between real-valued functions and the real part of complex valued functions and hence, a definite integral in the real plane can be transformed to a contour integral on the real-axis in the complex plane. The Residue Theorem requires a closed contour, so by creating a closed curve defined by the contour on the real axis and a half-circle in the upper or lower half-plane. If the integral along the half-circle goes to zero, then that leaves the integral along the real axis part which equals $2\pi i$ times the sum of the residues. But that contour integral along the real axis is equivalent to a definite integral in the real plane and now that definite integral has a solution by simple calculation.

3.1. Residue Theorem

THEOREM 3.1.1 (Cauchy's Residue Theorem). *Suppose the function f is analytic everywhere in a simply connected domain G, except for isolated singularities at $z_1, z_2, \ldots, z_n$. Let C be a positively oriented Jordan contour that does not pass through any singularity. Then*

$$\int_C f(z)dz \quad = \quad 2\pi i \sum_{j=1}^{n} \underset{z=z_j}{Res}\,[\,f(z)\,] \tag{3.1.1}$$

PROOF. By Theorem 1.5.6,

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + ... + \int_{C_n} f(z)dz$$

and since $f(z)$ is analytic there exists a Laurent series expansion at each isolated singular point z_j (for $j = 1, 2, \ldots, n$).

$$f(z) = \sum_{k=-\infty}^{\infty} a_k^{[j]}(z - z_j)^k$$

Then

$$\begin{aligned}\int_C f(z)dz &= \int_{C_1}\sum_{k=-\infty}^{\infty} a_k^{[1]}(z-z_1)^k dz + \int_{C_2}\sum_{k=-\infty}^{\infty} a_k^{[2]}(z-z_2)^k dz \\ &\qquad + \ldots + \int_{C_n}\sum_{k=-\infty}^{\infty} a_k^{[n]}(z-z_n)^k dz \\ &= \sum_{j=1}^{n}\sum_{k=-\infty}^{\infty} a_k^{[j]}\int_{C_j}(z-z_j)^k dz\end{aligned}$$

For each $C_j : |z - z_j|$

$$\begin{aligned}\int_{C_j}(z-z_j)^k dz &= \int_0^{2\pi}(R_j e^{i\theta})^k iR_j e^{i\theta}d\theta \\ &= i(R_j)^{k+1}\int_0^{2\pi} e^{(k+1)i\theta}d\theta\end{aligned}$$

$\Longrightarrow$

$$\int_{C_j}(z-z_j)^k dz = \left\{\begin{array}{ll} 2\pi i & \text{if } k=-1 \\ 0 & \text{otherwise}\end{array}\right\}$$

$\Longrightarrow$

$$\begin{aligned}\int_C f(z)dz &= \sum_{j=1}^{n}\sum_{k=-1}^{-1} a_k^{[j]}2\pi i \\ &= 2\pi i\sum_{j=1}^{n} a_{-1}^{[j]} \\ &= 2\pi i\sum_{j=1}^{n} \underset{z=j}{Res}\,[\,f(z)\,]\end{aligned}$$

$\square$

Cauchy's Principal Value can be useful when evaluating improper integrals. Often the principal value of an integral exists while the integral diverges. The basic idea is to restrict the bounds, evaluate the integral, and then take the limit.

DEFINITION 3.1.2 (Cauchy's Principal Value).

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx \tag{3.1.2}$$

For any even function, Definition 3.1.2 has a symmetric property (Property 3.1.1). The use of this property is shown in Example 3.1.3.

PROPERTY 3.1.1.

$$\begin{aligned}\int_0^R f(x)dx &= \frac{1}{2}\int_{-R}^{R} f(x)dx \\ \int_{-R}^{R} f(x)dx &= 2\int_0^R f(x)dx\end{aligned}$$

EXAMPLE 3.1.3. Evaluate

$$\int_0^\infty \frac{1}{(x^2+1)^2}dx$$

Let

$$F(z) = \frac{1}{(z^2+1)^2}\,,$$

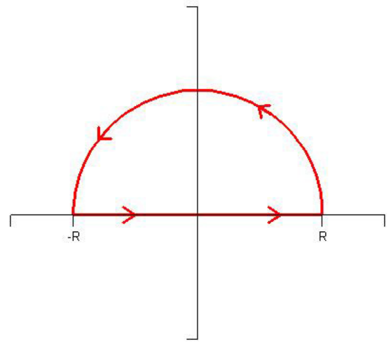


FIGURE 10. Contour for evaluating $\int_{-\infty}^{\infty} f(x)dx$

then by Theorem 3.1.1

$$\int_C F(z)dz = 2\pi i \operatorname*{Res}_{z=i}[\, f(z)\,]$$

Apply Theorem 1.5.6 and Property 3.1.1 for P.V.

$$\begin{aligned}\int_C F(z)dz &= \int_{-R}^{R} F(x)dx + \int_{C_R} F(z)dz \\ &= 2\pi i \operatorname*{Res}_{z=i}[\, F(z)\,]\end{aligned}$$

$$\Longleftrightarrow$$

$$2\pi i \operatorname*{Res}_{z=i}[\, F(z)\,] = 2\int_0^R F(x)dx + \int_{C_R} F(z)dz$$

$$\Longleftrightarrow$$

$$\int_0^R F(x)dx = \pi i \left[\operatorname*{Res}_{z=i}[\, F(z)\,]\right] - \frac{1}{2}\int_{C_R} \frac{dz}{(z^2+1)^2} \tag{3.1.3}$$

Calculate the residue at $z = i$:

$$\begin{aligned}\operatorname*{Res}_{z=i}[\, F(z)\,] &= \lim_{z\to i}\left[\frac{d}{dz}(z-i)^2 \frac{1}{(z-i)^2(z+i)^2}\right] \\ &= \lim_{z\to i}\left[\frac{-2}{(z+i)^3}\right] \\ &= \frac{-2}{(2i)^3} \\ &= \frac{1}{4i}\end{aligned}$$

Substitute the computed residue into Equation 3.1.3

$$\int_0^R \frac{dx}{(x^2+1)^2} = \frac{\pi}{4} - \frac{1}{2}\int_{C_R} \frac{dz}{(z^2+1)^2}$$

Show $\int_{C_R} F(z)dz$ vanishes as $R \to \infty$ (using Theorem 1.5.8)

For $z \in C_R$,

$$\begin{array}{rcl} \left|z^2+1\right| & \geq & \left|z^2\right| - 1 = R^2 - 1 \\ \dfrac{1}{|z^2+1|} & \leq & \dfrac{1}{R^2-1} \\ \dfrac{1}{(z^2+1)^2} & \leq & \dfrac{1}{(R^2-1)^2} \end{array}$$

Where $M = 1/(R^2-1)^2$ and $L = \pi R$ implies

$$\left|\int_{C_R} \frac{1}{(z^2+1)^2}\right| \leq \frac{1}{(R^2-1)^2}\pi R$$

But

$$\lim_{R\to\infty}\left[\frac{\pi R}{(R^2-1)^2}\right] = 0,$$

so

$$\lim_{R\to\infty}\left[\int_0^R \frac{dx}{(x^2+1)^2}\right] = \frac{\pi}{4} - 0$$

Hence,

$$\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4} \tag{3.1.4}$$

In the previous example, Theorem 1.5.8 was used to show $\int_{C_R} f(z)dz$ vanishes as R goes to infinity. The following theorem makes this task easier for rational functions where the degree of the denominator exceeds the degree of the numerator.

THEOREM 3.1.4. *Let $f(z) = P(z)/Q(z)$ be rational function such that the degree of $Q(z)$ is greater than the degree of $P(z)$ by at least one. Then*

$$\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$$

PROOF. Let

$$f(z) = \frac{a_m z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0}{b_n z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0}$$

where $n \geq m+2$, and

$$|b_n z^n + \cdots + b_0| \geq |b_n|\,|z^n| - \cdots - |b_0|$$

$\Longleftrightarrow$

$$\frac{1}{(|b_n z^n + \cdots + b_0|)} \leq \frac{1}{(|b_n|\,|z^n| - \cdots - |b_0|)},$$

which implies

$$\begin{array}{rcl} \left|\displaystyle\int_{C_R} f(z)dz\right| & \leq & \displaystyle\int_0^\pi \frac{|a_m|\,|z^m| + |a_{m-1}|\,\left|z^{m-1}\right| + \cdots + |a_1|\,|z| + |a_0|}{|b_n|\,|z^n| - |b_{n-1}|\,|z^{n-1}| - \cdots - |b_1|\,|z + b_0|}\,R\,d\theta \\ & = & \dfrac{\pi R\,(|a_m|\,R^m + \cdots + |a_0|)}{|b_n|\,R^n - \cdots - |b_0|} \qquad (3.1.5) \end{array}$$

As $R \longrightarrow \infty$, Equation 3.1.5 $\longrightarrow 0$, since $n \geq m+2$. Therefor

$$\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$$

$\square$

Example 3.1.3 showed the basic steps to evaluate an improper integral using Cauchy's Residue Theorem and principal value theorem. Integrals of this form have a general process to evaluate. Consider the integral $\int_{-\infty}^{\infty} f(x)dx$, then map $f(x)$ to the complex plane $(F(z))$ and apply Definition 3.1.2. Create contour C, that consists of the line segment on real axis from $[-R, R]$ and contour $C_R : z = Re^{i\theta}$ where $\theta \in [0, \pi]$. Then by the Residue Theorem

$$\int_C f(z) = 2\pi i \sum_{j=1}^{n} \underset{z=z_j}{Res} [\, f(z) \,]$$

$$\lim_{R\to\infty} \left[\int_C f(z) = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz \right],$$

which can be rewritten as

$$\lim_{R\to\infty} \left[\int_{-R}^{R} f(x)dx = 2\pi i \sum_{j=1}^{n} \underset{z=z_j}{Res} [\, f(z) \,] - \int_{C_R} f(z)dz \right].$$

Since finding the residue is not that difficult, then the task is to make $\int_{C_R} f(z)dz = 0$ and the job is done. This task is fairly easy by using Theorem 1.5.8 or Theorem 3.1.4. Suppose the $\lim_{R\to\infty} \int_{C_R} f(z)dz = 0$, then

$$\lim_{R\to\infty} \left[\int_{-R}^{R} f(x)dx = 2\pi i \sum_{j=1}^{n} \underset{z=z_j}{Res} [\, f(z) \,] - \int_{C_R} f(z)dz \right]$$

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)dx &= 2\pi i \sum_{j=1}^{n} \underset{z=z_j}{Res} [\, f(z) \,] - 0 \\ \int_{-\infty}^{\infty} f(x)dx &= 2\pi i \sum_{j=1}^{n} \underset{z=z_j}{Res} [\, f(z) \,] \end{aligned}$$

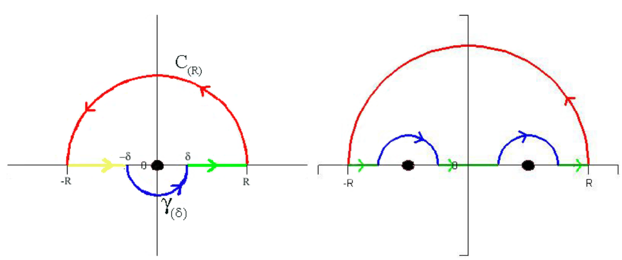


FIGURE 11. Singularity Inside/Outside Contour

If a singularity lies on the real axis, it is all right to bend around that singularity. The two ways to bend around the singularity are either to include it inside the contour or outside the contour (Figure 11). These integrals are commonly refered to as *indented integrals*

THEOREM 3.1.5. *Suppose that a function f is analytic in a simply connected domain D, except for an isolated singularity at z_0, and that if C is a positive oriented Jordan contour in D. Then*

$$\frac{1}{2\pi i} \int_C f(z)dz = \begin{bmatrix} a_{-1} & \text{if } z_0 \text{ lies inside } C \\ 0 & \text{if } z_0 \text{ lies outside } C \end{bmatrix}$$

The following theorem (Theorem 3.1.6) is taken directly from [**5**], and will be used for reference.

THEOREM 3.1.6. *Let $f(z) = P(z)/Q(z)$ where $P(x)$ and $Q(x)$ are polynomials with real coefficients of degree m and n, respectively, and $n \geq m+2$. If $Q(x)$ has simple zeros at the points $t_1, ..., t_n$ on the x-axis, then*

$$P.V. \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum_{j=1}^{k} \underset{z=z_j}{Res} \left[\frac{P(z)}{Q(z)} \right] + \pi i \sum_{j=1}^{l} \underset{z=t_j}{Res} \left[\frac{P(z)}{Q(z)} \right] \tag{3.1.6}$$

where $z_1, ..., z_k$ are the poles of $f(z)$ that lie in the upper half-plane

3.2. Jordan's Lemma

In the previous section, the difficult part was to make the contour integral around the half-circle vanish. Jordan's lemma makes this easier while at the same time more powerful. Jordan's lemma is a very useful tool when evaluating Fourier integrals, which are typically in the form:

$$\int_{-\infty}^{\infty} f(x) e^{i\lambda x}\, dx \tag{3.2.1}$$

$$\int_{-\infty}^{\infty} f(x) \cos \lambda x\, dx \tag{3.2.2}$$

$$\int_{-\infty}^{\infty} f(x) \sin \lambda x\, dx \tag{3.2.3}$$

(3.2.4)

where $f(x)$ is a rational function such that the degree of the denominator is greater than degree of the numerator. Suppose $f(x) = P(x)/Q(x)$ where the deg $(Q(x)) = n$, deg $(P(x)) = m$ and $n \geq m+1$.

LEMMA 3.2.1 (Jordan's Lemma). *Given a family of circular arcs*

$$\gamma_R : |z| = R,\ Im\, z \geq -a,$$

where a is a fixed real number, let $f(z)$ be a continuous function defined on every γ_R such that

$$\begin{aligned} \lim_{R\to\infty} M(R) &= \lim_{R\to\infty} \left[\max_{z\in\gamma_R} |f(z)| \right] \\ &= 0 \end{aligned}$$

then

$$\lim_{R\to\infty} \left[\int_{\gamma_R} f(z) e^{i\lambda z} dz \right] = 0$$

for every positive λ

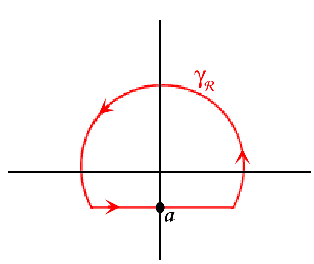


FIGURE 12. Jordan's Lemma

PROOF. Since γ_R is bounded by some function of R, let $M(R) = \max_{z\in\gamma_R} |f(z)|$ denote this function. Then $|f(z)| < M(R)$ and further,

$$\begin{aligned}
\int_{\gamma_R} f(z)e^{i\lambda z}dz &\leq \left|\int_{\gamma_R} f(z)e^{i\lambda z}dz\right| \\
&= \int_0^\pi \left|f(Re^{i\theta})\right| \left|Rie^{i\theta}\right| \left|e^{i\lambda z}\right| d\theta \\
&\leq R\int_0^\pi \left|f(Re^{i\theta})\right| \left|e^{i\lambda z}\right| d\theta \\
&\leq RM(R)\int_0^\pi e^{i\lambda z}d\theta \\
&= RM(R)\int_0^\pi e^{i\lambda Re^{i\theta}}d\theta
\end{aligned}$$

$$\left|e^{i\lambda Re^{i\theta}}\right| = e^{-\lambda R\sin\theta} \leq e^{-2\lambda R\theta/\pi}$$

$$\begin{aligned}
\int_{\gamma_R} f(z)e^{i\lambda z}dz &\leq RM(R)\int_0^\pi e^{-2\lambda R\theta/\pi}d\theta \\
&= \frac{\pi M(R)}{\lambda}(1-e^{-\lambda R})
\end{aligned}$$

Therefore

$$\lim_{R\to\infty}\int_{\gamma_R} f(z)e^{i\lambda z}dz = 0 \tag{3.2.5}$$

$\square$

3.2.1. Improper Trigonometric Integrals.

Encountering an integral like the ones in equation, 3.2.2, or 3.2.3 may be a little frightening or even overwhelming. Before giving up, try mapping equation 3.2.1 to the complex plane, then use Jordan's lemma (Lemma 3.2.1) to see how it may help. By considering the integral

$$\int_{-\infty}^{\infty}\frac{P(z)}{Q(z)}e^{i\lambda z}dz$$

that satisfies

$$\deg\left(\,Q(z)\,\right) > \deg\left(\,P(z)\,\right),$$

makes evaluating integrals involving sine and cosine functions very simple. Before a concrete example is shown, it would be good to derive Theorems 3.2.2 and 3.2.3. So also consider the contour C which defined by

$$C_R : Re^{i\lambda},\ \lambda\in[0,\pi] \text{ and } x\in[-R,R]$$

then by Theorem 3.1.1, the following can be obtained.

$$\begin{aligned}
\int_C \frac{P(z)}{Q(z)}e^{i\lambda z}dz &= \int_{-R}^{R}\frac{P(x)}{Q(x)}e^{i\lambda x}dx + \int_{C_R}\frac{P(z)}{Q(z)}e^{i\lambda z}dz \\
&= 2\pi i\sum_{j=1}^{n}\underset{z=z_j}{Res}\left[\frac{P(z)}{Q(z)}e^{i\lambda z}\right]
\end{aligned}$$

By letting $f(z) = P(z)/Q(z)$ and using Theorem 3.2.1,

$$z\in C_R \to \lim_{R\to\infty}|f(z)| = \lim_{R\to\infty}\left|\frac{P(z)}{Q(z)}\right| = 0$$

$$\Longrightarrow \lim_{R\to\infty}\int_{C_R}\frac{P(z)}{Q(z)}e^{i\lambda z}dz = 0$$

$$\Longrightarrow \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{i\lambda x} dx = 2\pi i \sum_{j=1}^{n} \underset{z=z_j}{Res} \left[\frac{P(z)}{Q(z)} e^{i\lambda z} \right] \tag{3.2.6}$$

Taking the real part (Property 1.4.2) of Equation 3.2.6, Theorem 3.2.2 is obtained

THEOREM 3.2.2. *If the degree of $Q(x)$ exceeds the degree of $P(x)$ by at least one, then*

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos \lambda x \, dx = Re \left[2\pi i \sum_{j=1}^{n} \underset{z=z_j}{Res} \left[\frac{P(z)}{Q(z)} e^{i\lambda z} \right] \right] \tag{3.2.7}$$

And by taking the imaginary part (Property 1.4.2) of Equation 3.2.6, Theorem 3.2.3 is obtained

THEOREM 3.2.3.

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin \lambda x \, dx = Im \left[2\pi i \sum_{j=1}^{n} \underset{z=z_j}{Res} \left[\frac{P(z)}{Q(z)} e^{i\lambda z} \right] \right] \tag{3.2.8}$$

Beautiful and Simple! Now that there is a nice easy way to evaluate such integrals, the following problem shows how to evaluate a Fourier integral involving the sine function.

EXAMPLE 3.2.4. Evaluate

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 4x + 20} dx$$

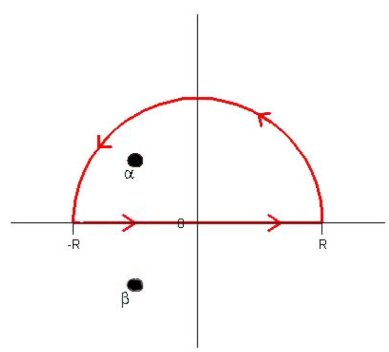


FIGURE 13

Let $f(z) = \dfrac{ze^{iz}}{z^2 + 4z + 20} = \dfrac{ze^{iz}}{(z-\alpha)(z-\beta)}$, where $\alpha = -2 + 4i$ and $\beta = -2 - 4i$ and let $C_R : \left\{ z = Re^{i\theta}, \theta \in [0, \pi] \right\}$ and $x \in [-R, R]$ (Figure 13)

NOTE 3.2.5. Lemma 3.2.1 can be used. $f(z) = (z/z^2 + 4z + 20)(e^{iz})$

$$\begin{aligned} \int_C f(z)dz &= \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz \\ &= Im[2\pi i \underset{z=\alpha}{Res} [\, f(z) \,]] \end{aligned}$$

Calculate the residue at $z = \alpha$:

$$\begin{aligned}
\underset{z=\alpha}{Res}\,[\,f(z)\,] &= \lim_{z\to\alpha}\left[(z-\alpha)\frac{ze^{iz}}{(z-\alpha)(z-\beta)}\right] \\
&= \frac{\alpha e^{i\alpha}}{\alpha-\beta} \\
&= \frac{(-2+4i)e^{i(-2+4i)}}{8i} \\
&= \frac{e^{2i}-2ie^{-2i}}{-4ie^4} \\
&= \frac{(\cos 2-2\sin 2)-i(\sin 2+2\cos 2)}{-4ie^4} \\
&= \frac{\cos 2-2\sin 2}{-4ie^4}-\frac{i(\sin 2+2\cos 2)}{-4ie^4} \\
&= \frac{(\sin 2+2\cos 2)}{4e^4}-\frac{\cos 2-2\sin 2}{-4ie^4}
\end{aligned}$$

So,

$$\underset{z=\alpha}{Res}\,[\,f(z)\,] = \frac{(2\sin 2-\cos 2)+i(\sin 2+2\cos 2)}{4ie^4}$$

and

$$\begin{aligned}
2\pi i\underset{z=\alpha}{Res}\,[\,f(z)\,] &= \frac{2\pi i(2\sin 2-\cos 2)+2\pi ii(\sin 2+2\cos 2)}{4ie^4} \\
&= \frac{\pi(2\sin 2-\cos 2)+\pi i(\sin 2+2\cos 2)}{2e^4}
\end{aligned}$$

Then by taking the imaginary part

$$\begin{aligned}
\operatorname{Im}\left[2\pi i\underset{z=\alpha}{Res}\,[\,f(z)\,]\right] &= \operatorname{Im}\left[\frac{\pi(2\sin 2-\cos 2)+i(\pi(\sin 2+2\cos 2))}{2e^4}\right] \\
&= \frac{\pi(\sin 2+2\cos 2)}{2e^4}
\end{aligned}$$

Now, as $R \to \infty$ and for $z \in C_R \longrightarrow \int_{C_R} f(z)dz \longrightarrow 0$ (Lemma 3.2.1)

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{x\sin x}{x^2+4x+20}dx+0 &= \frac{\pi(2\cos 2+\sin 2)}{2e^4} \\
&= \frac{\pi}{2e^4}(2\cos 2+\sin 2)
\end{aligned}$$

$$\int_{-\infty}^{\infty}\frac{x\sin x}{x^2+4x+20}dx = \frac{\pi}{2e^4}(2\cos 2+\sin 2) \tag{3.2.9}$$

Everything works out nicely! So, to summarize the process of evaluating an integral in the form

$$\int_{-\infty}^{\infty} f(x)\sin\lambda x dx \qquad \text{or} \qquad \int_{-\infty}^{\infty} f(x)\cos\lambda x dx,$$

where $f(x)$ is a rational function with no poles on the real axis.

(1) Let

$$\cos\lambda x = \sin\lambda x = e^{i\lambda z}$$

and make contour C be defined by a line $x \in [-R, R]$ (on real axis) and semicircular arc $C_R : Re^{i\lambda}$, $\lambda \in [0, \pi]$

(2) Let

$$\int_C F(z)e^{i\lambda z}dz = \int_{-R}^{R} F(x)dx + \int_{C_R} F(z)e^{i\lambda z}dz = 2\pi i \sum_{j=1}^{n} \underset{z=z_j}{Res}\,[\,F(z)\,]$$

(3) Find Residue(s) and let $R \longrightarrow \infty$

(4) Take the real part if the integral involved cosine or take the imaginary part if it involved sine

3.3. Definite Integrals of Functions of Sine and Cosine

Another common type of integral that can be difficult to evaluate is a definite proper integral involving rational functions of sine and cosine. Fortunately, the Residue Theorem can make it much easier. These integrals are typically in the form

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta$$

and can often be evaluated in a quick and easy manner.

By letting

$$z = e^{i\theta} \tag{3.3.1}$$

$$d\theta = \frac{dz}{iz}$$

$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right) \tag{3.3.2}$$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \tag{3.3.3}$$

The integral can be transformed into an integral along the unit circle

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta = \int_{|z|=1} F(z)\frac{dz}{iz} \tag{3.3.4}$$

The integral along the unit circle should be easy to evaluate using the Residue Theorem (Theorem 3.1.1). Example 3.3.1 is a concrete example that shows this very same process, along with how amazingly simple a difficult integral can be evaluated by using this technique.

Example 3.3.1.

$$\text{Evaluate } \int_0^{2\pi} \frac{d\phi}{a + \cos\phi} \text{ where } (a > 1)$$

Let $z = e^{i\phi} \rightarrow d\phi = \frac{dz}{iz}$, and , $\cos\phi = \frac{1}{2}(z + \frac{1}{z})$

$$\begin{aligned} \int_{|z|=1} \frac{1}{a + \frac{1}{2}(z + \frac{1}{z})} \frac{dz}{iz} &= \frac{2}{i} \int_{|z|=1} \frac{dz}{2az + z^2 + 1} \\ &= \frac{2}{i} \int_{|z|=1} \frac{dz}{(z-\alpha)(z-\beta)}, \end{aligned}$$

where $\alpha = -a + \sqrt{a^2 - 1}$ and $\beta = -a - \sqrt{a^2 - 1}$

Note 3.3.2. There is only one pole that lies within the unit circle, $z = \alpha$ (Figure 14)

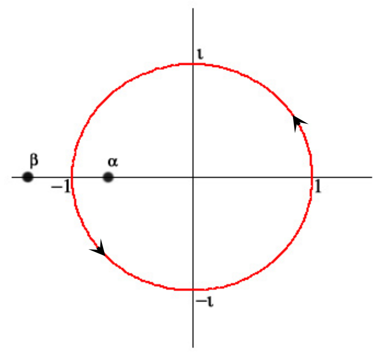


FIGURE 14

Calculate the residue at $z = -a + \sqrt{a^2-1}$:

$$\lim_{z\to(-a+\sqrt{a^2-1})}\left[\left[z-\left(-a+\sqrt{a^2-1}\right)\right]\frac{1}{(z-(-a+\sqrt{a^2-1}))(z-(-a-\sqrt{a^2-1}))}\right]$$
$$= \lim_{z\to(-a+\sqrt{a^2-1})}\left[\frac{1}{(z-(-a-\sqrt{a^2-1}))}\right]$$
$$= \frac{1}{-a+\sqrt{a^2-1}+a+\sqrt{a^2-1}}$$
$$= \frac{1}{2\sqrt{a^2-1}}$$

$$\operatorname*{Res}_{z=\alpha}\left[\frac{1}{2az+z^2+1}\right] = \frac{1}{2\sqrt{a^2-1}}$$

$$\int_0^{2\pi}\frac{d\phi}{a+\cos\phi} = 2\pi i\,\frac{2}{i}\,\frac{1}{2\sqrt{a^2-1}}$$
$$= \frac{2\pi}{\sqrt{a^2-1}} \tag{3.3.5}$$

3.4. Improper Integrals Involving Exponential Functions

THEOREM 3.4.1. *Let $f(x) = P(e^x)/Q(e^x)$ be rational function such that degree of $Q(x) = n$ and the degree of $P(x) = m$, where $n \geq m+2$, then*

$$\int_{-\infty}^{\infty}\frac{P(e^x)}{Q(e^x)}e^{ax}dx = \frac{2\pi i}{1-e^{2\pi ai}}\sum_{j=1}^{n}\operatorname*{Res}_{z=z_j}\left[\frac{P(e^z)}{Q(e^z)}e^{az}\right], \tag{3.4.1}$$

where $a = \alpha + i\beta \in \mathbb{C}$

PROOF. "Since the function $f(x) = P(e^x)/Q(e^x)$ is periodic of period $2\pi i$, that is, $f(x+2\pi i) =$ f(x) for all x, because $e^{x+2k\pi i} = e^x$, then it is convenient to choose a closed rectangular path, C" [**2**][246] (Figure 15), such that

$$-R \leq x \leq R, \qquad 0 \leq y = Im[z] \leq 2\pi$$

By Theorem 3.1.1,

$$\int_{-\infty}^{\infty}\frac{P(e^z)}{Q(e^z)}e^{az}dz = 2\pi i\sum_{j=1}^{n}\operatorname*{Res}_{z=z_j}\left[\frac{P(e^z)}{Q(e^z)}e^{az}\right],$$

and by Theorem 1.5.6,

$$\begin{aligned}\int_{\gamma_1} \frac{P(e^z)}{Q(e^z)} e^{az} dz + \int_{\gamma_2} \frac{P(e^z)}{Q(e^z)} e^{az} dz + \int_{\gamma_3} \frac{P(e^z)}{Q(e^z)} e^{az} dz \quad &+ \quad \int_{\gamma_4} \frac{P(e^z)}{Q(e^z)} e^{az} dz \\ &= \quad 2\pi i \sum_{j=1}^{n} \underset{z=z_j}{Res} \left[\frac{P(e^z)}{Q(e^z)} e^{az} \right]\end{aligned}$$

So as $R \longrightarrow \infty$

γ_1: $(z = x,\ x \in [-R, R])$

$$\int_{\gamma_3} \frac{P(e^z)}{Q(e^z)} e^{az} dz = \int_{-R}^{R} \frac{P(e^x)}{Q(e^x)} e^{ax} dx$$

$\Longrightarrow$

$$\int_{-R}^{R} \frac{P(e^x)}{Q(e^x)} e^{ax} dx \longrightarrow \int_{-\infty}^{\infty} \frac{P(e^x)}{Q(e^x)} e^{ax} dx$$

γ_2: $(z = R + iy,\ y \in [0, 2\pi])$

$$\begin{aligned}\left| \int_{\gamma_2} \frac{P(e^z)}{Q(e^z)} e^{az} dz \right| \quad &\leq \quad \int_0^{2\pi} \left| \frac{P(e^{R+iy})}{Q(e^{R+iy})} \right| \left| i e^{a(R+iy)} \right| dy \\ &\leq \quad \int_0^{2\pi} \left| \frac{P(e^{R+iy})}{Q(e^{R+iy})} \right| \left| i e^{(\alpha+i\beta)(R+iy)} \right| dy \\ &\leq \quad e^{-(n-m-\alpha)R} \int_0^{2\pi} i e^{-y\beta} dy \longrightarrow 0\end{aligned}$$

$$\Longrightarrow \int_{\gamma_2} \frac{P(e^z)}{Q(e^z)} e^{az} dz \longrightarrow 0$$

γ_3: $(z = x + 2\pi i,\ x \in [R, -R])$

$$\int_{\gamma_3} \frac{P(e^z)}{Q(e^z)} e^{az} dz = \int_{R}^{-R} \frac{P(e^{x+2\pi i})}{Q(e^{x+2\pi i})} e^{a(x+2\pi i)} dx$$

$\Longrightarrow$

$$\int_{R}^{-R} \frac{P(e^{x+2\pi i})}{Q(e^{x+2\pi i})} e^{a(x+2\pi i)} dx \longrightarrow e^{2\pi i a} \int_{\infty}^{-\infty} \frac{P(e^x)}{Q(e^x)} e^{ax} dx$$

γ_4: $(z = -R + iy,\ y \in [0, 2\pi])$ Then

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{P(e^x)}{Q(e^x)} e^{ax} dx + 0 - e^{2\pi i a} \int_{-\infty}^{\infty} \frac{P(e^x)}{Q(e^x)} e^{ax} dx \quad &+ \quad 0 \\ &= \quad 2\pi i \sum_{j=1}^{n} \underset{z=z_j}{Res} \left[\frac{P(e^z)}{Q(e^z)} e^{az} \right],\end{aligned}$$

which simplifies to

$$\int_{-\infty}^{\infty} \frac{P(e^x)}{Q(e^x)} e^{ax} dx = \frac{2\pi i}{1 - e^{2\pi i a}} \sum_{j=1}^{n} \underset{z=z_j}{Res} \left[\frac{P(e^z)}{Q(e^z)} e^{az} \right] \tag{3.4.2}$$

□

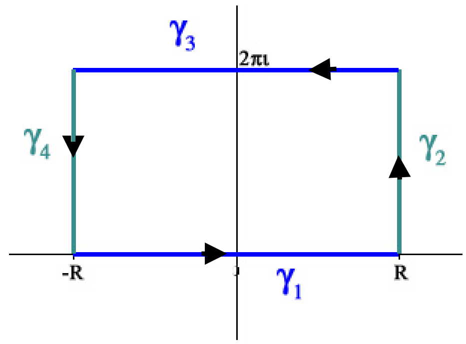


FIGURE 15

3.5. Integrals with Branch Points

Integrals with branch points can also be thought of as integrals involving multivalued functions. This section will cover different types of multivalued functions.

THEOREM 3.5.1 ([**5**]). *Let $P(x)$ and $Q(x)$ be polynomials of degree m and n, respectively, where $n \geq m+2$. If $Q(x) \neq 0$ for $x \geq 0$ and $Q(x)$ has a zero of order at most 1 at the origin, and $f(z) = z^\alpha\, P(z)/Q(z)$ where $0 \leq \alpha \leq 1$, then*

$$P.V. \int_0^\infty \frac{x^\alpha\, P(x)}{Q(x)} dx = \frac{2\pi i}{1 - e^{i\,\alpha\, 2\,\pi}} \sum_{j=1}^{n} \underset{z=z_j}{Res} \left[z^\alpha \frac{P(z)}{Q(z)} \right]$$

REMARK 3.5.2. Theorem 3.5.1 is also known as fractional powers or "rational functions times a power of x"[**2**]

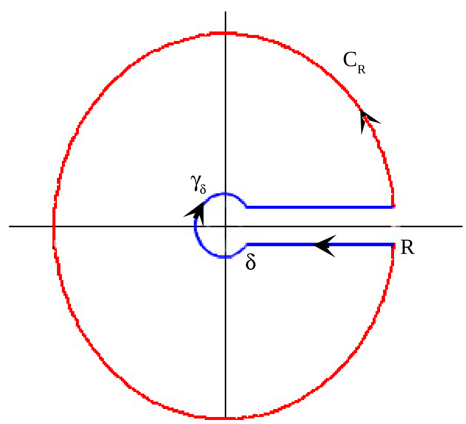


FIGURE 16. "Keyhole" Contour

These integrals can be evaluated along the "keyhole" contour (Figure 16) and by the Residue Theorem (Theorem 3.1.1)

PROOF.
(3.5.1)
$$\int_C f(z)dz = \int_\delta^R f(x)dx + \int_{C_R} f(z)dz + \int_R^\delta f(x)dx + \int_{\gamma_\delta} f(z)dz = 2\pi i \sum_{j=1}^{n} \underset{z=z_j}{Res}\,[\,f(z)\,]\,,$$

where

$$f(z) = z^{\alpha}[P(x)/Q(x)].$$

By letting $R \to \infty$ and $\delta \to 0$ implies

$$\int_{\delta}^{R} f(x)dx = \int_{0}^{\infty} f(x)dx,$$

$$\int_{R}^{\delta} f(x)dx = e^{2\pi\alpha i}\int_{\infty}^{0} f(x)dx,$$

$$\int_{C_R} f(z)dz = 0,$$

and

$$\int_{\gamma_\delta} f(z)dz = 0.$$

Then Equation 3.5.1 becomes

$$\int_{0}^{\infty} f(x)dx + 0 - e^{2\pi\alpha i}\int_{0}^{\infty} f(x)dx + 0 = 2\pi i\sum_{j=1}^{n} \underset{z=z_j}{Res}\,[\,f(z)\,],$$

and therefore

$$\int_{0}^{\infty} \frac{x^{\alpha}\,P(x)}{Q(x)}dx = \frac{2\pi i}{1-e^{i\,\alpha\,2\,\pi}}\sum_{j=1}^{n} \underset{z=z_j}{Res}\left[z^{\alpha}\frac{P(z)}{Q(z)}\right] \tag{3.5.2}$$

Remark 3.5.3. On line segment $[R, \delta]$, $z = xe^{2\pi i} \to z^{\alpha} = e^{2\pi\alpha i}x^{\alpha}$

□

Example 3.5.4. Prove that

$$\int_{0}^{\infty} \frac{x^{a-1}}{1+x}dx = \frac{\pi}{\sin(a\pi)}, (0 < a < 1)$$

$$\text{Let } F(z) = \frac{z^{a-1}}{1+z} = \frac{z^a}{z(z+1)}$$

Calculate the residue at $z = -1$

$$\begin{aligned}
\underset{z=-1}{Res}\,[\,F(z)\,] &= \lim_{z\to -1}\left[(z+1)\frac{z^a}{z(z+1)}\right] \\
&= \frac{(-1)^a}{-1} \\
&= \frac{(e^{i\pi})^a}{-1} \qquad \text{(Use of Property 1.4.1)} \\
&= \frac{e^{ia\pi}}{-1}
\end{aligned}$$

By Theorem 3.5.1

$$\begin{aligned}
\int_0^\infty \frac{x^{a-1}}{1+x}dx &= \frac{2\pi i}{1-e^{ia2\pi}}\left(-e^{ia\pi}\right) \\
&= \frac{-2\pi i e^{ia\pi}}{1-e^{ia2\pi}} \\
&= \frac{-2\pi i}{\frac{1}{e^{ia\pi}} - \frac{e^{ia2\pi}}{e^{ia\pi}}} \\
&= \frac{2\pi i}{e^{ia\pi} - e^{-ia\pi}} \\
&= \pi \frac{2i}{e^{ia\pi} - e^{-ia\pi}} \\
&= \pi \frac{1}{\sin a\pi} \qquad \text{(Use of Equation 1.4.7)}
\end{aligned}$$

$\Longrightarrow$

$$\int_0^\infty \frac{x^{a-1}}{1+x}dx = \frac{\pi}{\sin a\pi} \tag{3.5.3}$$

The following theorem considers $\int_0^\infty P(x)/Q(x)dx$ where $P(x)/Q(x)$ is not an even function. These can be evaluated directly in some cases, but it would be wise to use the Residue Theorem because it makes life much easier and more enjoyable. The derivation of the general formula to evaluate such integrals is shown in the proof of Theorem 3.5.5. The idea is to simply let $f(z) = [\, P(z) \,/\, Q(z) \,]\, Ln(z)$ and then use the "keyhole" contour as shown in Figure 16, and evaluate using our favorite theorem (Theorem 3.1.1).

THEOREM 3.5.5. *Let $P(x)/Q(x)$ be a rational function where the* $\deg(\,P(x)\,) = m$ *and the* $\deg(\,Q(x)\,) = n$ *such that $n \geq m+2$, then*

$$\int_0^\infty \frac{P(x)}{Q(x)}dx = -\sum_{j=1}^{n} \underset{z=z_j}{Res}\left[\frac{P(z)}{Q(z)}Ln(z)\right] \tag{3.5.4}$$

PROOF. Let

$$f(z) = \frac{P(z)}{Q(z)}Ln(z),$$

and applying Theorem 3.1.1 gives

$$\begin{aligned}
\int_C f(z)dz &= \int_{C_R} f(z)dz + \int_R^\delta f(z)dz + \int_{\gamma_\delta} f(z)dz + \int_\delta^R f(z)dz \\
&= 2\pi i \sum_{j=1}^{n} \underset{z=z_j}{Res}\left[\frac{P(z)}{Q(z)}Ln(z)\right]
\end{aligned}$$

As $R \longrightarrow \infty$ and $\delta \longrightarrow 0$

C_R: $\left(z \in C_R \Rightarrow z = Re^{i\theta}, \theta \in [0, 2\pi]\right)$

$$\begin{aligned}
\left|\int_{C_R} f(z)dz\right| \leq \left(\int_0^{2\pi} f(Re^{i\theta})Re^{i\theta}i\right) &\longrightarrow 0 \\
\Longrightarrow \int_{C_R} f(z)dz &\longrightarrow 0
\end{aligned}$$

γ_δ: $\left(z \in \gamma_\delta \Rightarrow z = \delta e^{i\theta}, \theta \in [2\pi, 0]\right)$

$$\begin{aligned}
\left|\int_{\gamma_\delta} f(z)dz\right| \leq \left(\int_{2\pi}^{0} f(\delta e^{i\theta})\delta e^{i\theta}i\right) &\longrightarrow 0 \\
\Longrightarrow \int_{\gamma_\delta} f(z)dz &\longrightarrow 0
\end{aligned}$$

$[\delta, R]$: $\left(z = xe^{0i}\right)$

$$\begin{aligned} \int_\delta^R f(z)dz &= \int_\delta^R \frac{P(x)}{Q(x)} Ln(x)dx \\ &= \int_\delta^R \frac{P(x)}{Q(x)} ln(x) + 0dx \\ &= \int_0^\infty \frac{P(x)}{Q(x)} ln(x)dx \end{aligned}$$

$[R, \delta]$: $\left(z = xe^{2\pi i}\right)$

$$\begin{aligned} \int_R^\delta f(z)dz &= \int_R^\delta \frac{P(x)}{Q(x)} Ln(xe^{2\pi i})dx \\ &= \int_R^\delta \frac{P(x)}{Q(x)} (ln(x) + 2\pi i + 0)dx \\ &= \int_\infty^0 \frac{P(x)}{Q(x)} (ln(x) + 2\pi i)dx \\ &= -\int_0^\infty \frac{P(x)}{Q(x)} (ln(x) + 2\pi i)dx \end{aligned}$$

Which implies

$$0 + \left(-\int_0^\infty \frac{P(x)}{Q(x)} \left(ln(x) + 2\pi i\right) dx\right) + \int_0^\infty \frac{P(x)}{Q(x)} ln(x)dx = 2\pi i \sum_{j=1}^n \underset{z=z_j}{Res} \left[\frac{P(z)}{Q(z)} Ln(z)\right]$$

$$-\int_0^\infty \frac{P(x)}{Q(x)} ln(x)dx - \int_0^\infty \frac{P(x)}{Q(x)} 2\pi i dx + \int_0^\infty \frac{P(x)}{Q(x)} ln(x)dx = 2\pi i \sum_{j=1}^n \underset{z=z_j}{Res} \left[\frac{P(z)}{Q(z)} Ln(z)\right]$$

$$\begin{aligned} -\int_0^\infty \frac{P(x)}{Q(x)} 2\pi i dx &= 2\pi i \sum_{j=1}^n \underset{z=z_j}{Res} \left[\frac{P(z)}{Q(z)} Ln(z)\right] \\ -2\pi i \int_0^\infty \frac{P(x)}{Q(x)} dx &= 2\pi i \sum_{j=1}^n \underset{z=z_j}{Res} \left[\frac{P(z)}{Q(z)} Ln(z)\right] \end{aligned}$$

and hence,

$$\int_0^\infty \frac{P(x)}{Q(x)} dx = -\sum_{j=1}^n \underset{z=z_j}{Res} \left[\frac{P(z)}{Q(z)} Ln(z)\right] \tag{3.5.5}$$

□

Equation 3.5.5 is a general strategy to evaluate a non-even rational function where the degree of the denominator is greater than the numerator by at least 1. To take it up a notch, consider the same indefinite integral (Equation 3.5.4) with $ln(x)$ in the numerator

$$\int_0^\infty \frac{P(x)}{Q(x)} \ln(x)dx,$$

which can also be evaluated by using the Residue theorem.

Theorem 3.5.6. *Let $P(x)/Q(x)$ be a rational function such that the* $\deg(P(x)) = m$ *and the* $\deg(Q(x)) = n$ *where $n \geq m + 2$, then*

$$\int_0^\infty \frac{P(x)}{Q(x)} \ln(x)dx = -\frac{1}{2} \sum_{j=1}^n \underset{z=z_j}{Res} \left[\frac{P(z)}{Q(z)} (Ln(z))^2\right] + i\pi \sum_{j=1}^n \underset{z=z_j}{Res} \left[\frac{P(z)}{Q(z)} Ln(z)\right] \tag{3.5.6}$$

PROOF. Let

$$\begin{aligned} f(z) &= \left(\frac{P(z)}{Q(z)}Ln(z)\right)Ln(z) \\ &= \frac{P(z)}{Q(z)}(Ln(z))^2 \end{aligned}$$

(Theorem 3.1.1)

$$\begin{aligned} \int_C f(z)dz &= \int_{C_R} f(z)dz + \int_R^\delta f(z)dz + \int_{\gamma_\delta} f(z)dz + \int_\delta^R f(z)dz \\ &= 2\pi i \sum_{j=1}^{n} \underset{z=z_j}{Res}\left[\frac{P(z)}{Q(z)}(Ln(z))^2\right] \end{aligned}$$

As $R \longrightarrow \infty$ and $\delta \longrightarrow 0$

C_R: $\left(z \in C_R \Rightarrow z = Re^{i\theta}, \theta \in [0, 2\pi]\right)$

$$\int_{C_R} f(z)dz \to 0$$

γ_δ: $\left(z \in \gamma_\delta \Rightarrow z = \delta e^{i\theta}, \theta \in [2\pi, 0]\right)$

$$\int_{\gamma_\delta} f(z)dz \to 0$$

$[\delta, R]$: $(z = xe^{0i})$

$$\begin{aligned} \int_\delta^R f(z)dz &= \int_\delta^R \frac{P(x)}{Q(x)}(Ln(x))^2 dx \\ &= \int_\delta^R \frac{P(x)}{Q(x)}(\ln(x))^2 dx \\ &= \int_0^\infty \frac{P(x)}{Q(x)}(\ln(x))^2 dx \end{aligned}$$

$[R, \delta]$: $(z = xe^{2\pi i})$

$$\begin{aligned} \int_R^\delta f(z)dz &= \int_R^\delta \frac{P(x)}{Q(x)}(Ln(xe^{2\pi i}))^2 dx \\ &= \int_R^\delta \frac{P(x)}{Q(x)}(\ln(x) + 2\pi i + 0)^2 dx \\ &= \int_\infty^0 \frac{P(x)}{Q(x)}\left[(\ln(x))^2 + 4\pi i \ln(z) + (2\pi i)^2\right] \\ &= -\int_0^\infty \frac{P(x)}{Q(x)}\left[(\ln(x))^2 + 4\pi i \ln(z) - 4\pi^2\right] \end{aligned}$$

Then

$$\begin{aligned} & \int_0^\infty \frac{P(x)}{Q(x)}(\ln(x))^2 dx - \int_0^\infty \frac{P(x)}{Q(x)}\left[(\ln(x))^2 + 4\pi i \ln(z) - 4\pi^2 dx\right] \\ = & \int_0^\infty \frac{P(x)}{Q(x)}(\ln(x))^2 dx - \int_0^\infty \frac{P(x)}{Q(x)}(\ln(x))^2 dx - 4\pi i \int_0^\infty \frac{P(x)}{Q(x)} \ln(z) dx + 4\pi^2 \int_0^\infty \frac{P(x)}{Q(x)} dx \\ = & -4\pi i \int_0^\infty \frac{P(x)}{Q(x)} \ln(z) dx + 4\pi^2 \int_0^\infty \frac{P(x)}{Q(x)} dx \end{aligned}$$

so

$$-4\pi i\int_0^{\infty}\frac{P(x)}{Q(x)}\ln(z)dx+4\pi^2\int_0^{\infty}\frac{P(x)}{Q(x)}dx=2\pi i\sum_{j=1}^{n}\mathop{Res}_{z=z_j}\left[\frac{P(z)}{Q(z)}(Ln(z))^2\right] \tag{3.5.7}$$

Now apply Theorem 3.5.5 to Equation 3.5.7 yields

(3.5.8)

$$-4\pi i\int_0^{\infty}\frac{P(x)}{Q(x)}\ln(z)dx-4\pi^2\sum_{j=1}^{n}\mathop{Res}_{z=z_j}\left[\frac{P(z)}{Q(z)}Ln(z)\right]=2\pi i\sum_{j=1}^{n}\mathop{Res}_{z=z_j}\left[\frac{P(z)}{Q(z)}(Ln(z))^2\right]$$

$\Longleftrightarrow$

$$-4\pi i\int_0^{\infty}\frac{P(x)}{Q(x)}\ln(z)dx=2\pi i\sum_{j=1}^{n}\mathop{Res}_{z=z_j}\left[\frac{P(z)}{Q(z)}(Ln(z))^2\right]+4\pi^2\sum_{j=1}^{n}\mathop{Res}_{z=z_j}\left[\frac{P(z)}{Q(z)}Ln(z)\right]$$

$\Longleftrightarrow$

$$\int_0^{\infty}\frac{P(x)}{Q(x)}\ln(z)dx=-\frac{1}{2}\sum_{j=1}^{n}\mathop{Res}_{z=z_j}\left[\frac{P(z)}{Q(z)}(Ln(z))^2\right]-\frac{\pi}{i}\sum_{j=1}^{n}\mathop{Res}_{z=z_j}\left[\frac{P(z)}{Q(z)}Ln(z)\right]$$

$\Longleftrightarrow$

$$\begin{aligned}&\int_0^{\infty}\frac{P(x)}{Q(x)}\ln(z)dx\\ &= -\frac{1}{2}\sum_{j=1}^{n}\mathop{Res}_{z=z_j}\left[\frac{P(z)}{Q(z)}(Ln(z))^2\right]+\pi i\sum_{j=1}^{n}\mathop{Res}_{z=z_j}\left[\frac{P(z)}{Q(z)}Ln(z)\right]\end{aligned} \tag{3.5.9}$$

□

Furthermore by dividing Equation 3.5.8 by $4\pi i$ and taking the Real parts the following can be obtained.

$$\int_0^{\infty}\frac{P(x)}{Q(x)}\ln(x)dx=-\frac{1}{2}Re\left[\sum_{j=1}^{n}\mathop{Res}_{z=z_j}\left[\frac{P(z)}{Q(z)}(Ln(z))^2\right]\right] \tag{3.5.10}$$

Example 3.5.7. Prove that

$$\int_0^{\infty}\frac{\ln(x)}{(x+a)^2+b^2}dx=\frac{\ln(\sqrt{a^2+b^2})}{b}\arctan(\frac{b}{a})$$

Let

$$f(z)=Ln(z)\frac{1}{(x+a)^2+b^2}\quad\text{and}\quad F(z)=\frac{(Ln(z))^2}{(z+a-ib)(z+a+ib)}$$

$f(z)$ has simple poles at $z=-a\pm ib$, (using Equation 3.5.10)
Residue at $z=-a+ib$:

$$\begin{aligned}\mathop{Res}_{z=-a+ib}[F(z)] &= \lim_{z\to -a+ib}\left[(z-(-a+ib))\frac{(Ln(z))^2}{(z+a-ib)(z+a+ib)}\right]\\ &= \lim_{z\to -a+ib}\left[\frac{[Ln(z)]^2}{(z+a+ib)}\right]\\ &= \frac{[Ln(-a+ib)]^2}{2ib}\end{aligned} \tag{3.5.11}$$

Residue at $z = -a - ib$:

$$\begin{aligned} \underset{z=-a-ib}{Res}\,[\,F(z)\,] &= \lim_{z\to -a-ib}\left[(z-(-a-ib))\frac{(Ln(z))^2}{(z-a-ib)(z+a+ib)}\right] \\ &= \lim_{z\to -a-ib}\left[\frac{[Ln(z)]^2}{(z-a-ib)}\right] \\ &= \frac{[Ln(-a-ib)]^2}{-2ib} \qquad (3.5.12) \end{aligned}$$

By Equation 3.5.10

$$\int_0^\infty \frac{\ln(x)}{(x+a)^2+b^2}dx = -\frac{1}{2}Re\left[\underset{z=-a+ib}{Res}\,[\,F(z)\,] + \underset{z=-a-ib}{Res}\,[\,F(z)\,]\right] \qquad (3.5.13)$$

$$\begin{aligned} \underset{z=-a-ib}{Res}\,[\,F(z)\,] + \underset{z=-a+ib}{Res}\,[\,F(z)\,] &= \frac{[Ln(-a+ib)]^2}{2ib} + \frac{[Ln(-a-ib)]^2}{-2ib} \\ &= \frac{[Ln(-a+ib)]^2 - [Ln(-a-ib)]^2}{2ib} \end{aligned}$$

(Equation 1.4.12)

$$\begin{aligned} &= \frac{\left[\ln(\sqrt{a^2+b^2}) + i\arctan(-\frac{b}{a})\right]^2 - \left[\ln(\sqrt{a^2+b^2}) + i\arctan(\frac{b}{a})\right]^2}{2ib} \\ &= \frac{[\ln(\sqrt{a^2+b^2})]^2 + 2i\ln(\sqrt{a^2+b^2})\arctan(-\frac{b}{a}) - [\arctan(-\frac{b}{a})]^2}{2ib} \\ &\quad - \frac{[\ln(\sqrt{a^2+b^2})]^2 - 2i\ln(\sqrt{a^2+b^2})\arctan(\frac{b}{a}) + [\arctan(\frac{b}{a})]^2}{2ib} \\ &= \left[\arctan(-\frac{b}{a}) - \arctan(\frac{b}{a})\right]\frac{\ln(\sqrt{a^2+b^2})}{b} + \frac{1}{i}\left[\frac{[\arctan(-\frac{b}{a})]^2 - [\arctan(\frac{b}{a})]^2}{2b}\right] \\ &= \left[-2\arctan(\frac{b}{a})\right]\frac{\ln(\sqrt{a^2+b^2})}{b} + \frac{1}{i}\left[\frac{[\arctan(-\frac{b}{a})]^2 - [\arctan(\frac{b}{a})]^2}{2b}\right] \qquad (3.5.14) \end{aligned}$$

Substituting Equation 3.5.14 back into Equation 3.5.13

$$\begin{aligned} \int_0^\infty \frac{\ln(x)}{(x+a)^2+b^2}dx &= -\frac{1}{2}Re\left[\frac{-2\arctan(\frac{b}{a})\ln(\sqrt{a^2+b^2})}{b} + \frac{1}{i}\left(\frac{[\arctan(-\frac{b}{a})]^2 - [\arctan(\frac{b}{a})]^2}{2b}\right)\right] \\ &= \frac{1}{2}Re\left[\frac{2\arctan(\frac{b}{a})\ln(\sqrt{a^2+b^2})}{b} + i\left(\frac{[\arctan(-\frac{b}{a})]^2 - [\arctan(\frac{b}{a})]^2}{2b}\right)\right] \\ &= \frac{1}{2}\left[\frac{2\arctan(\frac{b}{a})\ln(\sqrt{a^2+b^2})}{b}\right] \\ &= \frac{\arctan(\frac{b}{a})\ln(\sqrt{a^2+b^2}\,)}{b} \end{aligned}$$

$$\int_0^\infty \frac{\ln(x)}{(x+a)^2+b^2}dx = \frac{\arctan(\frac{b}{a})\ln(\sqrt{a^2+b^2})}{b} \qquad (3.5.15)$$

Conclusions

This capstone has focused on how to use complex analysis to evaluate various definite integrals in the real plane. Not only does this make computing certain integrals easier, but it also allows us to evaluate integrals of functions whose anti-derivative is unknown or impossible to find. Aside from evaluating integrals in the real plane, the amazing result of the Residue Theorem is the ability to evaluate contour integrals such that non-analytic points lie inside the closed contour.

The Residue Theorems included in this capstone are a small sample of all the Residue Theorems. The basic idea behind each Residue Theorem is the same, but each theorem holds its own power and beauty. Without the brilliant minds that contributed to the Residue theorems, the world would not be where it is today.

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