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L'HOSPITAL'S RULE FOR COMPLEX-VALUED FUNCTIONS

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L'Hospital's rule for real functions may be stated in the form:

Let $f(x)$ and $g(x)$, and their derivatives $f'(x)$, $g'(x)$, be continuous on an open interval $(0, a)$, and let

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0.$$

*If the ratio f'/g' is defined on $(0, a)$ (in the sense that g' does not vanish) and has a finite limit at $x=0$, then the ratio of functions, f/g, is defined and has the same limit.**

A simple example shows that this rule is not generally valid when f and g are complex-valued functions of a real variable. Taking $f=x$, $g=xe^{-i/x}$, we have

$$f'/g' = \frac{xe^{i/x}}{x+i},$$

which vanishes as $x\to 0$, while $f/g=e^{i/x}$ has no limit. The more pathological example $f=x$, $g=x(e^{-i/x}-1)$ shows that f/g need not even be defined.

However, by placing additional restrictions on the functions, we can obtain generalizations of the rule which cover many cases. For example, the rule is valid provided

I. *The ratio $|g'|/|g|'$ is defined (in the sense that the derivative $|g|'$ of $|g|$ exists and does not vanish) and bounded on $(0, a)$.*

This is the simplest of a class of conditions obtained by introducing the difference, $\delta(x)$, between f'/g' and its limit

$$L = \lim_{x\to 0} f'(x)/g'(x).$$

Multiplying the equation $\delta(x)=f'(x)/g'(x)-L$ by $g'(x)$ and integrating, we have

$$f(x) - Lg(x) = \int_0^x \delta(y)g'(y)dy,$$

in which the integral may be improper. On dividing by $g(x)$, it is clear that f/g has the limit L provided

* This rule is equally valid if f'/g' has an infinite limit at $x=0$, since the conditions on f and g are the same. This is not true of the conditions derived below for complex-valued functions. Here the roles of f and g must be interchanged explicitly to cover cases of infinite limits.

II. *The ratio*

$$R(x) \equiv \left[\int_0^x \delta(y) g'(y) dy\right] \Big/ g(x)$$

is defined on (0, a) *and vanishes as* $x \to 0$. We will show that each of the two following conditions, as well as I, implies Condition II and is therefore sufficient for validity:

III. *The ratio* $r(x) \equiv |\delta(x)||g'(x)|/|g(x)|'$ *is defined on* (0, a) *and vanishes as* $x \to 0$, *where* $|g|'$ *is the derivative of* $|g|$.

IV. $|g(x)|$ *is monotone and the real and imaginary parts of* $\delta(x)$ *are of bounded variation on* (0, a).

The first step in the proof is to show that g cannot vanish on (0, a). For Conditions I and III this follows immediately from Rolle's theorem and the fact that $|g|$ is continuous and vanishes at $x=0$, while $|g|'$ is defined and does not vanish. For Condition IV it follows from the monotonicity of $|g|$: If $g(\xi)=0$, $0<\xi<a$, then g vanishes identically on (0, ξ), which contradicts the hypothesis that $g' \neq 0$.

To show that III implies II, we note that $|g|'$ is given by $|g|' = (g\bar{g}' + g'\bar{g})/2|g|$, where the complex conjugate $\bar{g}$ enjoys the same continuity properties as g. Hence $|g|'$ is continuous; and if we define $r(0)=0$, it follows that $r(x)$ is continuous on the closed interval $[0, a/2]$. Let ρ be an upper bound for r on that interval. Then

$$\int_\epsilon^x |\delta(y)| \, |g'(y)| \, dy \leqq \rho \int_\epsilon^x |g(y)|' dy < \rho |g(x)|,$$

and on taking the limit $\epsilon \to 0$ we see that

$$\int_0^x |\delta(y)| \, |g'(y)| \, dy$$

is defined, continuous, and has the continuous derivative $|\delta||g'|$ on (0, $a/2$). Hence the ratio

$$R_1(x) = \left[\int_0^x |\delta(y)| \, |g'(y)| \, dy\right] \Big/ |g(x)|$$

satisfies the conditions of l'Hospital's rule for real functions. Applying the rule to R_1, we find that Condition II is satisfied, since $|R| \leqq |R_1|$.

Condition I clearly implies III, since $\delta(x)$ vanishes as $x \to 0$. Hence I implies II.

To show that IV implies II, we integrate the numerator of $R(x)$ by parts and use the monotonicity of $|g|$ to obtain the estimate

$$|R(x)| \leqq |\delta(x)| + \int_0^x |d\delta(y)|.$$

Hence R vanishes with x and II is satisfied.

Similar arguments show that Conditions I, III, and IV are also valid for the Stolz extension of l'Hospital's rule, in which the conditions

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$$

are replaced by

$$\lim_{x\to 0} |f(x)| = \lim_{x\to 0} |g(x)| = \infty.$$

ON A DETERMINANTAL INEQUALITY

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Recently L. K. Hua [2] proved the following interesting inequality:

Let A and B be n-square complex matrices and assume

$$I - A^*A \quad \text{and} \quad I - B^*B \tag{1}$$

are both positive semidefinite. Then

$$|d(I - A^*B)|^2 \geqq d(I - A^*A)d(I - B^*B) \tag{2}$$

where d is the determinant and A^* is the conjugate transpose of A.

We prove here an extension of the inequality (2). Let λ_j, α_j, β_j be respectively the eigenvalues of $I-A^*B$, A^*A and B^*B so indexed that

$$|\lambda_j| \geqq |\lambda_{j+1}|, \quad \alpha_j \geqq \alpha_{j+1}, \quad \beta_j \geqq \beta_{j+1} \quad \text{for } j = 1, \cdots, n-1.$$

THEOREM. *If $I-A^*A$ and $I-B^*B$ are both positive semidefinite then for each k satisfying $1 \leqq k \leqq n$,*

$$\prod_{j=1}^{k} |\lambda_{n-j+1}|^2 \geqq \prod_{j=1}^{k} (1 - \alpha_j)(1 - \beta_j). \tag{3}$$

We first establish an inequality.

LEMMA. *If u and v are complex n vectors and*

$$\|u + v\| \leqq 2 \tag{4}$$

then

$$|1 - (u, v)|^2 \geqq (1 - \|u\|^2)(1 - \|v\|^2). \tag{5}$$

* This work was completed under a National Research Council-National Bureau of Standards Postdoctoral Research Associateship. The author also wishes to acknowledge the helpful suggestions of Dr. Olga Taussky-Todd.