Section 3.5

IMPLICIT DIFFERENTIATION

If $f = \{(x, y) : y = 3x^3 - 5x^2 + x - 7\}$, then the equation

$$y = 3x^3 - 5x^2 + x - 7$$

defines the function f explicitly. However, not all functions are defined by such an equation. For example, if we have the equation:

$$3x^3 - x = 3y^3 + y^2 - 5y^7 + 9 \qquad (1)$$

One cannot solve for y in terms of x; however, Equation (1) may define one or more functions f such that if $y = f(x)$, Equation (1) is satisfied. That is

$$3x^3 - x = 3\big(f(x)\big)^3 + \big(f(x)\big)^2 - 5\big(f(x)\big)^7 + 9$$

holds for all x in the domain of f. In this case, we say that the function f is defined by equation (1) ***implicitly***.

If we assume that equation (1) defines at least one differentiable function y in terms of x, then the derivative of y with respect to x can be found by ***implicit differentiation***.

Since equation (1) can be written so that all the terms involving x are on one side of the equation and all the terms involving y are on the other side of the equation, such an equation is a special type of equations involving x and y. The left side of equation (1) is a function of x, and the right side is a function of y. Let F be the function defied by the left side, and G be the function defied by the right side. Thus,

$$F(x) = 3x^3 - x \qquad\qquad G(x) = 3y^3 + y^2 - 5y^7 + 9$$

where y is a function of x, say $y = f(x)$. Thus, equation (1) can be written as

$$F(x) = G(x)$$

This equation is satisfied by all values of x in the domain of f such that $G(f(x))$ exists. Therefore, for all values of x such that f is differentiable,

$$\frac{d}{dx}\big(3x^3 - x\big) = \frac{d}{dx}\big(3y^3 + y^2 - 5y^7 + 9\big) \qquad (2)$$

The derivative on the left side equation (2) is easily be found, and

$$\frac{d}{dx}(3x^3 - x) = 9x^2 - 1 \tag{3}$$

while the derivative on the right side of equation (2) is found by the chain rule.

$$\frac{d}{dx}\left(3y^3 + y^2 - 5y^7 + 9\right) = 9y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 35y^6\frac{dy}{dx} \tag{4}$$

Substituting the obtained values in (3) and (4) into (2), we get

$$9x^2 - 1 = \left(9y^2 + 2y - 35y^6\right)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{9x^2 - 1}{9y^2 + 2y - 35y^6}$$

As we can see from the last result we were able with the help of chain rule to find $\frac{dy}{dx}$ as an equation involves both variables x and y.

The above example leads us to the following definition

DEFINITION 3.5.1

The equation $F(x,y) = 0$ is said to define an explicit function $y = f(x)$ if the equation $F\left(x, f(x)\right) = 0$ is satisfied for every x in the domain of f.

The next illustration will clarify what we have been doing.

ILLUSTRATION

The equation

$$(x-2)^2 + y^2 - 9 = 0 \tag{5}$$

defines two functions $y_1 = \sqrt{9-(x-2)^2}$ and $y_2 = -\sqrt{9-(x-2)^2}$, see Figure 3.5.1.

Actually it defines more than two functions. Now, we show that y_1 and y_2 satisfy equation (5).

Substitute y_1 in (5) we obtain

$$\begin{aligned}(x-2)^2 + {y_1}^2 - 9 &= (x-2)^2 + \left(\sqrt{9-(x-2)^2}\right)^2 - 9 = 0 \\ &= (x-2)^2 + 9 - (x-2)^2 - 9 = 0\end{aligned}$$

Similarly, for y_2 .

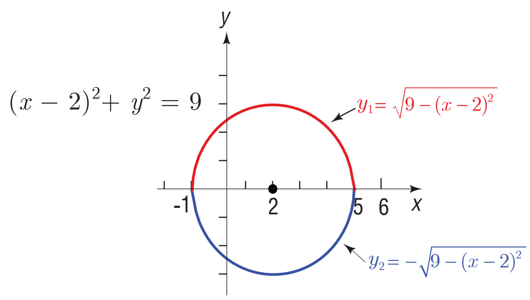

Figure 3.5.1

REMARK

It is out of the scope of this book to discuss under what conditions an equation defines a differentiable implicit function. Instead we will assume that in all examples and exercises.

Regarding differentiation of a given function, when it is explicitly expressed in the form $y = f(x)$, we apply the rules of differentiation directly to it and find its derivative as we have done in the previous sections. On the other hand, when the function is implicitly expressed, we use the method of implicit differentiation which is our goal in this section. The idea of implicit differentiation is based on the chain rule and it works as follows:

Guidelines for Implicit Differentiation

Let y be a function of x which is expressed implicitly in terms of an equation of the form $F(x,y)=0$. To find the derivative $\frac{dy}{dx}$, we proceed as follows:

- Differentiate both sides of the equation with respect to x.
- Write the terms involving y' (or dy/dx) on one side of the equation and the terms that do not involve $\frac{dy}{dx}$ on the other side of the equation, then solve for $\frac{dy}{dx}$.

Now, let us consider some examples to illustrate the method of implicit differentiation.

EXAMPLE 3.5.1 If each of the following equations determines an implicit differentiable function $y=f(x)$, find $\frac{dy}{dx}$.

a. $x^3+y^3=1+xy$

b. $y^2=x\cos y$

c. $xy^{2/3}+yx^{2/3}=x^2$

d. $\sqrt{3+\tan(xy)}-2=0$

Solution

a. Differentiating both sides of $x^3 + y^3 = 1 + xy$ with respect to x, regarding y as a function of x

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}\left(1 + xy\right)$$

$$\frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(y^3\right) = \frac{d}{dx}\left(1\right) + \frac{d}{dx}\left(xy\right)$$

Using the chain rule on the y^3 term and the product rule on the xy term, we obtain

$$3x^2 + 3y^2y' = xy' + y$$

Solving for y' yields

$$3y^2y' - xy' = y - 3x^2$$

$$\left(3y^2 - x\right)y' = y - 3x^2$$

$$y' = \frac{y - 3x^2}{3y^2 - x}$$

b. Differentiating both sides of the equation $y^2 = x\cos y$ with respect to x and using the product rule, we obtain

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x\cos y\right)$$

$$\frac{d}{dx}\left(y^2\right) = x\frac{d}{dx}\left(\cos y\right) + \cos y\frac{d}{dx}\left(x\right)$$

$$2yy' = x\left(-\sin y \cdot y'\right) + \cos y\left(1\right)$$

$$2yy' = -x\sin y \cdot y' + \cos y$$

Solving for y' yields

$$2yy' + x\sin yy' = \cos y$$

$$\left(2y + x\sin y\right)y' = \cos y$$

$$y' = \frac{\cos y}{2y + x\sin y}$$

c. Differentiating both sides of the equation $xy^{2/3} + yx^{2/3} = x^2$ with respect to x, we obtain

$$\frac{d}{dx}\left(xy^{2/3} + yx^{2/3}\right) = \frac{d}{dx}\left(x^2\right)$$

Apply product rule, we have

$$x\frac{d}{dx}\left(y^{2/3}\right) + y^{2/3}\frac{d}{dx}\left(x\right) + y\frac{d}{dx}\left(x^{2/3}\right) + x^{2/3}\frac{d}{dx}\left(y\right) = 2x$$

$$x\left(\frac{2}{3}y^{-1/3}y'\right) + y^{2/3}\left(1\right) + y\left(\frac{2}{3}x^{-1/3}\right) + x^{2/3}\left(y'\right) = 2x$$

$$\frac{2}{3}xy^{-1/3}y' + y^{2/3} + \frac{2}{3}yx^{-1/3} + x^{2/3}y' = 2x$$

Solving for y' yields

$$\left(\frac{2}{3}xy^{-1/3} + x^{2/3}\right)y' = 2x - \frac{2}{3}yx^{-1/3} - y^{2/3}$$

$$y' = \frac{2x - \frac{2}{3}yx^{-1/3} - y^{2/3}}{\frac{2}{3}xy^{-1/3} + x^{2/3}}$$

d. Differentiating both sides of the equation $\sqrt{3+\tan(xy)} - 2 = 0$ with respect to x, we obtain

$$\frac{d}{dx}\left(\sqrt{3+\tan(xy)} - 2\right) = \frac{d}{dx}(0)$$

Rewrite $\sqrt{3+\tan(xy)}$ as $\left(3+\tan(xy)\right)^{1/2}$, we obtain

$$\frac{d}{dx}\left(\left(3+\tan(xy)\right)^{1/2} - 2\right) = 0$$

Applying difference rule, we obtain

$$\frac{d}{dx}\left(\left(3+\tan(xy)\right)^{1/2}\right) - \frac{d}{dx}(2) = 0$$

$$\frac{1}{2}\left(3+\tan(xy)\right)^{-1/2}\frac{d}{dx}\left(3+\tan(xy)\right) - \frac{d}{dx}(2) = 0$$

$$\frac{1}{2}\left(3+\tan(xy)\right)^{-1/2}\left(0+\sec^2(xy)\frac{d}{dx}(xy)\right) - 0 = 0$$

$$\frac{1}{2}\left(3+\tan(xy)\right)^{-1/2}\left(\sec^2(xy)\left(xy'+y\right)\right) = 0$$

Solving for y' yields

$$\frac{1}{2}\left(3+\tan(xy)\right)^{-1/2}\left(\left(x\sec^2(xy)y' + y\sec^2(xy)\right)\right) = 0$$

$$\frac{1}{2}x\sec^2(xy)\left(3+\tan(xy)\right)^{-1/2}y' + \frac{1}{2}y\sec^2(xy)\left(3+\tan(xy)\right)^{-1/2} = 0$$

$$y' = \frac{-\frac{1}{2}y\sec^2(xy)\left(3+\tan(xy)\right)^{-1/2}}{\frac{1}{2}x\sec^2(xy)\left(3+\tan(xy)\right)^{-1/2}} = -\frac{y}{x}$$

RELATED PROBLEM 1 Find $\frac{dy}{dx}$ for each of the following

a. $8x^2 + y^2 = 10$

b. $\sin^2(3y) = x + y - 1$

c. $3xy = \left(x^3 + y^2\right)^{3/2}$

d. $\sqrt{1 + \sin^3\left(xy^2\right)} = y$

Answers

a. $\dfrac{-8x}{y}$

b. $\dfrac{1}{6\sin(3y)\cos(3y) - 1}$

c. $\dfrac{\frac{3}{2}x^2\left(x^3 + y^2\right)^{1/2} - y}{x - y\left(x^3 + y^2\right)^{1/2}}$

d. $\dfrac{3y^2\sin^2\left(xy^2\right)\cos\left(xy^2\right)}{2y - 6xy\sin^2\left(xy^2\right)\cos\left(xy^2\right)}$

The following example shows how to find an equation of the tangent line to the graph of functions that are defined implicitly.

EXAMPLE 3.5.2 Find an equation of the tangent line to the curve $y^3 + yx^2 + x^2 - 3y^2 = 0$ at the point $P\left(0,3\right)$.

Solution Note that $P\left(0,3\right)$ is on the graph since

$$\left(3\right)^3 + 3\left(0\right)^2 + \left(0\right)^2 - 3\left(3\right)^2 = 0$$

The slope m of the tangent line at $P\left(0,3\right)$ is the value of $\frac{dy}{dx}$ when $x = 0$ and $y = 3$. First let us find $\frac{dy}{dx}$ using implicit differentiation. Differentiating both sides of the equation with respect to x and using the product rule, we obtain

$$\frac{d}{dx}\left(y^3 + yx^2 + x^2 - 3y^2\right) = \frac{d}{dx}\left(0\right)$$

$$\frac{d}{dx}\left(y^3\right) + \frac{d}{dx}\left(yx^2\right) + \frac{d}{dx}\left(x^2\right) - \frac{d}{dx}\left(3y^2\right) = 0$$

$$3y^2\frac{dy}{dx} + y\frac{d}{dx}\left(x^2\right) + x^2\frac{d}{dx}\left(y\right) + 2x - 6y\frac{dy}{dx} = 0$$

$$3y^2\frac{dy}{dx} + 2xy + x^2\frac{dy}{dx} + 2x - 6y\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$ yields

$$3y^2\frac{dy}{dx} + x^2\frac{dy}{dx} - 6y\frac{dy}{dx} = -2xy - 2x$$

$$\left(3y^2 + x^2 - 6y\right)\frac{dy}{dx} = -2x(y + 1)$$

$$\frac{dy}{dx} = -\frac{2x(y+1)}{3y^2 + x^2 - 6y}$$

Thus, when $x = 0$ and $y = 3$,

$$m = \left.\frac{dy}{dx}\right|_{(0,3)} = -\frac{2(0)(3+1)}{3(3)^2 + (0)^2 - 6(3)} = 0$$

So, an equation of the tangent line at $(0,3)$ is

$$y = 0(x-0) + 3 \text{ or } y = 3,$$

see Figure 3.5.2.

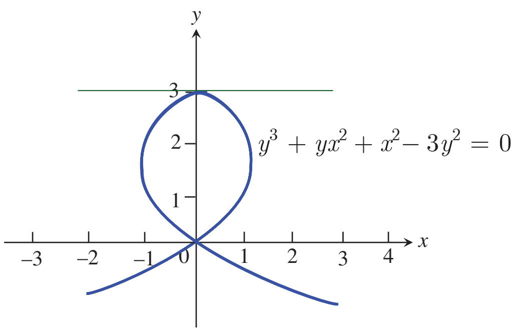

Figure 3.5.2

RELATED PROBLEM 2 Find an equation of the tangent line to the curve $x^2 + (y-x)^3 = 9$ at the point $P(1,3)$

Answer $y = \frac{5}{6}x + \frac{13}{6}$.

EXAMPLE 3.5.3 Given that $x \csc y = 2$, find $\left.\frac{dy}{dx}\right|_{(x,y)=\left(1,\frac{\pi}{6}\right)}$

Solution First let us find $\frac{dy}{dx}$ using implicit differentiation. Differentiating both sides of the equation with respect to x and using the product rule, we obtain

$$\frac{d}{dx}(x \csc y) = \frac{d}{dx}(2)$$

$$x\frac{d}{dx}(\csc y) + \csc y \frac{d}{dx}(x) = 0$$

$$x\left(-\csc y \cot y \frac{dy}{dx}\right) + \csc y(1) = 0$$

$$-x \csc y \cot y \frac{dy}{dx} + \csc y = 0$$

Solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = \frac{\csc y}{x \csc y \cot y} = \frac{1}{x \cot y}$$

Thus,

$$\left.\frac{dy}{dx}\right|_{(x,y)=\left(1,\frac{\pi}{6}\right)} = \frac{1}{(1)\cot\left(\frac{\pi}{6}\right)} = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

RELATED PROBLEM 3 Given that $x^2 \cos y + y^2 - 1 = 0$, find $\left.\frac{dy}{dx}\right|_{(x,y)=(0,1)}$.

Answer 0.

EXAMPLE 3.5.4 Find all points (x,y) on the graph of $x^{2/3} + y^{2/3} = 8$ where tangent to the graph at (x,y) have slope -1.

Solution Differentiate both sides of the equation, we have

$$\frac{d}{dx}\left(x^{2/3} + y^{2/3}\right) = \frac{d}{dx}(8)$$
$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

Thus,

$$y' = -\frac{y^{1/3}}{x^{1/3}}$$

Since the tangent to the graph will have slope -1, set $y' = -1$, we have

$$-\frac{y^{1/3}}{x^{1/3}} = -1 \Rightarrow y^{1/3} = x^{1/3} \Rightarrow y = x$$

Substitute this in the original equation $x^{2/3} + y^{2/3} = 8$, we obtain

$$x^{2/3} + x^{2/3} = 8 \Rightarrow 2x^{2/3} = 8 \Rightarrow x^{2/3} = 4$$
$$\left(x^{2/3}\right)^3 = (4)^3 \Rightarrow x^2 = 64 \Rightarrow x = \pm 8$$

If $x = 8$, then $y = 8$, and the tangent line passing throw the point $(8,8)$ has slope -1.

If $x = -8$, then $y = -8$, and the tangent line passing throw the point $(-8,-8)$ has slope -1.

RELATED PROBLEM 4 Find the coordinates of the point in the first quadrant at which the tangent line to the curve $x^3 - xy + y^3 = 0$ is horizontal.

Answer $\left(\dfrac{\sqrt[3]{2}}{3}, \dfrac{\sqrt[3]{4}}{3}\right)$.

EXERCISES 3.5

In Exercises 1-21, Find $\frac{dy}{dx}$ using implicit differentiation

1. $x^2 + y = 2xy$

2. $x^2 = y^2 + 1$

3. $x^2y + xy^2 = 3x$

4. $2x^2 - y^2 = 1$

5. $(2x + y)^2 = y$

6. $x^2 - y^2x = 1$

7. $x^3 - y^3 = 6xy$

8. $x^2y + 3xy^3 - x = 3$

9. $\sqrt{1 + xy} = y$

10. $\tan y = 3x^2$

11. $\cos(xy) = x$

12. $2\sin y = (x + 1)^2$

13. $x\sin y + y\sin x = 1$

14. $(x + 1)^2 + (y - 2)^2 = 9$

15. $\sqrt{x} + \sqrt{y} = 1$

16. $x = \sec(2y)$

17. $\sqrt{1 + \cos^2 y} = xy$

18. $x + y^2 = \cot(xy)$

19. $\frac{xy}{x^2 + y^2} = x + 1$

20. $(2x^2 + 3y^2)^{5/2} = x$

21. $\frac{1}{x} + \frac{1}{y} = 1$

In Exercises 22-30, find an equation of the tangent line at the given point.

22. $x^3y^2 = -3xy$, $(-1, -3)$

23. $x^2 - 4y^3 = 0$, $(2, 1)$

24. $x^4 = 8(x^2 - y^2)$, $(2, -\sqrt{2})$

25. $x^4 = 4(x^2 - y^2)$, $\left(1, \frac{\sqrt{3}}{2}\right)$

26. $\frac{x^2}{16} - \frac{y^2}{9} = 1$, $\left(-5, \frac{9}{4}\right)$

27. $\frac{x^2}{2} + \frac{y^2}{8} = 1$, $(1, 2)$

28. $x^{2/3} + y^{2/3} = 4$, $(-3\sqrt{3}, 1)$

29. $y^2 = x^3(2 - x)$, $(1, 1)$

30. $2(x^2 + y^2)^2 = 25(x^2 - y^2)$, $(3, 1)$

In Exercises 31-32, find the points at which the graph of the equation has a horizontal tangent line

31. $25x^2 + 16y^2 + 200x - 160y + 400 = 0$

32. $4x^2 + y^2 - 8x + 4y + 4 = 0$

33. Find all point(s) on the curve $x^2y^2 + xy = 2$ where the slope of the tangent line is -1.

34. Show that the equation of the tangent line to the parabola $y^2 = 4px$ at the point (a, b) is $by = 2p(a + x)$.