chapter 1. Utility theory and insurance.

1 Introduction:

Actuarial risk = losses in the futur.

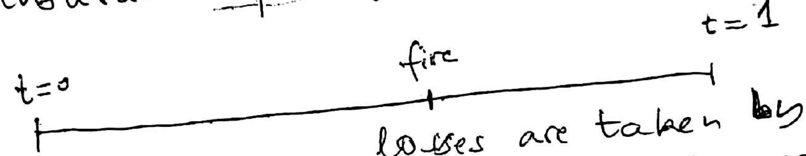
(car insurance, fire insurance, ...).

X = random Variable.

insured $\longleftrightarrow$ insurer (insurance company)

(individual) premium = price of the insurance policy.

Example: An individual takes an fire insurance policy for one year.



losses are taken by insurance company.

2 Premium Calculation:

Suppose we have one risk X.

2.1 Net premium:

the net premium of X is $E(X)$.

Example: Find the net premium for an insurance policy when the risk X is uniformly distributed on the interval (a,b)

$$E(X) = \frac{a+b}{2}$$

(2.2) Premium by utility function:

- A function $u: \mathbb{R} \longrightarrow \mathbb{R}$ is called an utility function if:

(1) u is increasing,

(2) u is concave

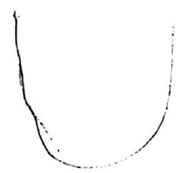
(3) u is twice differentiable.

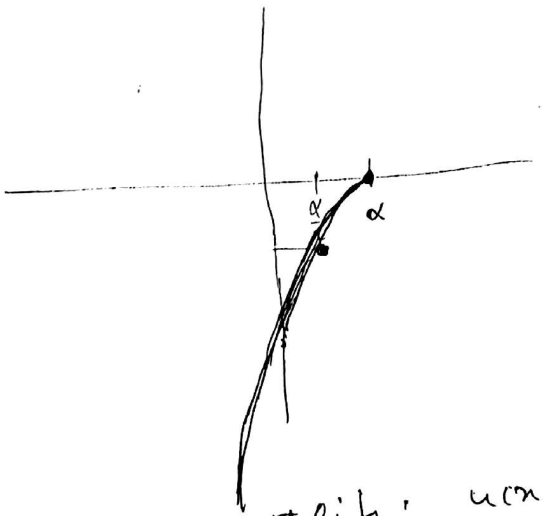
(4) $u(0) = 0$, $u'(0) = 1$.

Let a risk X,

- Five families of utility functions are used.

- Linear utility: $u(x) = x$.
- Quadratic utility: $u(x) = -(\alpha - x)^2$, $x \le \alpha$.





- Logarithmic utility: $u(x) = \log(\alpha + x)$; $x > -\alpha$
- Exponential utility: $u(x) = -\alpha e^{-\alpha x}$; $\alpha > 0$.
- Power utility: $u(x) = x^c$, $x > 0$, $0 < c < 1$.

- We define the premium P against risk X, as the solution of the equation:

$$E\,u(W-X) = u(W-P);$$

W = wealth or budget.

- we write $P = P^+$ for an insured's utility, and $P = P^-$ for an insurer's utility.

($P^+ = 200$, $P^- = 150$).

Example: Suppose that an insurer has an exponential utility function with parameter α.

a) what is the minimum premium P^- to be asked for a risk X?

b) Suppose $X \sim Exp(\lambda)$. Find P^-?

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a) $u(x) = -\alpha e^{-\alpha x}$.

$$E\,u(W-X) = u(W-P).$$

$$E\,{-\alpha}\, e^{-\alpha(W-X)} = -\alpha e^{-\alpha(W-P)}.$$

$$E\, e^{\alpha X} = e^{\alpha P} \qquad m_X(t) = E\,e^{tX}$$

$$\ln(E\, e^{\alpha X}) = \alpha P.$$

$$P = \frac{1}{\alpha}\ln(E\, e^{\alpha X}).$$

$$\boxed{P^- = \frac{1}{\alpha}\ln(m_X(\alpha))}.$$

b) $X \sim exp(\lambda)$; $m_X(t) = E\, e^{tX} = \int_0^\infty e^{tx}\lambda e^{-\lambda x}\,dx.$

$$= \int_0^\infty \lambda e^{-(\lambda-t)x}\,dx$$

$$= \begin{cases} +\infty & t \geq \lambda \\ \dfrac{\lambda}{\lambda - t} & t < \lambda \end{cases}$$

$$P^- = \frac{1}{\alpha}\ln\left(\frac{\lambda}{\lambda-\alpha}\right) \quad ; \quad \underline{\alpha<\lambda}$$

Example ②: Suppose for a wealth $w<\alpha$, the insured's utility function $u(x) = -(\alpha-x)^2$.

a) compute the premium P^+ against risk X

b) Suppose $X = 1$ with probability $\frac{1}{2}$, $\alpha=5$, what happens to this premium if w increases?

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a)
$$E\,u(w-X) = u(w-P).$$
$$E\,-(\alpha-w+X)^2 = -(\alpha-w+P)^2.$$
$$\alpha-w+P = \sqrt{E(\alpha-w+X)^2}.$$
$$P^+ = -\alpha+w+\sqrt{E(\alpha-w+X)^2}.$$
$$\boxed{P^+ = -\alpha+w+\sqrt{(\alpha-w)^2+E(X^2)+2(\alpha-w)E(X)}}$$

b)
$$X = \begin{cases} 1 & 1/2 \\ 0 & 1/2 \end{cases}, \quad E(X)=\frac{1}{2}, \quad E(X^2)=\frac{1}{2}.$$
$$P(w)=P^+ = -5+w+\sqrt{(5-w)^2+\frac{1}{2}+(5-w)}$$
$$P'(w) = 1+\frac{-2(5-w)-1}{2\sqrt{(5-w)^2+\frac{1}{2}+(5-w)}}$$
$$= 1+\frac{2w-11}{2\sqrt{\cdots}} \geq 0.$$
$$P^+ = -5+w+\sqrt{25+w^2-10w+\frac{1}{2}+5-w}$$
$$= -5+w+\sqrt{w^2-11w+30.5}$$
$$= -5+w+\sqrt{\left(w-\frac{11}{2}\right)^2+0.25}.$$

③ Premium approximation:

we consider a loss X with mean μ and variance σ^2.

$$E\,u(W-X) = u(W-P).$$

* let f a function, twice differentiable:

$$f(x) = f(x_0) + (x-x_0) f'(x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \varepsilon.$$

$x = W - X$; $x_0 = W - \mu$.

$$u(W-X) = u(W-\mu) - (X-\mu)\,u'(W-\mu) + \frac{1}{2}(X-\mu)^2 u''(W-\mu) + \varepsilon$$

$$\simeq u(W-\mu) - (X-\mu)\,u'(W-\mu) + \frac{1}{2}(X-\mu)^2 u''(W-\mu).$$

$x = W - P$; $x_0 = W - \mu$.

$$u(W-P) = u(W-\mu) - (P-\mu)\,u'(W-\mu) + \varepsilon.$$

$$\simeq u(W-\mu) - (P-\mu)\,u'(W-\mu).$$

$$E\left(u(W-\mu) - (X-\mu)\,u'(W-\mu) + \frac{1}{2}(X-\mu)^2 u''(W-\mu)\right)$$

$$\simeq u(W-\mu) - (P-\mu)\,u'(W-\mu).$$

$E(X-\mu) = E(X) - \mu = 0$; $E(X-\mu)^2 = \sigma^2$.

$$\frac{1}{2}\sigma^2 u''(W-\mu) \simeq -(P-\mu)\,u'(W-\mu).$$

$$\boxed{P \simeq \mu - \frac{1}{2}\sigma^2 \frac{u''(W-\mu)}{u'(W-\mu)}}$$

$r(x) = -\dfrac{u''(x)}{u'(x)}$ risk aversion coefficient.

Example: Consider the exponential utility function u. Compute the approximative value of the premium P against a risk $X = 100$ with probability $\frac{1}{2}$.

$*$ $E(X) = 100 \times \frac{1}{2} = 50$.

$Var(X) = 100 \times \frac{1}{2} \times \frac{1}{2} = 25$.

$u(x) = -\alpha e^{-\alpha x}$; $u'(x) = \alpha^2 e^{-\alpha x}$, $u''(x) = -\alpha^3 e^{-\alpha x}$.

$r(x) = -\frac{-\alpha^3 e^{-\alpha x}}{\alpha^2 e^{-\alpha x}} = \alpha$.

$X = 100 - \frac{1}{2}$

$\alpha = 0.5$ P_1

$\alpha = 3$ P_2

$$\boxed{P \simeq \mu + \frac{\alpha \sigma^2}{2}}$$

$P \simeq 50 + 12.5\alpha$.

$\alpha = 1 \longrightarrow P \simeq 62.5$.

$P \simeq \frac{1}{\alpha} \ln(m_X(\alpha)) = \ln(m_X(1))$.

$m_X(t) = \frac{1}{2} e^{100t} + \frac{1}{2} \longrightarrow P = \ln\left(\frac{1}{2} e^{100} + \frac{1}{2}\right) =$

$= 99.3$

(1)

4) Reinsurance:

individual (insured) $\xleftrightarrow[\text{policy}]{\text{insurance}}$ insurance company (insurer)

+ budget
+ big losses
+ Regulations

insurance company (insurer) $\longleftrightarrow$ Reinsurance company (Reinsurer).

* stop-loss reinsurance: a fixed limit d called retention or priority is agreed on.

X:
- $X \leq d \longrightarrow$ insurer pays the losses
- $X > d \longrightarrow$ d is paid by the insurer. $X-d$ by the reinsurer.

stop-loss premium: $\left(\begin{array}{l} 2_+ = 2 \\ (-2)_+ = 0 \end{array}\right)$

$$\pi_X(d) = E(X-d)_+$$

* $$\pi_X(d) = \int_d^{\infty} (1 - F_X(x))\, dx.$$

Example: $X \sim \text{Uniform}(2,4)$. Compute $\pi_X(d)$?

$$\pi_X(d) = E(X-d)_+ = \int_2^4 (x-d)_+ \frac{1}{2}\, dx$$

$\underline{\underline{d<2}}$: $$\pi_X(d) = \frac{1}{2}\int_2^4 (x-d)\, dx = \frac{1}{4} (x-d)^2 \Big|_2^4$$

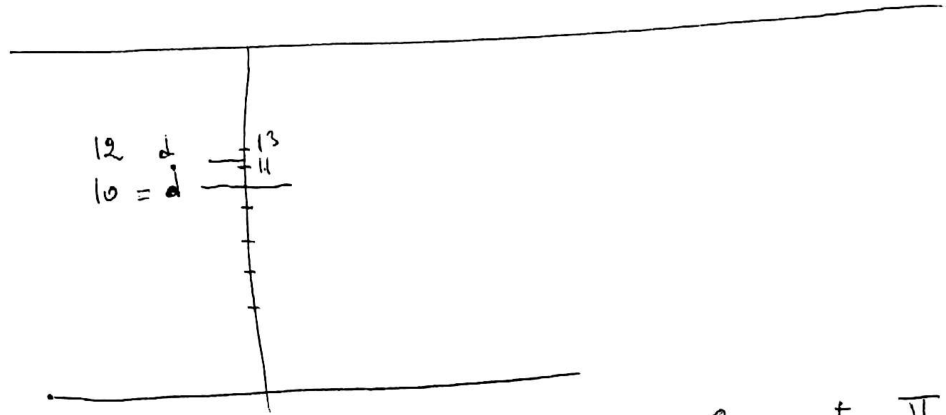
$$= \frac{1}{4}\left((4-d)^2 - (2-d)^2\right).$$

$2 \le d < 4$: $\quad \pi_X(d) = \frac{1}{2}\int_2^d (x-d)_+\,dx + \frac{1}{2}\int_d^4 (x-d)_+\,dx$

$$= \frac{1}{4}(4-d)^2.$$

$d \geq 4$: $\quad \pi_X(d) = 0.$

$$\pi_X(d) = \begin{cases} \frac{1}{4}\left((4-d)^2 - (2-d)^2\right); & d < 2 \\ \frac{1}{4}(4-d)^2, & 2 \le d < 4 \\ 0 & d \geq 4. \end{cases}$$



Example: Suppose $X \sim Exp(2)$. Compute $\pi_X(1)$?

$$\pi_X(1) = E(X-1)_+ = \int_0^\infty (x-1)_+\, 2e^{-2x}\,dx.$$

$$= \int_0^1 2(x-1)_+ e^{-2x}\,dx + \int_1^\infty 2(x-1)e^{-2x}\,dx.$$

$u = x-1 \longrightarrow du = 1dx$

$dv = 2e^{-2x}dx \rightarrow v = -e^{-2x}$

$\int u\,dv = uv - \int v\,du.$

$$= -(x-1)e^{-2x}\Big|_1^\infty + \int_1^\infty e^{-2x}\,dx.$$

$$= -\frac{1}{2}e^{-2x}\Big|_1^\infty = \frac{1}{2}e^{-2} \simeq 0.06.$$

$\frac{x-1}{e^{2x}} \rightarrow \frac{1}{2e^{2x}} \rightarrow 0.$