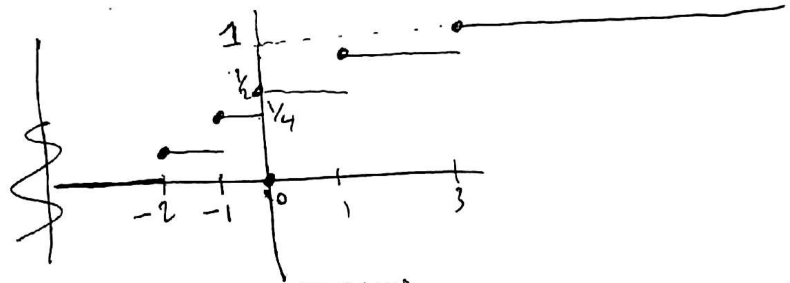
Chapter (2): Individual risk model.

From the insurer point view, they need a model to estimate the losses for a portfolio of insurance policies.

(1) Mixed distribution:

distribution < discrete → mass function; continuous → density fonctn

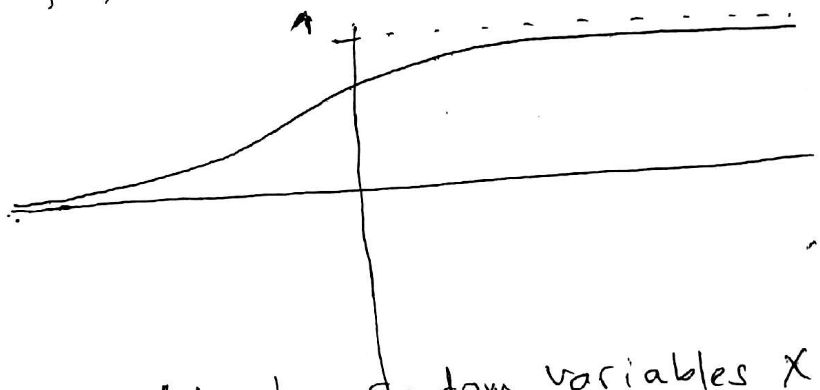
Discrete:



$$f(t) = F(t) - F(t^-)$$

$$f(0) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Continuous:



Definition: let two random variables X and Y. we call Z the q-mixture of X and Y if:

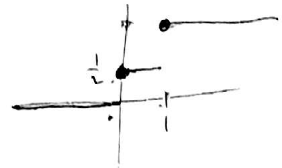
$$F_Z(t) = q\,F_X(t) + (1-q)\,F_Y(t). \quad 0 \le q \le 1.$$

Example: Suppose $X \sim \text{Exp}(1)$, $Y \sim \text{Bernoulli}(0.5)$. Define Z to be the q-mixture of X and Y.

* Compute the cdf of Z?

$$F_X(t) = \begin{cases} 1 - e^{-x} & ; x \geq 0 \\ 0 & ; x \leq 0 \end{cases}$$

$$F_Y(t) = \begin{cases} 1 & x \geq 1 \\ 1/2 & 0 \leq x < 1 \\ 0 & x < 0 \end{cases}$$



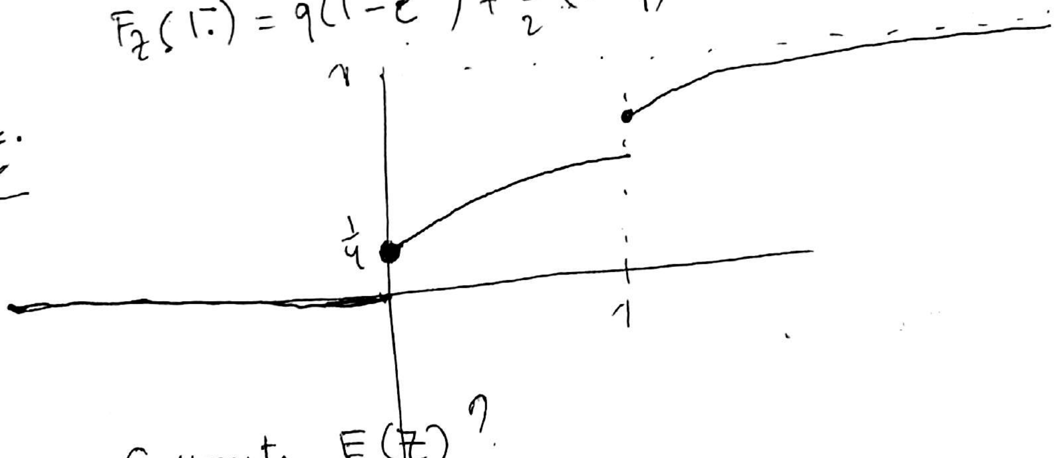
$$F_Z(t) = \begin{cases} 0 & x < 0 \\ q(1-e^{-x}) + \frac{1}{2}(1-q) & 0 \leq x < 1 \\ q(1-e^{-x}) + (1-q) & x \geq 1 \end{cases}$$

$F_Z(0) = \frac{1}{2}(1-q)$, $F_Z(0^-) = 0$.

$F_Z(1) = q(1-e^{-1}) + (1-q)$

$F_Z(1^-) = q(1-e^{-1}) + \frac{1}{2}((1-q)$.

$\underline{q = \frac{1}{2}.}$



* Compute $E(Z)$?

$$E(Z) = \int_{-\infty}^{\infty} t \, dF_Z(t).$$

$$E(Z) = \int_{-\infty}^{0^-} t \, dF_Z(t) + \int_{0^-}^{0} t \, dF_Z(t) + \int_{0}^{1^-} t \, dF_Z(t) + \int_{1^-}^{1} t \, dF_Z(t) + \int_{1}^{\infty} t \, dF_Z(t)$$

$$= 0 + 0 + \int_0^{1^-} t f_Z(t) \, dt + \boxed{\int_{1^-}^{\infty} t \, dF_Z(t)} + 1 . f(1) + \int_1^{\infty} t f_Z(t) \, dt.$$

* Let Z be the q-mixture of X and Y: there exists a random variable I taking values $\{0, 1\}$ with $P(I=1) = q$ and

$$Z = IX + (1-I)Y \; /.$$

$I, (X, Y)$ are independent.

• $$E\,g(Z) = E\,g(IX + (1-I)Y)$$
$$= E\,E(g(IX + (1-I)Y) \mid I)$$

$$\boxed{E(X) = E\,E(X \mid Y)}$$

$$= E(g(IX + (1-I)Y) \mid I=1)\,P(I=1)$$
$$+ E(g(IX + (1-I)Y) \mid I=0)\,P(I=0)$$
$$= q\,E(g(X) \mid I=1) + (1-q)\,E(g(Y) \mid I=0).$$

$$\boxed{E\,g(Z) = q\,E\,g(X) + (1-q)\,E\,g(Y).}$$

* $g(x) = x^m.$
$$E(Z^m) = q\,E(X^m) + (1-q)\,E(Y^m).$$

* $g(x) = e^{tx}.$
$$m_Z(t) = q\,m_X(t) + (1-q)\,m_Y(t).$$

Example ①: Z is the q-mixture of X and Y,
$X \sim Bin(n,p)$, $Y \sim N(0,1)$.
Compute the mean, the variance and the mgf of Z

* $E(Z) = q\,E(X) + (1-q)\,E(Y)$

$= qnp + 0 = qnp$

* $E(Z^2) = q\,E(X^2) + (1-q)\,E(Y^2)$

$= q\left(np(1-p) + n^2p^2\right) + (1-q).$

$Var(Z) = E(Z^2) - (E(Z))^2$

$= qnp(1-p) + qn^2p^2 + (1-q) - q^2n^2p^2.$

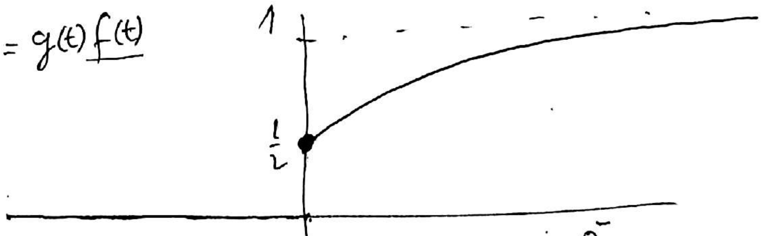
* $m_Z(t) = q\,m_X(t) + (1-q)\,m_Y(t)$

$= q\,(pe^t + 1 - p)^n + (1-q)\,e^{t^2/2}.$

Example ②: let Z with

$$F_Z(t) = \begin{cases} 0 & ; \ t<0 \\ 1 - \frac{1}{2}e^{-t} & ; \ t \geq 0 \end{cases}$$

$\int_{t^-}^{t} g(x)\,dF(x) = g(t)\,\underline{f(t)}$



* $E(Z) = \int_{-\infty}^{\infty} t\,dF(t) = \int_{-\infty}^{0^-} t\,dF(t) + \int_{0^-}^{0} t\,dF(t) + \int_{0}^{\infty} t\,dF(t)$

$= 0 + 0 + \int_0^\infty t\,\frac{1}{2}e^{-t}\,dt = \frac{1}{2}\int_0^\infty t e^{-t}\,dt = \frac{1}{2}.$

$\left(X \sim Exp(\lambda) \ ; \ E(X) = \int_0^\infty t\,\lambda e^{-\lambda t}\,dt = 1/\lambda \right)$

* $m_Z(t) = E\,e^{tZ} = \int_{-\infty}^{\infty} e^{tx}\,dF(x) = \int_{-\infty}^{0^-} e^{tx}\,dF(x)$

$+ \int_{0^-}^{0} e^{tx}\,dF(x) + \int_0^\infty e^{tx}\,dF(x) = 0 + \frac{1}{2} + \frac{1}{2}\int_0^\infty e^{tx}\,e^{-x}\,dx$

$$= \frac{1}{2} + \frac{1}{2}\,\frac{1}{1-t} \quad ; \quad t < 1.$$

$q = \frac{1}{2}$, $Y \sim Exp(1)$; $X = 0$ with Prob one.

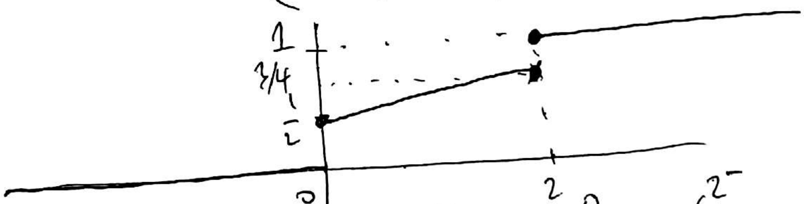
$$F_X(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

$$F_Y(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-t} & t \geq 0 \end{cases}$$

$$F_Z(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2} + \frac{1}{2}(1 - e^{-t}) = 1 - \frac{1}{2}e^{-t} , & t \geq 0 \end{cases}$$

Example (3) : let Z :

$$F(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{8}(t+4) \, , & 0 \leq t < 2 \\ 1 \, ; & t \geq 2 \end{cases}$$



$$E(Z) = \int_{-\infty}^{\infty} t \, dF(t) = \int_{-\infty}^{0^-} + \int_{0^-}^{0} + \int_{0}^{2^-} + \int_{2^-}^{2} + \int_{2}^{\infty}$$

$$= 0 + 0 + \int_0^{2^-} \frac{t^2}{8} dt + 2 * \frac{1}{4} + 0$$

$$= \left. \frac{t^3}{24} \right|_0^2 + \frac{1}{2} = \frac{1}{3} + \frac{1}{2} = 5/6 .$$

$$* \quad m_Z(t) = \int_{-\infty}^{\infty} e^{tx} dF(x) = \int_{-\infty}^{0^-} + \int_{0^-}^{0} + \int_{0}^{2^-} + \int_{2^-}^{2} + \int_{2}^{\infty}$$

$$= 0 + 1 \times \frac{1}{2} + \int_0^2 \frac{1}{8} e^{tx} dx + \frac{1}{4} e^{2t} + 0$$

$$\int_0^2 x e^{tx} dx = \left. \frac{x}{t} e^{tx} \right|_0^2 - \int_0^2 \frac{1}{t} e^{tx} dx = \left. \frac{x}{t} e^{tx} - \frac{1}{t^2} e^{tx} \right|_0^2$$

$$= \left(\frac{2}{t} e^{2t} - \frac{1}{t^2} e^{2t} \right) + \frac{1}{t^2} .$$

(5)

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Portfolio: $X_1, X_2, \dots, X_n$.

$S = X_1 + X_2 + \dots + X_n$.

We suppose that the X_i's are independent and identically distributed (iid).

(3) Convolution:

Suppose X and Y are independent: what is the distribution of $X+Y$?

- $$F_{X+Y}(t) = P(X+Y \le t)$$
$$= \int \underbrace{P(X+y \le t)}\, dF_Y(y)$$
$$= \int F_X(t-y)\, dF_Y(y)$$
$$= F_X * F_Y(t).$$

$*$ is called the convolution product.

- X and Y are discrete.
$$f_{X+Y}(t) = \sum_s f_X(t-s) f_Y(s).$$

- X and Y are continuous:
$$f_{X+Y}(t) = \int f_X(t-y) f_Y(y)\, dy.$$

Example (1): Suppose $X \sim Exp(1)$, $Y \sim Exp(2)$, with $X \perp\!\!\!\perp Y$ (independent).

Compute the distribution of $X+Y$?

$$f_X(t) = \begin{cases} \lambda e^{-\lambda t} ; & t \ge 0 \\ 0 ; & t < 0 \end{cases}$$

$$f_{X+Y}(t) = \int e^{-(t-y)} 1_{(t-y \ge 0)}\, 2e^{-2y} 1_{(y \ge 0)}\, dy.$$
$$= \int_0^t 2e^{-t-y}\, dy = 2e^{-t}(1-e^{-t}); \quad t \ge 0.$$

Example ② : X:

x	0	1	2
f_X	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

, Y:

y	1	2
f_Y	$\frac{1}{3}$	$\frac{2}{3}$

$X \perp\!\!\!\perp Y$.

Find the distribution of $X+Y$?

t	f_X	f_Y	f_{X+Y}
0	1/4		
1	1/4	1/3	1/12
2	1/2	2/3	$\frac{1}{12}+\frac{2}{12}=\frac{3}{12}=\frac{1}{4}$.
3			$\frac{2}{12}+\frac{1}{6}=\frac{4}{12}=\frac{1}{3}$.
4			$\frac{2}{6}=1/3$.

$x\ y$

$1 \leftarrow (0,1)$

$2 \leftarrow (1,1),(0,2)$

$3 \leftarrow (1,2),(2,1)$

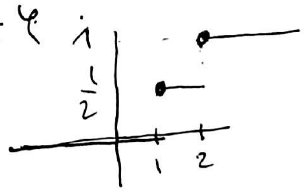
$4 \leftarrow (2,2)$

Example: X:

x	1	2
f_X	$\frac{1}{2}$	$\frac{1}{2}$

, $Y \sim U(0,1)$. $X \perp\!\!\!\perp Y$.

Find the distribution of $X+Y$



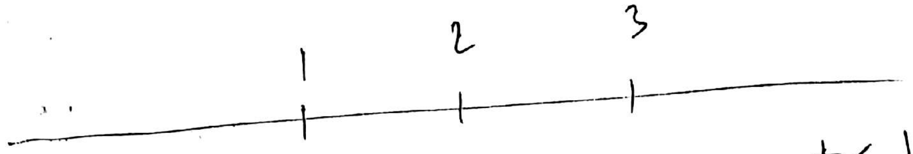
$$F_{X+Y}(t) = \int F_Y(t-x)\, dF_X(x)$$

$$= F_Y(t-1) f_X(1) + F_Y(t-2) f_X(2)$$

$$= \text{[illegible]}$$

$$Y \sim U(0,1) \longrightarrow F_Y(t) = \begin{cases} 0 & t<0 \\ t & 0 \le t < 1 \\ 1 & t \ge 1 \end{cases}$$

$$= \frac{1}{2}\begin{cases} 0 & t<1 \\ t-1 & 1\le t<2 \\ 1 & t>2 \end{cases} + \frac{1}{2}\begin{cases} 0 & t<2 \\ t-2 & 2\le t<3 \\ 1 & t\ge 3 \end{cases}$$



$$= \begin{cases} 0 & t<1 \\ \frac{1}{2}(t-1) & 1\le t<2 \\ \frac{1}{2}+\frac{1}{2}(t-2) & 2\le t<3 \\ 1 & t\ge 3 \end{cases}$$

$$F_{X+Y}(t) = \int F_X(t-y)\, dF_Y(y)$$

$$= \int_0^1 F_X(t-y)\, dy.$$

$t-y<1$
$y>t-1.$

$$F_X(t) = \begin{cases} 0 & t<1 \\ 1/2 & 1 \le t < 2 \\ 1 & t \ge 2 \end{cases}$$

- $X_1, X_2, \ldots, X_n$ independent:

$$F_S(t) = F_{X_1} * F_{X_2} * \cdots * F_{X_n}$$

(4) Transformations:

- moment generating function:

$$m_X(t) = E\, e^{tX}.$$

$$\rightarrow \mu_k = E(X^k) = \frac{d^k}{dt^k} m_X(t) \Big|_{t=0}$$

- Probability generating function:

$$g_X(t) = E\, t^X = m_X(\ln(t)).$$

$$P(X=k) = \frac{1}{k!} \frac{d^k}{dt^k} g_X(t) \Big|_{t=0}$$

- Cumulant generating function:

$$K_X(t) = \ln(m_X(t)).$$

$$K_3 = E(X-E(X))^3 = \frac{d^3}{dt^3} K_X(t) \Big|_{t=0}$$

- $X \perp\!\!\!\perp Y$, $m_{X+Y}(t) = E\, e^{t(X+Y)} = E\, e^{tX+tY}$

$$= E\, e^{tX} e^{tY} = E e^{tX} E e^{tY}$$

$$= m_X(t) \cdot m_Y(t).$$

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Example: $X \perp\!\!\!\perp Y$. Compute the distribution of $X+Y$ in the following cases:

(1) $X \sim N(0,1)$, $Y \sim N(0,2)$.

(2) $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\delta)$.

(3) $X \sim \text{Gamma}(\alpha, \beta)$, $Y \sim \text{Gamma}(\alpha', \beta)$.

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(1) $$m_{X+Y}(t) = m_X(t)\, m_Y(t) = e^{t^2/2}\, e^{2t^2} = e^{\frac{1}{2} \times 5 t^2} \sim N(0,5)$$

(2) $$m_{X+Y}(t) = m_X(t)\, m_Y(t) = e^{\lambda(e^t-1)}\, e^{\delta(e^t-1)} = e^{(\lambda+\delta)(e^t-1)} \sim \text{Poisson}(\lambda+\delta).$$

(3) $$m_{X+Y}(t) = \left(\frac{\beta}{\beta-t}\right)^{\alpha} \left(\frac{\beta}{\beta-t}\right)^{\alpha'} = \left(\frac{\beta}{\beta-t}\right)^{\alpha+\alpha'}$$

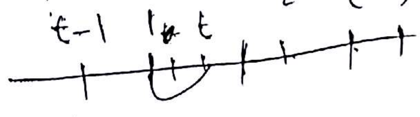
$$\sim \text{Gamma}(\alpha+\alpha', \beta)$$

Exercise: Suppose $X \sim U(0,1)$; $Y \sim U(1,2)$, $X \perp\!\!\!\perp Y$. Compute the distribution of $X+Y$?

- $$m_{X+Y}(t) = \frac{e^t - 1}{t} \cdot \frac{e^{2t} - e^t}{t} \quad ?$$

- $$f_{X+Y}(t) = \int f_X(t-y)\, f_Y(y)\, dy$$
$$= \int 1_{(0 \le t-y \le 1)}\, 1_{(1 \le y \le 2)}\, dy.$$

$0 \le t-y \le 1 \Rightarrow t-1 \le y \le t$. $\quad$? $(t-1, t) \cap (1,2) =$



$t < 1$: $\quad f_{X+Y}(t) = 0$.

$1 \le t < 2$: $\quad f_{X+Y}(t) = \int_1^t dy = t-1$.

$2 \le t < 3$: $\quad f_{X+Y}(t) = 3-t$.

$t \ge 3$: $\quad f_{X+Y}(t) = 0$

Example: ~~X~~, X and Y iid:

x	0	1	3	4
f	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$

$y \backslash x$	0	1	3	4
0	0	1	3	4
1	1	2	4	5
3	3	4	6	7
4	4	5	7	8

Find the distribution of $X+Y$?

t	f_X	f_Y	f_{X+Y}
0	$1/8$	$1/8$	$1/64$
1	$1/4$	$1/4$	$\frac{1}{32}+\frac{1}{32}=\frac{1}{16}$
2	[illegible]	—	$\frac{1}{16}$
3	$1/8$	$1/8$	$\frac{1}{64}+\frac{1}{64}=\frac{1}{32}$
4	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{16}+\frac{1}{16}+\frac{1}{32}+\frac{1}{32}=\frac{3}{16}$
5			$\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$
6			$\frac{1}{64}$
7			$\frac{1}{16}+\frac{1}{16}=\frac{1}{8}$
8			$\frac{1}{4}$

$$P(X+Y>5) = \frac{1}{64}+\frac{1}{8}+\frac{1}{4} = \frac{25}{64}$$

Example: X, Y iid $\sim$ Geometric(p).

$X+Y \sim$?

(5) Approximation:

(5.1) Normal approximation:

- Central limit theorem (CLT):

$X_1, X_2, \ldots X_n$, iid; with mean μ and variance σ^2. For large values of n $(n \geq 30)$:

$$S := X_1 + \cdots + X_n \sim N(n\mu; n\sigma^2).$$

- Example: 1,000 people take a one year life insurance policy. The Probability of dying during this year is 0.01 ~~and~~ the payment for ~~every~~ death is 1.

Find the probability that the total payment is at least 4 using NA.

one individual: $X = \begin{cases} 1 & 0.01 \\ 0 & 0.99. \end{cases}$

$X \sim \text{Bernoulli}(0.01)$.

$S = X_1 + \cdots + X_{1000} \sim \text{Bin}(1{,}000; 0.01)$.

$\mu = E(X) = 0.01$, $\sigma^2 = 0.01 * 0.99 = 0.0099$.

NA: $S \sim N(10, 9.9)$.

$$P(S \geq 4) = P\left(\frac{S-10}{\sqrt{9.9}} \geq \frac{4-10}{\sqrt{9.9}}\right)$$

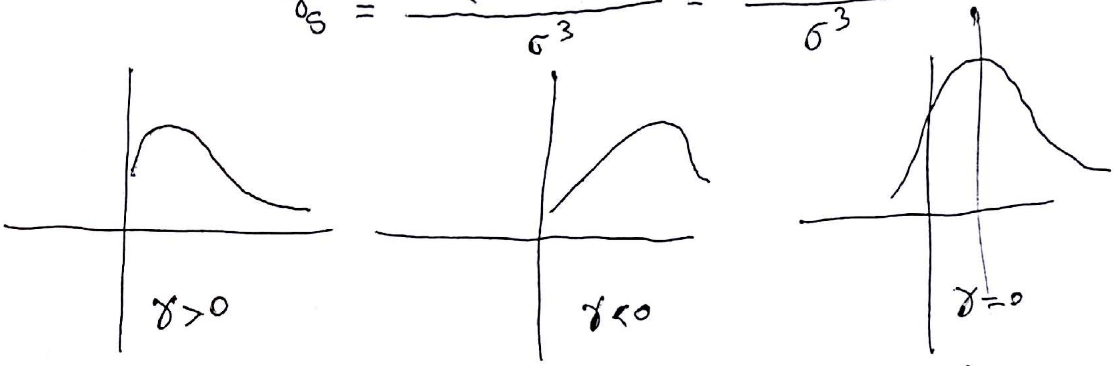
$$\simeq 1 - \phi\left(\frac{-6}{\sqrt{9.9}}\right) = 1 - \phi(-1.9)$$

$$= 0.9713.$$

5.7 Translated gamma approximation:

Let S a random Variable with mean μ, variance σ^2 and skewness γ.

$$\gamma_S = \frac{E(S-E(S))^3}{\sigma^3} = \frac{K_3(S)}{\sigma^3}.$$



$$S \longleftrightarrow x_0 + G \sim \text{Gamma}(\alpha,\beta).$$

$$\begin{cases} E(S) = E(x_0+G) \\ Var(S) = Var(x_0+G) \\ \gamma(S) = \gamma(x_0+G) \end{cases}$$

$$\begin{cases} \mu = x_0 + \alpha/\beta \Rightarrow \\ \sigma^2 = \alpha/\beta^2 \Rightarrow \\ \gamma = 2/\sqrt{\alpha} \Rightarrow \end{cases} \qquad \boxed{\begin{aligned} x_0 &= \mu - \frac{2\sigma}{\gamma} \\ \beta &= \frac{\sqrt{\alpha}}{\sigma} = \frac{2}{\sigma\gamma} \\ \alpha &= \frac{4}{\gamma^2} \end{aligned}}$$

Example: Suppose $S \sim$ Poisson (1), Compute $P(S \geq 3.5)$ using TGA?

$\mu = \sigma^2 = \gamma = 1.$

$\alpha = 4,\ \beta = 2,\ x_0 = -1.$

$P(S \geq 3.5) \simeq P(G - 1 \geq 3.5)$, $G \sim$ Gamma $(4,2)$

$= P(G \geq 4.5)$

$= 1 - F_G(4.5).$

5.3 Normal power approximation:

Let S a random variable with mean μ, variance σ^2 and skewness γ.

For $x \geq 1$:

$$P\left(\frac{S-\mu}{\sigma} \leq x\right) \simeq \phi\left(\sqrt{\frac{9}{\gamma^2} + \frac{6x}{\gamma} + 1} - \frac{3}{\gamma}\right).$$

For $u \geq 1$:

$$P\left(\frac{S-\mu}{\sigma} \leq u + \frac{\gamma}{6}(u^2-1)\right) \simeq \phi(u).$$

ϕ is the cdf of a $N(0,1)$.

Example: Let $S \sim$ Poisson (1). Find $P(S \geq 3.5)$.

$\mu = \sigma^2 = \gamma = 1$.

$$P(S \geq 3.5) = P\left(\frac{S-1}{1} \geq \frac{3.5-1}{1}\right) \simeq 1 - \phi\left(\sqrt{\frac{9}{1} + \frac{6 \times 2.5}{1} + 1} - \frac{3}{1}\right)$$

$$\simeq 1 - \phi(2)$$
$$= 1 - 0.9772$$
$$= 0.0228.$$

Example: Let S the total claim with: $\mu = 10{,}000$; $\sigma = 1{,}000$; $\gamma = 1$.

Calculate the minimum capital that covers lose S with probability 95%, using the NP approximation ?.

B: $P(S \leq B) = 0.95$.

$$P\left(\frac{S-\mu}{\sigma} \leq A\right) \simeq \phi(u) = 0.95 \longrightarrow u = 1.645$$

$u = 1.645.$

$A = u + \frac{\gamma}{6}(u^2 - 1).$

$P\left(\frac{S-\mu}{\sigma} \le A\right) = 0.95.$

$P(S \le \boxed{\sigma A + \mu}) = 0.95.$

$B = \mu + \sigma A = \mu + \sigma\left(u + \frac{\gamma}{6}(u^2 - 1)\right).$

$= 10{,}000 + 1{,}000\left(1.645 + \frac{1}{6}\left((1.645)^2 - 1\right)\right).$

$= 11{,}929.34.$

6 Application to reinsurance:

Example:

Insured amount	Number of policies.
1	10,000
2	5,000
3	5,000

one-year
$X = b$

$P(\text{dying in one year}) = 0.01.$

$d = 2.$

Compute the capital B that cover losses under reinsurance with probability 95%.

insurance

Ins. amount	Nber
1	10,000
2	5,000
2	5,000

$\left.\begin{matrix}5{,}000\\5{,}000\end{matrix}\right\} 10{,}000$

Reinsurance.

ins. amount	Nber of policies
1	5,000.

$S_1 = X_1 + X_2 + \cdots + X_{10,000}$ $\quad X \sim \text{Bernoulli}(0.01)$

$S_2 = Y_1 + \cdots + Y_{10,000}$ $\quad Y \sim 2\ \text{Bernoulli}(0.01)$

$S_3 = Z_1 + \cdots + Z_{5,000}$, $\quad Z \sim \text{Bernoulli}(0.01)$.

$S = S_1 + S_2 + E(S_3)$

stop-loss premium.

$\mu_S = \mu_{S_1} + \mu_{S_2} + \mu_{S_3}$

$\mu_{S_1} = 10,000\, \mu_X = 10,000 * 0.01 = 100.$

$\mu_{S_2} = 10,000\, \mu_Y = 10,000 * 2 * 0.01 = 200.$

$\mu_{S_3} = 5,000\, \mu_Z = 5000 * 0.01 = 50.$

$\mu_S = 350.$

$Var(S) = Var(S_1) + Var(S_2).$

$Var(S_1) = 10,000\ Var(X) = 10,000 * 0.01 * 0.99 = 99.$

$Var(S_2) = 10,000\ Var(Y) = 10,000 * 2^2 * 0.01 * 0.99$

$= 396.$

$Var(S) = 495.$

$$\gamma(S) = \frac{K_3(S)}{\sigma^3(S)} = \frac{E(S - E(S))^3}{\sigma^3(S)}.$$

$\boxed{\gamma(S) \neq \gamma(S_1) + \gamma(S_2)}$

$K_3(S) = K_3(S_1) + K_3(S_2)$

$K_3(S_1) = 10,000\ K_3(X) = 10,000 * 0.01 * 0.99 * 0.98.$ $= 97.02$

$K_3(S_2) = 10,000 * 2^3 * 0.01 * 0.99 * 0.98$

$= 776.16.$

$K_3(S) = 873.18$, $\quad \sigma(S) = \sqrt{495} = 22.24.$

$$\gamma(S) = \frac{873.18}{(22.24)^3} = 0.079.$$

$$P\left(\frac{S-\mu}{\sigma} \leq u+\frac{\gamma}{6}(u^2-1)\right) \simeq \phi(u)=0.95.$$

$$u=1.645.$$

$$B = \mu+\sigma\left(u+\frac{\gamma}{6}(u^2-1)\right)$$

$$= 350+22.24\left(1.645+\frac{0.079}{6}\left((1.645)^2-1\right)\right)$$

$$= 387.08.$$

Example: Let consider a portfolio of insurance policies where the losses are exponentially distributed with parameter λ.

Number of insured	parameter λ
1,000	1
2,000	2

(1) Compute the probability that the total loss exceeds ~~1,200~~ 2060, using the NP approximation

$X \sim Exp(1)$, $S_1 = X_1+X_2+\dots+X_{1,000}$.

$Y \sim Exp(2)$, $S_2 = Y_1+Y_2+\dots+Y_{2,000}$

$S = S_1+S_2$.

- $\mu_S = \mu_{S_1}+\mu_{S_2}$ / $\mu_{S_1} = 1,000\, E(X) = 1,000$
$\mu_{S_2} = 2,000\, E(Y) = 1,000.$

$\mu_S = 1000+1000$
$= 2000.$

- $Var(S) = Var(S_1)+Var(S_2).$
$= 1000\, Var(X) + 2000\, Var(Y)$
$= 1000 + \frac{2000}{4} = 1,500.$

$\sigma(S) = \sqrt{1500} = 38.73.$

$x \sim Exp(\lambda)$

$K_3(x) = \frac{2}{\lambda^3}$.

- $K_3(S) = K_3(S_1) + K_3(S_2)$

$= 1{,}000\, K_3(X) + 2{,}000\, K_3(Y)$.

$= 2{,}000 + 500 = 2500$.

- $\gamma(S) = \frac{K_3(S)}{\sigma^3(S)} = \frac{2500}{(1500)^{3/2}} = 0.043$.

- $P(S \geq 1{,}200) = P\left(\frac{S-\mu_S}{\sigma_S} \geq \frac{\cancel{2260}\; 2000}{\sqrt{1500}}\right)$

$= P\left(\frac{S-\mu}{\sigma_s} \geq + \frac{60}{38.73}\right)$

$= P\left(\frac{S-\mu}{\sigma_s} \geq + 1.5\right)$

$\simeq 1 - \phi\left(\sqrt{\frac{9}{(0.043)^2} + \frac{6 \times 1.5}{0.043} + 1} - \frac{3}{0.043}\right)$

$= 1 - \phi(1.5)$

$= 1 - 0.9332 = \cancel{0.1268}\ 0.0668$

(2) . . . . under reinsurance with

$d = 2$.

Reinsurance	insurance
$(Z-d)_+$,	$Z - (Z-d)_+$.

$Z \sim Exp(\lambda) \xrightarrow{d=2}$

$\hat{X}_i = X_i - (X_i - d)_+$

$\hat{Y}_i = Y_i - (Y_i - d)_+$

$S_1 = X_1 + \cdots + X_{1{,}000}$

$S_2 = Y_1 + \cdots + Y_{2{,}000}$

$S_3 = (X_1 - d)_+ + \cdots + (X_{1{,}000} - d)_+$

$S_4 = (Y_1 - d)_+ + \cdots + (Y_{2{,}000} - d)_+$

$S = S_1 + S_2 + E(S_3) + E(S_4)$.

$$\tilde{z} = (z-d)_+ \quad , \quad \hat{z} = z - (z-d)_+$$

$$E(\tilde{z}) = E(z-d)_+ = \int_0^\infty (x-d)_+ \lambda e^{-\lambda x} dx$$

$$= \int_d^\infty \underbrace{(x-d)}_{u} \underbrace{\lambda e^{-\lambda x} dx}_{dv}$$

$$= -(x-d) e^{-\lambda x} \Big|_d^\infty + \int_d^\infty e^{-\lambda x} dx$$

$$= -(x-d) e^{-\lambda x} - \frac{1}{\lambda} e^{-\lambda x} \Big|_d^\infty$$

$$= \cancel{-\frac{1}{\lambda} e^{-\lambda d} - \left(d - \frac{1}{\lambda} \right)}$$

$$= + \frac{1}{\lambda} e^{-\lambda d}$$

$$E(\tilde{z}^2) = \int_d^\infty (x-d)^2 \lambda e^{-\lambda x} dx$$

$$= \cancel{-(x-d)^2 e^{-\lambda x} \Big|_d^\infty} + \int_d^\infty 2(x-d) e^{-\lambda x} dx$$

$$= \frac{2}{\lambda} \frac{1}{\lambda} e^{-\lambda d} = \frac{2}{\lambda^2} e^{-\lambda d}$$

$$var(\tilde{z}) = \frac{2}{\lambda^2} e^{-\lambda d} - \frac{1}{\lambda^2} e^{-2\lambda d}$$

$\gamma(\tilde{z})$, $E(\hat{z})$, $var(\hat{z})$, $\gamma(\hat{z})$.