Chapter (5): Collective risk model

* Individual risk model: $X_1, \dots, X_n$ iid.
  total loss $S = X_1 + \dots + X_n$.
* in Collective risk model, we will take the number of risks random.
  $X_1, X_2, \dots\dots$ iid.
  N integer-valued random variable.
  we define the total risk:
  $$S = X_1 + X_2 + \dots + X_N.$$
  we call S a compound random variable.

(1) Compound distribution:

let a function g, then

$$E\,g(S) = E\,g(X_1 + \dots + X_N).$$
$$= \sum_{n=0}^{\infty} E\,g(X_1 + \dots + X_n)\, f_N(n).$$

* $g(x) = x$:
$$E(S) = \sum_{n=0}^{\infty} E(X_1 + \dots + X_n)\, f_N(n).$$
$$= \sum_{n=0}^{\infty} n\, E(X)\, f_N(n) = E(X) \underbrace{\sum_{n=0}^{\infty} n f_N(n)}$$

$$\boxed{E(S) = E(X)\, E(N)}$$

* $g(x) = x^2$:
$$E(S^2) = \sum_{n=0}^{\infty} E(X_1 + \dots + X_n)^2\, f_N(n)$$
$$= \sum_{n=0}^{\infty} \left[Var(X_1 + \dots + X_n) + \left(E(X_1 + \dots + X_n)\right)^2 \right] f_N(n)$$
$$= \sum_{n=0}^{\infty} \left[n\, Var(X) + n^2 (E(X))^2 \right] f_N(n)$$

$$= \text{var}(X)\, E(N) + (E(X))^2\, E(N^2).$$

$$\text{Var}(S) = E(S^2) - (E(S))^2$$

$$= \text{Var}(X)\, E(N) + (E(X))^2 E(N^2) - (E(X))^2 (E(N))^2.$$

$$\boxed{\text{Var}(S) = \text{var}(X)\, E(N) + \text{Var}(N)\,(E(X))^2}$$

- $g(x) = e^{tx}$.

$$m_S(t) = E e^{tS} = \sum_{n=0}^{\infty} E\, e^{t(X_1+\cdots+X_n)}\, f_N(n).$$

$$= \sum_{n=0}^{\infty} (m_X(t))^n f_N(n).$$

$$= \sum_{n=0}^{\infty} e^{n \ln(m_X(t))} f_N(n).$$

$$\boxed{m_S(t) = m_N(\ln(m_X(t)))}$$

Example: $X \sim Exp(\lambda)$; $N \sim Geo(p)$.
Compute the mean, variance, mgf and cdf of S.

- $E(S) = E(X)\, E(N) = \frac{1}{\lambda} \frac{1-p}{p}$.
- $\text{Var}(S) = \text{var}(X)\, E(N) + \text{Var}(N)\, (E(X))^2$.

$$= \frac{1}{\lambda^2} \frac{1-p}{p} + \frac{1-p}{p^2} \frac{1}{\lambda^2}.$$

$$= \frac{1}{\lambda^2} \frac{1-p}{p} \left(1 + \frac{1}{p}\right).$$

- $m_S(t) = m_N(\ln(m_X(t))) = m_N\left(\ln\left(\frac{\lambda}{\lambda - t}\right)\right)$, $t<\lambda$

$$= \frac{p}{1-(1-p)\frac{\lambda}{\lambda-t}} = \frac{p(\lambda-t)}{\lambda-t-(1-p)\lambda}$$

$$m_S(t) = q + (1-q)\boxed{\frac{\beta}{\beta - t}}.$$

$x = 0.$

$m_X(t) = e^0 = 1.$

$$= \frac{q(\beta - t) + (1-q)\beta}{\beta - t}.$$

$$= \frac{p(\lambda - t)}{p\lambda - t} = \frac{p}{p}\,\frac{p(\lambda - \cancel{\beta} t)}{p\lambda - t}$$

$$= \frac{p\lambda - pt}{p\lambda - t} = \frac{p\lambda + p(p\lambda - t) - p^2\lambda}{p\lambda - t}$$

$$= p + \frac{p\lambda - p^2\lambda}{p\lambda - t}$$

$$= p + (1-p)\frac{p\lambda}{p\lambda - t}.$$

S is a p-mixture of $Y \sim \mathrm{Exp}(p\lambda)$ and $Z = 0$.

$$F_S(t) = p\,F_Z(t) + (1-p)\,F_Y(t).$$

$$= p\begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} + (1-p)\begin{cases} (1 - e^{-p\lambda t}) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$= \begin{cases} 0 & t < 0 \\ p + (1-p)(1 - e^{-p\lambda t}). & t \geq 0 \end{cases}$$

Example: Let $X \sim$ logarithmic(c), $N \sim \mathrm{Poi}(\lambda)$.

$X \in \{1, 2, \dots\}$, $f_X(k) = \dfrac{c^k}{k\,h(c)}$, $h(c) = -\ln(1-c)$.

$*$ $E(X) = \displaystyle\sum_{k=1}^{\infty} k\,f_X(k) = \sum_{k=1}^{\infty} \cancel{k}\,\frac{c^k}{\cancel{k}\,h(c)} = \frac{c}{(1-c)h(c)}.$

$*$ $E(X^2) = \displaystyle\sum_{k=1}^{\infty} \frac{k\,c^k}{h(c)} = \frac{c}{(1-c)^2 h(c)}.$

$*$ $m_X(t) = \displaystyle\sum_{k=1}^{\infty} e^{tk}\frac{c^k}{k\,h(c)} = \sum_{k=1}^{\infty}\frac{(e^t c)^k}{k\,h(c)} = \frac{h(ce^t)}{h(c)}$

* $E(S) = E(X)E(N) = \dfrac{c}{(1-c)h(c)} \lambda.$

* $m_S(t) = M_N(\ln(m_X(t))) = e^{\lambda(m_X(t)-1)}$

$= e^{\lambda(h(ce^t)/h(c) - 1)}.$

$= e^{\boxed{-\frac{\lambda}{h(c)}} \ln(1-ce^t)}$

$= e^{-\lambda} (1-ce^t)^{-\lambda/h(c)}$

$= e^{-\lambda} \left(\dfrac{1}{(1-ce^t)}\right)^{\lambda/h(c)}$

$= \left(\dfrac{e^{-h(c)}}{1-ce^t}\right)^{\lambda/h(c)}.$

$= \left(\dfrac{1-c}{1-c e^t}\right)^{\lambda/h(c)}$

$S \sim \text{Neg-Binomial}\left(\underbrace{\frac{\lambda}{h(c)}}_{r}, 1-c\right).$

* $f_S(k) = \binom{r+k-1}{k}(1-c)^r c^k.$

Ex: let $X \sim \text{Unif}(0,1)$; $N \sim \text{Bernoulli}(p)$.

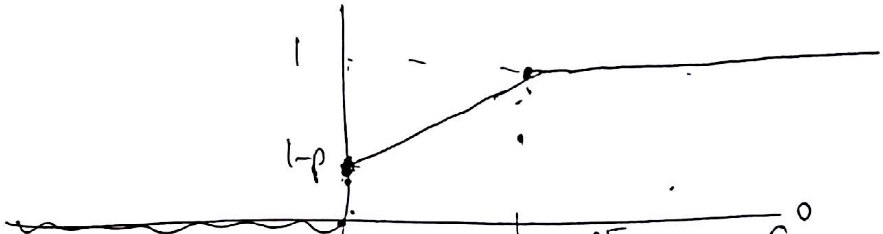
- $E(S) = E(X)E(N) = \frac{1}{2}p.$
- $Var(S) = var(X)E(N) + Var(N)(E(X))^2$
  $= \frac{1}{12}p + p(1-p)\frac{1}{4}.$
- $m_S(t) = m_N(\ln(m_X(t))) = \, ?$
  $m_N(t) = Ee^{tN} = e^t p + (1-p).$; $m_X(t) = \dfrac{e^t-1}{t}.$
  $m_S(t) = p\left(\dfrac{e^t-1}{t}\right) + 1 - p.$

S is a p-mixture of $X \sim Unif(0,1)$ and $Y = 0$.

- $F_S(t) = p\,F_X(t) + (1-p)\,F_Y(t)$.

$$= p\begin{cases} 0 & t<0 \\ t & 0\le t<1 \\ 1 & t\ge 1 \end{cases} + (1-p)\begin{cases} 0 & t<0 \\ 1 & \ge 0 \end{cases}$$

$$= \begin{cases} 0 & t<0 \\ pt+(1-p) & 0\le t<1 \\ 1 & t\ge 1 \end{cases}$$



$$E(S) = \int_{-\infty}^{\infty} t\,dF_S(t) = \int_{-\infty}^{0^-} + \int_{0^-}^{0} + \int_0^1 + \int_1^\infty$$

$$= 0 + 0 + \int_0^1 pt\,dt + 0$$

$$= pt^2/2\Big|_0^1 = \frac{p}{2}.$$

Example: let $X \sim Gamma(\alpha,\beta)$; $N \sim Bin(2,p)$.

* $E(S) = E(X)E(N) = \frac{\alpha}{\beta} * 2p$

* $m_S(t) = m_N(\ln(m_X(t))) = (1-p+p\,m_X(t))^2$

$$= \left(1-p+p\left(\frac{\beta}{\beta-t}\right)^{\alpha}\right)^2, \quad t<\beta.$$

$$= (1-p)^2 + 2(1-p)p\left(\frac{\beta}{\beta-t}\right)^{\alpha} + p^2\left(\frac{\beta}{\beta-t}\right)^{2\alpha}.$$

S is a mixture of $X \sim Gamma(\alpha,\beta)$, $Y \sim Gamma(2\alpha,\beta)$, $Z = 0$.

Example: $X \sim \text{Exp}(\lambda)$; $N \sim$ Neg-Binomial (r, p).

$$S = X_1 + X_2 + \cdots + X_N$$

$E(S)$, $Var(S)$, $m_S(t)$, $F_S(t) = ?$

- $E(S) = E(X)\,E(N) = \frac{1}{\lambda} \cdot \frac{r(1-p)}{p}$.

- $Var(S) = Var(X)\,E(N) + Var(N)\,E(X)^2$

$$= \frac{1}{\lambda^2}\,\frac{r(1-p)}{p} + \frac{r(1-p)}{p^2}\,\frac{1}{\lambda^2}$$

$$= \frac{1}{\lambda^2}\,\frac{r(1-p)}{p}\left(1 + \frac{1}{p}\right).$$

- $m_S(t) = m_N(\ln(m_X(t)))$ $\qquad m_N(t) = \left(\frac{p}{1-qe^t}\right)^r$.

$$= \left(\frac{p}{1-(1-p)\,m_X(t)}\right)^r$$

$$= \left(\frac{p}{1-(1-p)\frac{\lambda}{\lambda-t}}\right)^r. \qquad t < \lambda.$$

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② Convolution formula:

- $X_1, X_2, \dots,$ are iid.
  N independent with the X_i's.

  $S = X_1 + X_2 + \dots + X_N$.

  $N = 0 \longrightarrow S = 0$.

- CDF of S:

$$F_S(t) = P(S \le t) = P(X_1 + \dots + X_N \le t)$$

$$= \sum_{n=0}^{\infty} P(X_1 + \dots + X_n \le t)\, f_N(n).$$

$$\boxed{F_S(t) = \sum_{n=0}^{\infty} F_X^{*n}(t)\, f_N(n).}$$

$$F_X^{*0}(t) = \begin{cases} 0 & t < 0 \\ 1 & t \ge 0. \end{cases}$$

$$F_X^{*1}(t) = F_X(t).$$

$$F_X^{*2}(t) = F_X(t) * F_X(t), \dots$$

Example ① Let $X \sim Exp(\lambda)$; $N \sim Geo(p)$.

$$F_X(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-\lambda t}; & t \ge 0. \end{cases}$$

$$f_N(n) = p \cdot q^n, \quad q = 1 - p, \; n = 0, 1, \dots$$

$$\left\{ \begin{array}{l} Y = X_1 + X_2 + \dots + X_n \sim Gamma(n, \lambda). \\ m_Y(t) = (m_X(t))^n = \left(\frac{\lambda}{\lambda - t}\right)^n. \end{array} \right.$$

$$F_S(t) = \sum_{n=0}^{\infty} F_X^{*n}(t)\, f_N(n).$$

$$= F_X^{*0}(t)\, f_N(0) + \sum_{n=1}^{\infty} F_X^{*n}(t)\, f_N(n).$$

$t \geq 0$ $$= p + \sum_{n=1}^{\infty} \int_0^t f_G(x)\,dx\; pq^n \quad ; \; G \sim \text{Gamma}(n, \lambda)$$

$$= p + \sum_{n=1}^{\infty} \int_0^t \frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x}\,dx\; pq^n.$$

$\boxed{\Gamma(n) = (n-1)!}$ $$= p + \int_0^t \sum_{n=1}^{\infty} \frac{\lambda^n x^{n-1} q^n}{(n-1)!}\, p e^{-\lambda x}\,dx.$$

$j = n-1$ $$= p + \int_0^t \sum_{j=0}^{\infty} \frac{\lambda^{j+1} x^j q^{j+1}}{j!}\, p e^{-\lambda x}\,dx$$

$\boxed{e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}}$ $$= p + \int_0^t \boxed{\sum_{j=0}^{\infty} \frac{(\lambda x q)^j}{j!}}\, \lambda pq\, e^{-\lambda x}\,dx$$

$$= p + \int_0^t e^{\lambda q x}\, \lambda pq\, e^{-\lambda x}\,dx$$

$$= p + \int_0^t \lambda pq\, e^{-\lambda p x}\,dx.$$

$$= p - q e^{-\lambda p x}\Big|_0^t = p - q e^{-\lambda p t} + q$$

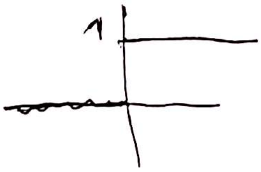
$$= 1 - q e^{-\lambda p t}.$$

Example: $X \sim \text{Poisson}(\lambda)$; $N = \begin{cases} 0 & 0.2 \\ 1 & 0.3 \\ 2 & 0.5 \end{cases}$.

S is discrete random variable.

$$\boxed{f_S(t) = \sum_{n=0}^{\infty} f_X^{*n}(t)\, f_N(t)}$$

$$f_S(t) = f_X^{*0}(t)\, f_N(0) + f_X^{*1}(t)\, f_N(1) + f_X^{*2}(t)\, f_N(2).$$



$$f_X^{*0}(t) = \begin{cases} 1 & t = 0. \\ 0 & t \neq 0. \end{cases}$$

$$f_X^{*1}(t) = f_X(t) = e^{-\lambda} \frac{\lambda^t}{t!}$$

$$f_X^{*2}(t) = e^{-2\lambda} \frac{(2\lambda)^t}{t!}.$$

$t \geq 1$ $$f_S(t) = (0.3)\, e^{-\lambda} \frac{\lambda^t}{t!} + (0.5)\, e^{-2\lambda} \frac{(2\lambda)^t}{t!}.$$

$t = 0$ $$f_S(0) = (0.2) + (0.3)\, e^{-\lambda} + (0.5)\, e^{-2\lambda}.$$

Example: $X = \begin{cases} 1 & 0.4 \\ 2 & 0.6 \end{cases}$; $N = \begin{cases} 0 & 1/10 \\ 1 & 2/10 \\ 2 & 3/10 \\ 3 & 4/10 \end{cases}$.

$$S = X_1 + X_2 + \cdots + X_N.$$

$$X_1 + X_2 + X_3 \rightarrow (2+2+2) = 6.$$

s	f_X^{*0}	f_X^{*1}	f_X^{*2}	f_X^{*3}		f_S
0	1					0.1
1		0.4				0.08
2		0.6	$(0.4)^2$	0		$0.12 + (0.3)(0.4)^2 = 0.16$
3			$2*0.4*0.6$	$(0.4)^3$		.
4			$(0.6)^2$	$2*(0.4)^2(0.6) + (0.6)(0.4)^2$		.
5				$(0.4)(0.6)^2 + (0.6)^2*2*(0.4)$		.
6				$(0.6)^3$		.
n	0	1	2	3		
$f_N(n)$	0.1	0.2	0.3	0.4		

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Example: let $X \sim Gamma(\alpha, \beta)$; $N \sim Bernoulli(p)$.

$$f_S(t) = f_X^{*0}(t) f_N(0) + f_X^{*1}(t) f_N(1).$$

$t>0$:
$$f_S(t) = p \frac{\beta^{\alpha}}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t}.$$

$t=0$:
$$f_S(0) = 1-p + 0.$$

$$F_S(t) = F_X^{*0}(t) f_N(0) + F_X^{*1}(t) f_N(1)$$

$t \geq 0$
$$= \begin{cases} 1-p+p\,F_X(t). & t \geq 0 \\ 0 & t<0. \end{cases}$$

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(3) Compound Poisson distribution:

we suppose that $N \sim Poisson(\lambda)$.

$*$ $E(S) = E(X)\,E(N)$

$$\boxed{E(S) = \lambda\,E(X)}.$$

$*$ $Var(S) = Var(X)\,E(N) + Var(N)\,(E(X))^2$

$$= \lambda\,Var(X) + \lambda\,(E(X))^2$$

$$\boxed{Var(S) = \lambda\,E(X^2)}$$

$*$ $k_3(S) = E(S-E(S))^3 = \lambda\,E(X^3).$

$*$ $m_S(t) = m_N(\ln(m_X(t)))$

$$\boxed{m_S(t) = e^{\lambda(m_X(t)-1)}}$$

Example: let $X \sim$ Bernoulli(p), $N \sim$ Poisson(λ).
Compute $E(S)$, $var(S)$, $K_3(S)$, $m_S(t)$?

$E(S) = \lambda E(X) = \lambda p$.

$var(S) = \lambda E(X^2) = \lambda(pq + p^2) = \lambda p$ / $E(X^n) = p$.

$K_3(S) = \lambda E(X^3) = \lambda p$.

$E(X^3) = p$

$m_S(t) = e^{\lambda(q + pe^t - 1)}$

$= e^{\lambda(pe^t - p)}$

$= e^{\lambda p(e^t - 1)}$

$S \sim$ Poisson (λp).