Chapter (5): Collective risk model

* Individual risk model: $X_1, \dots, X_n$ iid.
  total loss $S = X_1 + \dots + X_n$.
* in Collective risk model, we will take the number of risks random.
  $X_1, X_2, \dots\dots$ iid.
  N integer-valued random variable.
  we define the total risk:
  $$S = X_1 + X_2 + \dots + X_N.$$
  we call S a compound random variable.

(1) Compound distribution:

let a function g, then

$$E\,g(S) = E\,g(X_1 + \dots + X_N).$$
$$= \sum_{n=0}^{\infty} E\,g(X_1 + \dots + X_n)\, f_N(n).$$

* $g(x) = x$:
$$E(S) = \sum_{n=0}^{\infty} E(X_1 + \dots + X_n)\, f_N(n).$$
$$= \sum_{n=0}^{\infty} n\,E(X)\, f_N(n) = E(X) \sum_{n=0}^{\infty} n f_N(n)$$
$$\boxed{E(S) = E(X)\,E(N)}$$

* $g(x) = x^2$:
$$E(S^2) = \sum_{n=0}^{\infty} E(X_1 + \dots + X_n)^2\, f_N(n)$$
$$= \sum_{n=0}^{\infty} \left[Var(X_1 + \dots + X_n) + \left(E(X_1 + \dots + X_n)\right)^2 \right] f_N(n)$$
$$= \sum_{n=0}^{\infty} \left[n\,Var(X) + n^2 (E(X))^2 \right] f_N(n)$$

$$= \operatorname{var}(X)\, E(N) + (E(X))^2\, E(N^2).$$

$$\operatorname{Var}(S) = E(S^2) - (E(S))^2$$

$$= \operatorname{Var}(X)\, E(N) + (E(X))^2\, E(N^2) - (E(X))^2 (E(N))^2.$$

$$\boxed{\operatorname{Var}(S) = \operatorname{var}(X)\, E(N) + \operatorname{Var}(N) (E(X))^2}$$

- $g(x) = e^{tx}.$

$$m_S(t) = E e^{tS} = \sum_{n=0}^{\infty} E e^{t(X_1 + \cdots + X_n)} f_N(n).$$

$$= \sum_{n=0}^{\infty} (m_X(t))^n f_N(n).$$

$$= \sum_{n=0}^{\infty} e^{n \ln(m_X(t))} f_N(n).$$

$$\boxed{m_S(t) = m_N(\ln(m_X(t)))}$$

Example: $X \sim \operatorname{Exp}(\lambda)$; $N \sim \operatorname{Geo}(p)$.
Compute the mean, variance, mgf and cdf of S.

- $E(S) = E(X)\, E(N) = \frac{1}{\lambda} \frac{1-p}{p}.$
- $\operatorname{Var}(S) = \operatorname{var}(X)\, E(N) + \operatorname{Var}(N)\, (E(X))^2.$

$$= \frac{1}{\lambda^2} \frac{1-p}{p} + \frac{1-p}{p^2} \frac{1}{\lambda^2}.$$

$$= \frac{1}{\lambda^2} \frac{1-p}{p} \left(1 + \frac{1}{p}\right).$$

- $m_S(t) = m_N(\ln(m_X(t))) = m_N\left(\ln\left(\frac{\lambda}{\lambda - t}\right)\right),\ t < \lambda$

$$= \frac{p}{1 - (1-p)\frac{\lambda}{\lambda - t}} = \frac{p(\lambda - t)}{\lambda - t - (1-p)\lambda}$$

$$m_S(t) = q + (1-q)\boxed{\frac{\beta}{\beta-t}}$$

$x=0$.

$m_X(t) = e^0 = 1$.

$$= \frac{q(\beta-t)+(1-q)\beta}{\beta-t}$$

$$= \frac{p(\lambda-t)}{p\lambda - t} = \frac{p(\lambda - \cancel{\beta} t)}{p\lambda - t}$$

$$= \frac{p\lambda - pt}{p\lambda - t} = \frac{p\lambda + p(p\lambda - t) - p^2\lambda}{p\lambda - t}$$

$$= p + \frac{p\lambda - p^2\lambda}{p\lambda - t}$$

$$= p + (1-p)\frac{p\lambda}{p\lambda - t}.$$

S is a p-mixture of $Y \sim Exp(p\lambda)$ and $Z=0$.

$$F_S(t) = p\,F_Z(t) + (1-p)\,F_Y(t).$$

$$= p\begin{cases} 1 & t\geq 0 \\ 0 & t<0 \end{cases} + (1-p)\begin{cases} (1-e^{-p\lambda t}) & t \geq 0 \\ 0 & t<0 \end{cases}$$

$$= \begin{cases} 0 & t<0 \\ p + (1-p)(1-e^{-p\lambda t}) & t\geq 0 \end{cases}$$

Example: Let $X \sim$ logarithmic(c), $N \sim Poi(\lambda)$.

$X \in \{1,2,\dots\}$, $f_X(k) = \frac{c^k}{k\,h(c)}$, $h(c) = -\ln(1-c)$.

* $E(X) = \sum_{k=1}^{\infty} k\, f_X(k) = \sum_{k=1}^{\infty} \cancel{k}\frac{c^k}{\cancel{k}\,h(c)} = \frac{c}{(1-c)h(c)}$

* $E(X^2) = \sum_{k=1}^{\infty} \frac{k\,c^k}{h(c)} = \frac{c}{(1-c)^2 h(c)}$.

* $m_X(t) = \sum_{k=1}^{\infty} e^{tk}\frac{c^k}{k\,h(c)} = \sum_{k=1}^{\infty}\frac{(e^t c)^k}{k\,h(c)} = \frac{h(ce^t)}{h(c)}$

* $E(S) = E(X)E(N) = \frac{c}{(1-c)h(c)} \cdot \lambda$.

* $m_S(t) = m_N(\ln(m_X(t))) = e^{\lambda(m_X(t)-1)}$

$$= e^{\lambda\left(h(ce^t)/h(c) - 1\right)}.$$

$$= e^{\boxed{-\frac{\lambda}{h(c)}} \ln(1-ce^t)} e^{-\lambda}$$

$$= e^{-\lambda} (1-ce^t)^{-\lambda/h(c)}$$

$$= e^{-\lambda} \left(\frac{1}{(1-ce^t)}\right)^{\lambda/h(c)}$$

$$= \left(\frac{e^{-h(c)}}{1-ce^t}\right)^{\lambda/h(c)}.$$

$$= \left(\frac{1-c}{1-ce^t}\right)^{\lambda/h(c)}$$

$S \sim \text{Neg-Binomial}\left(\overset{r}{\frac{\lambda}{h(c)}}, 1-c\right)$.

* $f_S(k) = \binom{r+k-1}{k}(1-c)^r c^k$.

Ex: let $X \sim \text{Unif}(0,1)$; $N \sim \text{Bernoulli}(p)$.

- $E(S) = E(X)E(N) = \frac{1}{2}p$.
- $Var(S) = var(X)E(N) + var(N)(E(X))^2$

$$= \frac{1}{12}p + p(1-p)\frac{1}{4}.$$

- $m_S(t) = m_N(\ln(m_X(t))) = \,?$

$m_N(t) = Ee^{tN} = e^t p + (1-p)$; $m_X(t) = \frac{e^t-1}{t}$.

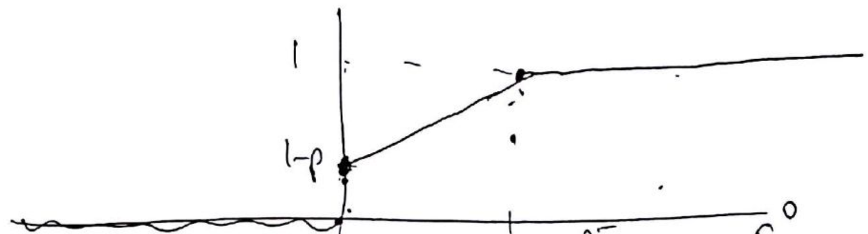
$m_S(t) = p\left(\frac{e^t-1}{t}\right) + 1-p$.

S is a p-mixture of $X \sim \mathrm{Unif}(0,1)$ and $Y = 0$.

$$F_S(t) = p\,F_X(t) + (1-p)\,F_Y(t).$$

$$= p \begin{cases} 0 & t<0 \\ t & 0 \le t < 1 \\ 1 & t \geq 1 \end{cases} + (1-p) \begin{cases} 0 & t<0 \\ 1 & \geq 0 \end{cases}$$

$$= \begin{cases} 0 & t<0 \\ pt + (1-p) & 0 \le t < 1 \\ 1 & t \geq 1 \end{cases}$$



$$E(S) = \int_{-\infty}^{\infty} t\, dF_S(t) = \int_{-\infty}^{0^-} + \int_{0^-}^{0} + \int_{0}^{1} + \int_{1}^{\infty}$$

$$= 0 + 0 + \int_0^1 pt\,dt + 0$$

$$= pt^2/2 \Big|_0^1 = \frac{p}{2}.$$

Example: Let $X \sim \mathrm{Gamma}(\alpha, \beta)$; $N \sim \mathrm{Bin}(2, p)$.

* $E(S) = E(X)\,E(N) = \frac{\alpha}{\beta} * 2p$.

* $m_S(t) = m_N(\ln(m_X(t))) = \left(1-p+p\,m_X(t)\right)^2$

$$= \left(1-p+p\left(\frac{\beta}{\beta-t}\right)^{\alpha}\right)^2, \quad t<\beta.$$

$$= (1-p)^2 + 2(1-p)p\left(\frac{\beta}{\beta-t}\right)^{\alpha} + p^2\left(\frac{\beta}{\beta-t}\right)^{2\alpha}.$$

S is a mixture of $X \sim \mathrm{Gamma}(\alpha,\beta)$, $Y \sim \mathrm{Gamma}(2\alpha,\beta)$, $Z = 0$.

Example: $X \sim Exp(\lambda)$; $N \sim$ Neg-Binomial (r, p).

$$S = X_1 + X_2 + \cdots + X_N$$

$E(S)$, $Var(S)$, $m_S(t)$, $F_S(t) = ?$

- $E(S) = E(X)\,E(N) = \frac{1}{\lambda} \cdot \frac{r(1-p)}{p}$.

- $Var(S) = Var(X)\,E(N) + Var(N)\,E(X)^2$

$$= \frac{1}{\lambda^2}\,\frac{r(1-p)}{p} + \frac{r(1-p)}{p^2}\,\frac{1}{\lambda^2}.$$

$$= \frac{1}{\lambda^2}\,\frac{r(1-p)}{p}\left(1 + \frac{1}{p}\right).$$

- $m_S(t) = m_N(\ln(m_X(t)))$ $\qquad m_N(t) = \left(\frac{p}{1-qe^t}\right)^r.$

$$= \left(\frac{p}{1-(1-p)m_X(t)}\right)^r$$

$$= \left(\frac{p}{1-(1-p)\frac{\lambda}{\lambda - t}}\right)^r. \qquad t < \lambda.$$

② Convolution formula:

- $X_1, X_2, \ldots,$ are iid.

  N independent with the X_i's.

  $S = X_1 + X_2 + \cdots + X_N$.

  $N=0 \longrightarrow S=0$.

- CDF of S:

$$F_S(t) = P(S \le t) = P(X_1 + \cdots + X_N \le t)$$

$$= \sum_{n=0}^{\infty} P(X_1 + \cdots + X_n \le t)\, f_N(n).$$

$$\boxed{F_S(t) = \sum_{n=0}^{\infty} F_X^{*n}(t)\, f_N(n).}$$

$$F_X^{*0}(t) = \begin{cases} 0 & t<0 \\ 1 & t \geqslant 0. \end{cases}$$

$$F_X^{*1}(t) = F_X(t).$$

$$F_X^{*2}(t) = F_X(t) * F_X(t), \ldots$$

Example ① Let $X \sim Exp(\lambda)$; $N \sim Geo(p)$.

$$F_X(t) = \begin{cases} 0 & t<0 \\ 1-e^{-\lambda t}; & t \geqslant 0. \end{cases}$$

$$f_N(n) = p \cdot q^n, \quad q = 1-p, \; n = 0, 1, \ldots$$

$Y = X_1 + X_2 + \cdots + X_n \sim Gamma(n, \lambda)$.

$$m_Y(t) = (m_X(t))^n = \left(\frac{\lambda}{\lambda - t}\right)^n.$$

$$F_S(t) = \sum_{n=0}^{\infty} F_X^{*n}(t)\, f_N(n).$$

$$= F_X^{*0}(t)\, f_N(0) + \sum_{n=1}^{\infty} F_X^{*n}(t)\, f_N(n).$$

$t \geq 0$

$$= p + \sum_{n=1}^{\infty} \int_0^t f_G(x)\, dx \; pq^n \quad ; \; G \sim \text{Gamma}(n, \lambda)$$

$$= p + \sum_{n=1}^{\infty} \int_0^t \frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x} dx \; pq^n.$$

$\boxed{\Gamma(n) = (n-1)!}$

$$= p + \int_0^t \sum_{n=1}^{\infty} \frac{\lambda^n x^{n-1} q^n}{(n-1)!} p e^{-\lambda x} dx.$$

$j = n-1$

$$= p + \int_0^t \sum_{j=0}^{\infty} \frac{\lambda^{j+1} x^j q^{j+1}}{j!} p e^{-\lambda x} dx$$

$\boxed{e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}}$

$$= p + \int_0^t \boxed{\sum_{j=0}^{\infty} \frac{(\lambda x q)^j}{j!}} \lambda pq e^{-\lambda x} dx$$

$$= p + \int_0^t e^{\lambda q x} \lambda pq\, e^{-\lambda x} dx$$

$$= p + \int_0^t \lambda pq\, e^{-\lambda p x} dx.$$

$$= p - q e^{-\lambda p x}\Big|_0^t = p - q e^{-\lambda p t} + q$$

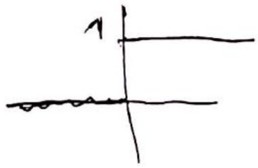
$$= 1 - q e^{-\lambda p t}.$$

Example: $X \sim \text{Poisson}(\lambda)$; $N = \begin{cases} 0 & 0.2 \\ 1 & 0.3 \\ 2 & 0.5 \end{cases}$.

S is discrete random variable.

$$\boxed{f_S(t) = \sum_{n=0}^{\infty} f_X^{*n}(t)\, f_N(t)}$$

$$f_S(t) = f_X^{*0}(t)\, f_N(0) + f_X^{*1}(t)\, f_N(1) + f_X^{*2}(t)\, f_N(2).$$



$$f_X^{*0}(t) = \begin{cases} 1 & t = 0. \\ 0 & t \neq 0. \end{cases}$$

$$f_X^{*1}(t) = f_X(t) = e^{-\lambda} \frac{\lambda^t}{t!}$$

$$f_X^{*2}(t) = e^{-2\lambda} \frac{(2\lambda)^t}{t!}.$$

$t \geq 1$ $\quad f_S(t) = (0.3)\, e^{-\lambda} \dfrac{\lambda^t}{t!} + (0.5)\, e^{-2\lambda} \dfrac{(2\lambda)^t}{t!}.$

$t = 0$ $\quad f_S(0) = (0.2) + (0.3)\, e^{-\lambda} + (0.5)\, e^{-2\lambda}.$

Example : $X = \begin{cases} 1 & 0.4 \\ 2 & 0.6 \end{cases}$; $N = \begin{cases} 0 & 1/10 \\ 1 & 2/10 \\ 2 & 3/10 \\ 3 & 4/10 \end{cases}.$

$$S = X_1 + X_2 + \cdots + X_N.$$

$X_1 + X_2 + X_3 \rightarrow (2+2+2) = 6.$

s	f_X^{*0}	f_X^{*1}	f_X^{*2}	f_X^{*3}	f_S
0	1				0.1
1		0.4			0.08
2		0.6	$(0.4)^2$	0	$0.12 + (0.3)(0.4)^2 = 0.16?$
3			$2*0.4*0.6$	$(0.4)^3$	.
4			$(0.6)^2$	$2*(0.4)^2(0.6) + (0.6)(0.4)^2$	.
5				$(0.4)(0.6)^2 + (0.6)^2 * 2 * (0.4)$	.
6				$(0.6)^3$	.

n	0	1	2	3
$f_N(n)$	0.1	0.2	0.3	0.4

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Example: let $X \sim \text{Gamma}(\alpha, \beta)$; $N \sim \text{Bernoulli}(p)$.

$$f_S(t) = f_X^{*0}(t) f_N(0) + f_X^{*1}(t) f_N(1).$$

t>0:

$$f_S(t) = p \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t}.$$

t=0 : $f_S(0) = 1-p + 0$.

$$F_S(t) = F_X^{*0}(t) f_N(0) + F_X^{*1}(t) f_N(1)$$

$t \geq 0$

$$= \begin{cases} 1-p + p F_X(t). & t \geq 0 \\ 0 & t < 0. \end{cases}$$

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3 Compound Poisson distribution:

we suppose that $N \sim \text{Poisson}(\lambda)$.

* $E(S) = E(X) E(N)$

$$\boxed{E(S) = \lambda E(X)}.$$

* $Var(S) = Var(X) E(N) + Var(N) (E(X))^2$

$$= \lambda Var(X) + \lambda (E(X))^2$$

$$\boxed{Var(S) = \lambda E(X^2)}$$

* $k_3(S) = E(S - E(S))^3 = \lambda E(X^3)$.

* $m_S(t) = m_N(\ln(m_X(t)))$

$$\boxed{m_S(t) = e^{\lambda(m_X(t) - 1)}}$$

Example: let $X \sim \text{Bernoulli}(p)$, $N \sim \text{Poisson}(\lambda)$.
Compute $E(S)$, $var(S)$, $K_3(S)$, $m_S(t)$?

$$E(S) = \lambda E(X) = \lambda p .$$
$$var(S) = \lambda E(X^2) = \lambda(pq + p^2) = \lambda p \qquad / \; E(X^n) = p .$$
$$K_3(S) = \lambda E(X^3) = \lambda p .$$
$$E(X^3) = p$$
$$m_S(t) = e^{\lambda(q + pe^t - 1)}$$
$$= e^{\lambda(pe^t - p)}$$
$$= e^{\lambda p(e^t - 1)}$$

$S \sim \text{Poisson}(\lambda p)$.

- Let $S_1, S_2, \ldots, S_n$ be independent compound Poisson distributions with respective Poisson distributions parameters $\lambda_1, \ldots, \lambda_n$ and claim distributions $F_1, \ldots, F_n$. Let $S = S_1 + S_2 + \cdots + S_n$. Then S is a compound Poisson distribution with parameter $\lambda = \lambda_1 + \cdots + \lambda_n$ and claim distribution $F(x) = \frac{\lambda_1}{\lambda} F_1(x) + \cdots + \frac{\lambda_n}{\lambda} F_n(x)$.

<u>Proof</u>: $n = 2$:

$$m_S(t) = m_{S_1+S_2}(t) = m_{S_1}(t)\, m_{S_2}(t)$$
$$= e^{\lambda_1(m_X(t) - 1)}\, e^{\lambda_2(m_Y(t) - 1)}$$
$$= e^{\lambda_1(m_X(t) - 1) + \lambda_2(m_Y(t) - 1)}$$
$$\lambda = \lambda_1 + \lambda_2 \qquad = e^{\lambda\left(\frac{\lambda_1}{\lambda} m_X(t) + \frac{\lambda_2}{\lambda} m_Y(t) - 1\right)}$$
$$F(x) = \frac{\lambda_1}{\lambda} F_1(x) + \frac{\lambda_2}{\lambda} F_2(x).$$

- Sparse vector algorithm: Assume S is a Compound Poisson distribution with parameter λ and claim distribution, taking values $x_1, \dots, x_m$ with Probabilities $\pi(x_1), \dots, \pi(x_m)$. Then:

$$S = x_1 N_1 + x_2 N_2 + \dots + x_m N_m,$$

where $N_1, \dots, N_m$ are independent Poisson random variables with parameters $\lambda_1, \dots, \lambda_m$, where $\lambda_i = \lambda \pi(x_i)$.

Example: let $S \sim$ Comp. Poisson (λ), $\lambda = 4$,

$X = 1, 2, 3$, $f_X(1) = \frac{1}{4}$, $f_X(2) = \frac{1}{2}$, $f_X(3) = \frac{1}{4}$.

Compute $f_S(s)$ for $s = 0, 1, 2, 3$.

$$S = N_1 + 2N_2 + 3N_3, \quad N_1 \sim Poi(1)$$

$$N_2 \sim Poi(2)$$

$N \sim Poi(\delta)$, $P(N=x) = e^{-\delta} \frac{\delta^x}{x!}$ $\quad N_3 \sim Poi(1)$

$$f_S(0) = P(S=0) = P(N_1=0;\ N_2=0;\ N_3=0).$$

$$= P(N_1=0)\,P(N_2=0)\,P(N_3=0)$$

$$= e^{-1} \cdot e^{-2} \cdot e^{-1} = e^{-4}.$$

$$f_S(1) = P(S=1) = P((1,0,0)) = e^{-1}\,e^{-2}\,e^{-1} = e^{-4}.$$

$$f_S(2) = P(S=2) = P((2,0,0),(0,1,0))$$

$$= \frac{1}{2}e^{-1}\,e^{-2}\,e^{-1} + e^{-1}\,2e^{-2}\,e^{-1} = 2.5\,e^{-4}.$$

$$f_S(3) = P(S=3) = P((3,0,0),(1,1,0),(0,0,1))$$

$$= \left(\frac{1}{6} + 2 + 1\right)e^{-4} = \frac{19}{6}e^{-4}$$

Ex: $S \sim$ Comp. Poisson (6),

$f_X(1) = \frac{1}{6}$; $f_X(3) = \frac{1}{3}$, $f_X(4) = \frac{1}{2}$.

Compute $f_S(s)$ for $s = 0, 1, 10$.

$S = N_1 + 3N_2 + 4N_3$.

$N_1 \sim Poi(1)$

$N_2 \sim Poi(2)$

$N_3 \sim Poi(3)$.

$f_S(0) = P(0,0,0) = e^{-6}$.

$f_S(1) = P(1,0,0) = e^{-6}$

$$f_S(10) = P\big((10,0,0), (7,1,0), (6,0,1), (4,2,0), (3,1,1), (2,0,2), (0,2,1), (1,3,0)\big).$$

$$= e^{-6}\left(\frac{1}{10!} + \frac{2}{7!} + \frac{3}{6!} + \frac{1}{4!}2 + \frac{1}{3!}2*3 + \frac{1}{2}\frac{9}{2} + \frac{4}{2}*3 + \frac{8}{3!}\right) =$$

$$= e^{-6}\left(0 + 0 + 0 + \frac{1}{12} + 1 + \frac{9}{4} + 6 + \frac{8}{6}\right)$$

$= 10.5\, e^{-6}$.

④ Panjer recursion:

Consider a Compound distribution with with x taking integer-values, $p(x) = f_X(x)$ and $q_n = f_N(n)$.

we suppose that:

$$\frac{q_{n}}{q_{n-1}} = a + \frac{b}{n} \quad \text{for } n = 1, 2, \ldots \text{ and } a, b \in \mathbb{R}.$$

The the mass function of $S = X_1 + \cdots + X_N$, is given by:

$$f(0) = \begin{cases} q_0 & \text{if } p(0) = 0 \\ m_N(\ln(p(0))) & \text{if } p(0) > 0. \end{cases}$$

for $s = 1, 2, \ldots$

$$f(s) = \frac{1}{1 - a\,p(0)} \sum_{h=1}^{s} \left(a + \frac{bh}{s}\right) p(h)\, f(s-h).$$

Example: Compute the panjer recursive formula for:

(1) $N \sim \text{Poisson}(\lambda)$:

$$q_n = e^{-\lambda} \frac{\lambda^n}{n!}.$$

$$\frac{q_{n+1}}{q_n} = \frac{\cancel{e^{-\lambda}} \frac{\lambda^{n+1}}{(n+1)!}}{\cancel{e^{-\lambda}} \frac{\lambda^n}{n!}} = \frac{\lambda}{n+1}$$

$$\frac{q_n}{q_{n-1}} = \frac{\lambda}{n} \quad ; \quad a = 0 \; ; \; b = \lambda.$$

$s = 1, 2, \ldots$

$$f(s) = \sum_{h=1}^{s} \frac{\lambda h}{s}\, p(h)\, f(s-h).$$

Suppose now that $X \in \begin{cases} 1 & 0.4 \\ 2 & 0.1 \\ 3 & 0.5 \end{cases}$

$$f(0) = q_0 = e^{-\lambda}$$

$$f(1) = \lambda\, p(1) f(0) = 0.4 \lambda e^{-\lambda}$$

$$f(2) = \frac{\lambda}{2} p(1) f(1) + \lambda\, p(2) f(0)$$

$$= \frac{\lambda}{2} (0.4)(0.4 \lambda e^{-\lambda}) + \lambda (0.1) e^{-\lambda}$$

$$= 0.08 \lambda^2 e^{-\lambda} + 0.1 \lambda e^{-\lambda}$$

$$f(3) = \frac{\lambda}{3} p(1) f(2) + \frac{2\lambda}{3} p(2) f(1) + \lambda\, p(3) f(0)$$

$$= \frac{\lambda}{3} (0.4) (0.08 \lambda^2 e^{-\lambda} + 0.1 \lambda e^{-\lambda}) + \frac{2\lambda}{3} (0.1)(0.4 \lambda e^{-\lambda}) + \lambda (0.5) e^{-\lambda} = \rule{2cm}{0.4pt}$$

$s = 4, 5, \ldots$

$$f(s) = \sum_{h=1}^{3} \frac{\lambda h}{s} p(h) f(s-h)$$

$$= \frac{\lambda}{s} (0.4) f(s-1) + \frac{2\lambda}{s} (0.1) f(s-2) + \frac{3\lambda}{s} (0.5) f(s-3).$$

(a) $N \sim \text{Binomial}(m, p)$.

? $\frac{\cancel{(m-n+1)}!}{\cancel{(m-n)}!}$

$$q_n = \binom{m}{n} p^n q^{m-n}.$$

$$\frac{q_n}{q_{n-1}} = \frac{\binom{m}{n} p^n q^{m-n}}{\binom{m}{n-1} p^{n-1} q^{m-n+1}} = \frac{(m-n+1)}{n} \frac{p}{q}$$

$$= \frac{p}{q} \left(\frac{(m+1)}{n} - 1 \right) = -\frac{p}{q} + \frac{p/q\,(m+1)}{n}.$$

$$a = -p/q \;; \quad b = \frac{p}{q}(m+1).$$

$s = 1, 2, \ldots$

$$f(s) = \frac{1}{1 + \frac{p}{q} p(0)} \sum_{h=1}^{s} \left(-\frac{p}{q} + \frac{p}{q}(m+1) \frac{h}{s} \right) p(h) f(s-h)$$

Suppose now that:

$$X = \begin{cases} 0 & 0.2 \\ 10 & 0.8 \end{cases}$$

$$f(0) = (0.2p + 1 - p)^m. \qquad m_N(t) = (pe^t + 1 - p)^m.$$

$$f(1) = \cdots = f(9) = 0.$$

$$f(10) = \frac{1}{1 + \frac{p}{q}(0.2)} \left(-\frac{p}{q} + \frac{p}{q}(m+1) \right) p(10)\, f(0).$$

$s = 11, 12, \ldots$

$$f(s) = \frac{1}{1 + \frac{p}{q}(0.2)} \left(-\frac{p}{q} + \frac{p}{q}(m+1)\frac{10}{s} \right) p(10)\, f(s-10).$$

? (3) $N \sim$ Neg-Binomial (r, p).

$$q_n = \binom{r+n-1}{n} p^r q^n$$

$$\frac{q_n}{q_{n-1}} = \frac{\binom{r+n-1}{n} \cancel{p^r} q^n}{\binom{r+n-2}{n-1} \cancel{p^r} q^{n-1}} = \frac{r+n-1}{n} q$$

$$= q\left(1 + \frac{r-1}{n}\right) = q + \frac{q(r-1)}{n}.$$

$$a = q; \quad b = q(r-1).$$

$s = 1, 2, \ldots$

$$f(s) = \frac{1}{1 - q\,p(0)} \sum_{h=1}^{s} \left(q + \frac{q(r-1)h}{s}\right) p(h)\, f(s-h).$$

Suppose now $X = \begin{cases} 1 & 0.5 \\ 2 & 0.5 \end{cases}$?

$$f(0) = q_0 = p^r.$$

$$f(1) = (q + q(r-1))\, p(1)\, f(0) = qr(0.5)p^r.$$

$$f(2) = \left(q + \frac{q(r-1)}{2}\right) p(1)\, f(1) + \left(q + q(r-1)\right) p(2)\, f(0)$$

$s = 3, 4, \ldots$

$$f(s) = \left(q + \frac{q(r-1)}{s}\right) p(1)\, f(s-1) + \left(q + \frac{q(r-1)2}{s}\right) p(2)\, f(s-2)$$

(5) Application to stop-loss premium:

For an integer-valued random variable S, we define the stop-loss premium by:

$$\pi(d) = E(S-d)_+ .$$

Then: $$\pi(d) = \sum_{n=0}^{\infty} (n-d)_+ f_S(n).$$

d is an integer: $$= \sum_{n=d}^{\infty} (n-d) f_S(n)$$

$$= \sum_{n=d}^{\infty} \left[(n-(d-1)) f_S(n) - f_S(n) \right].$$

$$= \pi(d-1) - (1 - F_S(d-1)).$$

$$\pi(d) = \pi(d-1) - 1 + F_S(d-1).$$

Ex: let $S \sim$ Comp-Poisson (1), $X = \begin{cases} 1 & 0.4 \\ 2 & 0.1 \\ 3 & 0.5 \end{cases}$.

Compute $\pi(4)$? $\quad \pi(0) = E(S) = 2.1$

s	$f_S(s)$	$F_S(s)$ (e^{-1})	$\pi(s)$
			2.1
0	e^{-1}	$1\ e^{-1}$	$1.1 + e^{-1}$
1	$0.4\, e^{-1}$	$1.4\ e^{-1}$	$0.1 + 2.4\, e^{-1}$
2	$0.18\, e^{-1}$	$1.58\, e^{-1}$	$3.98\, e^{-1} - 0.9$
3	$0.529\, e^{-1}$	$2.109\, e^{-1}$	$6.1\, e^{-1} - 1.9 = \pi(4)$.

(6) Approximation for Compound Poisson distribution

let $S^\lambda \sim$ Compound-Poisson (λ).

For large values of λ, we have:

$$\frac{S^\lambda - E(S^\lambda)}{\sigma(S^\lambda)} \sim N(0,1).$$

Example: Suppose $S \sim$ Comp. Poisson (12) and $X \sim$ Unif (0,1).
Approximate $P(S<10)$ using the normal approximation.

$$E(S) = \lambda E(X) = 12 * \frac{1}{2} = 6.$$
$$\sigma^2(S) = \lambda E(X^2) = 12\left(\frac{1}{4} + \frac{1}{12}\right) = 4.$$
$$\sigma(S) = 2.$$
$$P(S<10) = P\left(\frac{S-6}{2} < \frac{10-6}{2}\right)$$
$$\simeq \phi(2) = 0.9772.$$

Example: Suppose $S \sim$ Comp. Poisson ($\lambda = 20$).

$$X = \begin{cases} 1 & 0.4 \\ 2 & 0.1 \\ 3 & 0.5 \end{cases}.$$

Approximate (and calculate) $P(S \leq 3)$?

$$P(S \leq 3) = 2.109\, e^{-1} = 0.776.$$

$\longrightarrow$ $E(X) = 2.1$ $\quad E(S) = 20 * 2.1 = 42$

$\longrightarrow$ $E(X^2) = 5.3$ $\quad Var(S) = 20 * 5.3 = 106$

$\quad \sigma(S) = \sqrt{106} = 10.29$

$$P(S \leq 3) = P\left(\frac{S-42}{10.3} \leq \frac{3-42}{10.3}\right)$$
$$\simeq \phi\left(-\frac{39}{10.3}\right) = \phi(-3.786) = 0.00008.$$