Chapter (4): Generating new distributions:

(1) Scalar multiplication:

let X be a continuous random variable.
let c a positive real number.
let $Y = cX$. with c a positive

- $F_Y(y) = P(Y \le y) = P(cX \le y) \le P(X \le \frac{y}{c})$
  $= F_X(\frac{y}{c})$.
- $f_Y(y) = \frac{1}{c} f_X(y/c)$.

Example: let X with Pareto distribution with parameters α and θ:

$$F_X(x) = 1 - \theta^\alpha (x+\theta)^{-\alpha} \quad ; \quad x \ge 0.$$

Find the distribution of $Y = cX$; $c > 0$.

$$\rightarrow \quad F_Y(y) = F_X(\tfrac{y}{c}) = 1 - \theta^\alpha (\tfrac{y}{c} + \theta)^{-\alpha}$$
$$= 1 - \theta^\alpha \frac{(y + c\theta)^{-\alpha}}{c^{-\alpha}}.$$
$$= 1 - (c\theta)^\alpha (y + c\theta)^{-\alpha}.$$

τ

$\rightarrow$ Y has a Pareto distribution with parameters α and $c\theta$.

(2) Powers of random variables:

let X be a continuous random variable.
Let $Y = X^{1/\tau}$ or $Y = X^{-1/\tau}$.

- $Y = X^{1/\tau}$ is called transformed (of X).
- $Y = X^{-1/\tau}$ is called inverse transformed (of X).
- $Y = X^{-1}$ is called the inverse of X.

- $Y = X^{1/\tau}$:

$$F_Y(y) = P(X^{1/\tau} \le y) = P(X \le y^{\tau}) = F_X(y^{\tau}).$$

$$f_Y(y) = \tau y^{\tau-1} f_X(y^{\tau}).$$

($x^{-\tau}$ is decreasing)

- $Y = X^{-1/\tau}$:

$$F_Y(y) = P(X^{-1/\tau} \le y) = P(X \ge y^{-\tau})$$

$$= 1 - F_X(y^{-\tau}).$$

$$f_Y(y) = \tau y^{-\tau-1} f_X(y^{-\tau}).$$

- $Y = X^{-1}$: $F_Y(y) = 1 - F_X(y^{-1})$

$$f_Y(y) = y^{-2} f_X(y^{-1}).$$

Example: let $X \sim Gamma(\alpha, \beta)$.

$$f_X(x) = \frac{\Gamma(\alpha)}{\beta^{\alpha}} x^{\alpha-1} e^{-\beta x} \; ; \; x \ge 0.$$

- Gamma-transformed:

$$f_Y(y) = \tau y^{\tau-1} \frac{\Gamma(\alpha)}{\beta^{\alpha}} (y^{\tau})^{\alpha-1} e^{-\beta y^{\tau}}, \; y \ge 0$$

$$= \frac{\tau \Gamma(\alpha)}{\beta^{\alpha}} y^{\tau\alpha-1} e^{-\beta y^{\tau}}.$$

- Gamma-inverse-transformed:

$$f_Y(y) = \tau y^{-\tau-1} \frac{\Gamma(\alpha)}{\beta^{\alpha}} (y^{-\tau})^{\alpha-1} e^{-\beta y^{-\tau}} \; ; \; y > 0.$$

$$= \tau \frac{\Gamma(\alpha)}{\beta^{\alpha}} y^{-\tau\alpha-1} e^{-\beta y^{-\tau}}.$$

- Inverse-Gamma:

$$f_Y(y) = \frac{\Gamma(\alpha)}{\beta^{\alpha}} y^{-\alpha-1} e^{-\beta y^{-1}}, \; y \ge 0.$$

③ Exponentiation of random variables:

Let X be a continuous random variable.

Let $Y = e^X$.

- $F_Y(y) = P(e^X \le y) = P(X \le \ln(y))$; $y>0$.
$= F_X(\ln(y))$.

- $f_Y(y) = \frac{1}{y} f_X(\ln(y))$.

Example: $X \sim N(\mu, \sigma^2)$; $Y = e^X$.

1) $f_Y(y) = \frac{1}{y}\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln(y)-\mu)^2}{2\sigma^2}}$; $y>0$.

2) $E(Y) = Ee^X = m_X(1) = e^{\mu+\sigma^2/2}$

$E(Y^2) = Ee^{2X} = m_X(2) = e^{2\mu+2\sigma^2}$

$Var(Y) = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$.

④ Limiting distributions:

Let X^α be a continuous random variable.

Let Y: $F_Y(y) = \lim_{\alpha\to\alpha_0} F_{X^\alpha}(y)$.

Example: $X \sim \text{Binomial}(n,p)$. $e^{\lambda(e^t-1)}$

$m_X(t) = (1-p+pe^t)^n$.

$= (1+p(e^t-1))^n$.

$= (1+\frac{\lambda}{n}(e^t-1))^n \longrightarrow 1^\infty$ not defined.

$\lambda = np \longrightarrow$

$\ln(m_X(t)) = \frac{\ln(1+\frac{\lambda}{n}(e^t-1))}{1/n}$

$\simeq \frac{\lambda(e^t-1)}{1+\frac{\lambda}{n}(e^t-1)} \longrightarrow \lambda(e^t-1)$

$m_X(t) \longrightarrow e^{\lambda(e^t-1)} \sim \text{Poisson}(\lambda)$.

⑤ Linear exponential family:

- A random variable X belongs to the linear exponential family if the pdf of X is in the form:

$$f_X(x,\theta) = \frac{p(x)}{q(\theta)} e^{r(\theta)x}.$$

Example: $X \sim Bin(n,p)$.

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}.$$

$$= \binom{n}{x} e^{\ln(p)x} e^{\ln(1-p)(n-x)}$$

$$= \binom{n}{x} (1-p)^n e^{(\ln(p)-\ln(1-p))x}.$$

$$= \frac{\overbrace{\binom{n}{x}}^{p(x)}}{\underbrace{(1-p)^{-n}}_{q(p)}} e^{\overbrace{(\ln(p)-\ln(1-p))}^{r(p)}x}.$$

Example: $X \sim Gamma(\alpha,\beta)$.

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}.$$

$$p(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^{-\alpha}} e^{-\beta x}$$

$r(\beta)$

$q(\beta)$

- $E(X) = \dfrac{q'(\theta)}{r'(\theta)q(\theta)} = \mu(\theta)$; $Var(X) = \dfrac{\mu'(\theta)}{r'(\theta)}$.

Example: $X \sim Bin(n,p)$.

$q(p) = (1-p)^{-n}$; $r(p) = \ln(p) - \ln(1-p)$

$q'(p) = n(1-p)^{-n-1}$; $r'(p) = \frac{1}{p} + \frac{1}{1-p}$.

$$E(X) = \frac{n(1-p)^{-1}}{\frac{1}{p}+\frac{1}{p-1}} = \frac{n}{\frac{1-p}{p}+1} = \frac{np}{1-p+p} = np.$$

(6) The $(a, b, 0)$ class of discrete distributions

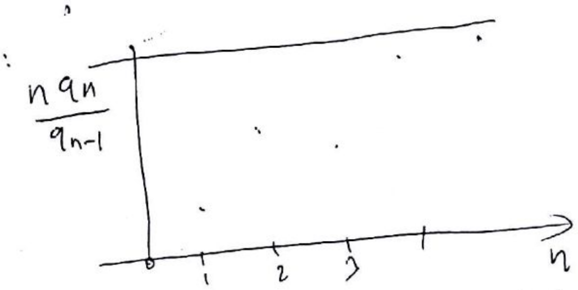
- An element of $(a, b, 0)$ class is any distribution satisfying:

$$\frac{q_n}{q_{n-1}} = a + \frac{b}{n}, \quad n = 1, 2, \ldots$$

Distribution	a	b
Poisson	0	λ
Neg. Binomial	q	$(r-1)\,q$
Geometric	$\frac{\beta}{1+\beta}$	0
Binomial	$-\frac{q}{1-q}$	$(m+1)\frac{q}{1-q}$

- $$n \frac{q_n}{q_{n-1}} = a n + b.$$

Ex:



Example: let N has a distribution:

$$n \frac{q_n}{q_{n-1}} = \frac{3}{4} n + 3, \quad n = 1, 2, \ldots$$

Determine the distribution of N?

$$\tfrac{3}{4} = q = 1 - p, \quad 3 = (r-1) q.$$

$$p = 1/4, \quad r = 5.$$

$$N \sim \text{Neg-Binomial}(5, 1/4).$$

(7) Zero-modified distribution:

Let N be a discrete random variable with $p_n = P(N = n)$, $n = 0, 1, 2, \ldots$

We define the zero-modified distribution by:

p_0^M is given; $p_n^M = c\, p_n$ for $n = 1, 2, \ldots$

$$1 = p_0^M + \sum_{n=1}^{\infty} p_n^M = p_0^M + \sum_{n=1}^{\infty} c\, p_n$$
$$= p_0^M + c \sum_{n=1}^{\infty} p_n = p_0^M + c\,(1 - p_0)$$
$$\boxed{c = \frac{1 - p_0^M}{1 - p_0}}.$$

Example: let $N \sim \text{Poisson}(\lambda)$. Define N^M to be the zero-modified Poisson random variable, with $p_0^M = \alpha$.

a) Compute p_n^M, $n = 1, 2, \cdots$

b) Compute $E(N^M)$.

c) Compute the mgf of N^M.

---

a) $p_n = e^{-\lambda} \frac{\lambda^n}{n!}$; $p_0 = e^{-\lambda}$.
$$p_n^M = c\, p_n = \frac{1 - p_0^M}{1 - p_0} p_n = \frac{1-\alpha}{1 - e^{-\lambda}} e^{-\lambda} \frac{\lambda^n}{n!}, \; n \geq 1.$$

b) $$E(N^M) = \sum_{n=0}^{\infty} n\, p_n^M = 0 + \sum_{n=1}^{\infty} n\, c\, p_n.$$
$$= c \sum_{n\geq 1} n\, p_n = c\, E(N) = c\lambda$$
$$= \frac{1-\alpha}{1 - e^{-\lambda}} \lambda.$$

c) $$m_{N^M}(t) = E\, e^{tN^M} = \sum_{n=0}^{\infty} e^{tn} p_n^M$$
$$= p_0^M + \sum_{n=1}^{\infty} e^{tn} c\, p_n = p_0^M + c \sum_{n\geq 1} e^{tn} p_n$$
$$= \alpha + c \left(\sum_{n\geq 0} e^{tn} p_n - p_0 \right).$$
$$= \alpha + c\,(m_N(t) - p_0)$$
$$= \alpha + c\left(e^{\lambda(e^t - 1)} - p_0 \right).$$
$$= \alpha + \frac{1-\alpha}{1 - e^{-\lambda}} \left(e^{\lambda(e^t-1)} - e^{-\lambda} \right).$$

d) $$E(N^M)^2 = - - - - = c\, E(N^2) = c(\lambda + \lambda^2)$$
$$Var(N^M) = c(\lambda + \lambda^2) - c^2 \lambda^2.$$

⑧ Zero-truncated distribution:

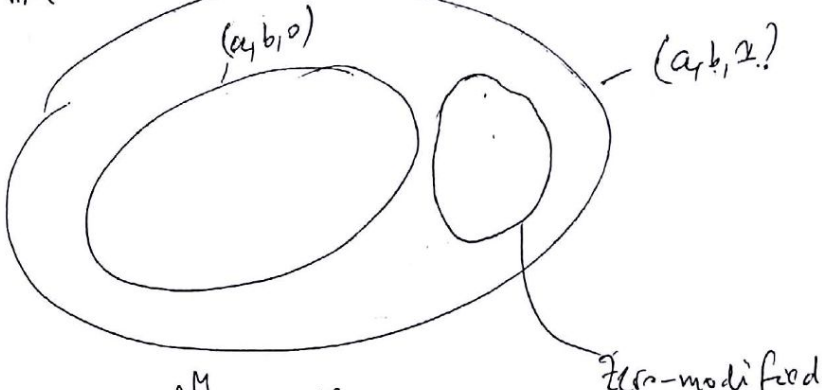
A zero-truncated distribution (p^T), associated to a distribution (p) is defined by:

$$p_0^T = 0 \ ; \quad p_n^T = \frac{1}{1-p_0}\, p_n \ , \ n \geq 1 .$$

⑨ The $(a,b,1)$-class of distributions:

An element of the $(a,b,1)$-class distribution is any distribution satisfying:

$$\frac{q_n}{q_{n-1}} = a + \frac{b}{n} \ , \qquad n = 2, 3, \dots$$



$$n \geq 2 \quad p_n^M = c\, p_n \ , \quad \frac{p_n^M}{p_{n-1}^M} = \frac{\cancel{c}\, p_n}{\cancel{c}\, p_{n-1}} = a + \frac{b}{n} .$$

⑩ Extended truncated negative-Binomial distribution

$$p_0 = 0 \ ;$$

$$n \geq 2 \qquad p_n = \left(a + \frac{b}{n}\right) p_{n-1} \ , \qquad a = q = 1-p, \quad b = (r-1)q .$$

$$p_1 = \ ?$$

$$1 = \sum_{n=0}^{\infty} p_n = p_1 + \sum_{n=2}^{\infty} \left[\left(a+\frac{b}{n}\right)\left(a+\frac{b}{n-1}\right)\cdots\left(a+\frac{b}{2}\right)\right] p_1 .$$

$$p_1 = \frac{1}{1 + \sum_{n=2} \left(a+\frac{b}{n}\right)(\ \ )\cdots\left(a+\frac{b}{2}\right)} .$$