SOLUTION

1

TIME: 90 min
M - 107

KING SAUD UNIVERSITY

FULL MARKS: 40

DEPARTMENT OF MATHEMATICS

II MID TERM EXAM (SEM II) 1431-1432

NOTE: Attempt all Questions.

Question: 1 . For the given points $A(3,4,2), B(4,2,0)$ and $C(1,3,1)$

(a) Using dot product check if the triangle ABC is a right angle triangle, [9]
(b) Find the component of $\overrightarrow{BA}$ along $\overrightarrow{BC}$ also projection of $\overrightarrow{BA}$ along $\overrightarrow{BC}$,
(c) Find the distance of point A from the line through point B and C, and

Question: 2 . (a)Let A(1, -1, 2), B(2,-3,1), and C(-1,5,-2) be the points in a plane, find the area of triangle ABC. [4]

(b) Find a, if the planes $ax + 2y + z = 4$ and $2x - y - az = 1$ are perpendicular to each other, [4]

Question: 3 .(i) Check whether lines

$L_1 : x = 1 + 2t, \ y = -2 + 3t, \ z = 4 - t$

$L_2 : x = 2s, \quad y = 3 + s, \quad z = -3 + 4s$

intersect, if they do not intersect find the distance between the lines. [6]

(ii) Identify the surface $9x^2 + 4y^2 + 9z^2 = 36$. Find its traces on the coordinate planes and sketch the surface. [4]

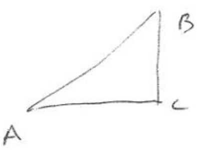
Question: 4 (a) Find the equation of the tangent line to the curve

$r(t) = (1+t)i + e^t j + e^{-t} k$ at the point $t = 0$. [5]

(b) If the acceleration of a moving particle at any time t is given by
$a(t) = 2i - 2j - 6tk$ and at time $t = 1$,
the velocity and position are $2i$ and $3k$ respectively find the position of the particle at any time t. [5]

(c) Find the point on the curve $y = \frac{1}{2}x^2 - x + 1$ where the curvature is 1. [3]

Q1.(a) $A(3,4,2)$, $B(4,2,0)$, $C(1,3,1)$



$\overrightarrow{AB} = \langle 1, -2, -2 \rangle$

$\overrightarrow{AC} = \langle 2, -1, -1 \rangle$

$\overrightarrow{BC} = \langle 3, -1, -1 \rangle$

$\overrightarrow{AB} \cdot \overrightarrow{AC} = 2 + 2 + 2 \neq 0$

$\overrightarrow{BC} \cdot \overrightarrow{AC} = 6 + 1 + 1 \neq 0$

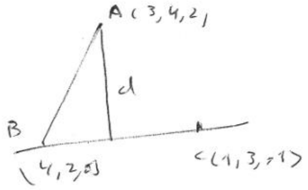
$\overrightarrow{BC} \cdot \overrightarrow{AB} = 3 + 2 - 2 \neq 0$ ABC is not $\perp$ triangle

(b) $\overrightarrow{BA} = \langle -1, 2, 2 \rangle$, $\overrightarrow{BC} = \langle 3, -1, -1 \rangle$

$$\mathrm{Comp}_{\overrightarrow{BC}} \overrightarrow{BA} = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BC}|} = \frac{\cancel{4}-3-2-2}{\sqrt{9+1+1}} = \frac{-7}{\sqrt{11}}$$

$$\mathrm{Proj}_{\overrightarrow{BC}} \overrightarrow{BA} = -\frac{7}{\sqrt{11}} \cdot \frac{\langle 3, -1, -1 \rangle}{\sqrt{11}} = \frac{-7 \langle 3, -1, -1 \rangle}{11}$$

(c)



$$d = \frac{\|\overrightarrow{BA} \times \overrightarrow{BC}\|}{\|\overrightarrow{BC}\|}$$

$$\overrightarrow{BA} \times \overrightarrow{BC} = \begin{vmatrix} i & j & k \\ -1 & 2 & 2 \\ 3 & -1 & -1 \end{vmatrix} = \langle 0, 5, -5 \rangle$$

$\|\overrightarrow{BA} \times \overrightarrow{BC}\| = \sqrt{50}$

$d = \sqrt{\frac{50}{11}}$ unit

Q 2. a.

$\overrightarrow{AB} = \langle 1, -2, -1 \rangle$ (1)

$\overrightarrow{AC} = \langle -2, 6, -4 \rangle$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 1 & -2 & -1 \\ -2 & 6 & -4 \end{vmatrix} = \langle 8+6, -(-4-2), 6-4 \rangle \quad (2)$$

$$= \langle 14, +6, 2 \rangle$$

Area of $\triangle ABC = \frac{1}{2}\sqrt{196+36+4} = \frac{1}{2}\sqrt{236}$ unit² (1)

(b) $ax + 2y + z = 4$, P_1

$2x - y - az = 1$, P_2

$n_1 = \langle a, 2, 1 \rangle$, $n_2 = \langle 2, -1, -a \rangle$ (2)

$n_1 \cdot n_2 = 0$

$2a - 2 - a = 0$ (2)

$a - 2 = 0 \Rightarrow a = 2.$

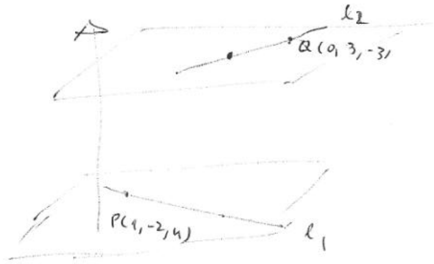
Q3. (a) If lines intersects, then (x_0, y_0, z_0) be point on lines

$$\begin{aligned} x_0 &= 1+2t_0 \\ y_0 &= -2+3t_0 \\ z_0 &= 4-t_0 \end{aligned} \quad \text{and} \quad \begin{aligned} x_0 &= 2s_0 \\ y_0 &= 3+s_0 \\ z_0 &= -3+4s_0 \end{aligned} \Rightarrow \begin{aligned} 1+2t_0 &= 2s_0 \\ -2+3t_0 &= 3+s_0 \\ 4-t_0 &= -3+4s_0 \end{aligned}$$

$$\Rightarrow \begin{aligned} 2t_0 - 2s_0 &= 1 \to E_1 \\ 3t_0 - s_0 &= 5 \to E_2 \\ -t_0 + 4s_0 &= -7 \to E_3 \end{aligned}$$

solving E_2 and E_3 $s_0 = \frac{22}{15}$, $t_0 = \frac{35}{13}$ which do not satisfy E_1

$\Rightarrow$ The lines do not intersect. lines are in $\parallel$ planes



Line L_1 Point $P(1,-2,4)$

vector $a = \langle 2, 3, -1 \rangle$

Line L_2 Point $Q(0,3,-3)$,

vector $b = \langle 2, 1, 4 \rangle$

$$n = a \times b = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = \langle 13, -10, -4 \rangle, \quad \vec{PQ} = \langle 1, -5, 7 \rangle$$

$$d = \text{Comp}_n^{PQ} = \frac{PQ \cdot n}{\|n\|} = \frac{13+50-28}{\sqrt{169+100+16}} = \frac{35}{\sqrt{285}}$$

(b) $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{4} = 1$ is ellipsoid

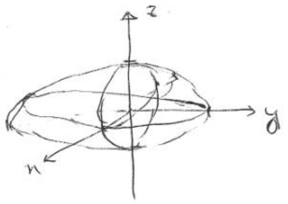
Traces

xy - plane $\frac{x^2}{4} + \frac{y^2}{9} = 1$ Ellipse

yz - plane $\frac{y^2}{9} + \frac{z^2}{4} = 1$ Ellipse

xz - plane $\frac{x^2}{4} + \frac{z^2}{4} = 1$ Ellipse

Sketch



Q.4(a) Find the equation of the tangent line to the curve

$$r(t) = (1+t)i + e^t j + e^{-t} k$$

At the point $t = 0$.

Solution: The point on the curve is $P(1,1,1)$. We have $r'(t) = i + e^t j - e^{-t} k$

Vector along the tangent line at P is $r'(0) = i + j - k$. The equation of the tangent line is

$$x = 1 + t, y = 1 + t, z = 1 - t$$

Q.4(b) If the acceleration of a moving particle at any time t is given by

$$a(t) = 2i - 2j - 6tk$$

And at time t=1, the velocity and position are 2i and 3k respectively find the position of the particle at any time t.

Solution: We have $v(t) = \int a(t)dt = 2\,ti - 2tj - 3t^2 k + L$

To find the constant L we use $v(1) = 2i$ that is $2i - 2j - 3k + L = 2i$ which gives $L = 2j + 3k$ and

$$v(t) = 2ti + 2(1-t)j + 3(1-t^2)k$$

Which on integration gives

$$r(t) = (t^2)i - (1-t)^2 j + 3\left(t - {}^1\!/_3\, t^3\right)k + M$$

And using $r(1) = 3k$ to find the constant M, we get $3k = i + 2k + M$ that is $M = -i + k$. Thus

$$r(t) = (t^2 - 1)i - (1-t)^2 j + (1 + 3t - t^3)k$$

Q. 4(c) Find the point on the curve $y = {}^1\!/_2\, x^2 - x + 1$ where the curvature is 1.

Solution: We have $\frac{dy}{dx} = x - 1$ and $\frac{d^2y}{dx^2} = 1$ and this gives

$$\kappa = \frac{1}{[1 + (x-1)^2]^{3/2}} = 1$$

That is $x = 1$ and putting this value in the equation of the curve we get $y = \frac{1}{2}$ and thus the point on the curve is $\left(1, \frac{1}{2}\right)$.