Manual for SOA Exam FM/CAS Exam 2. Chapter 1. Basic Interest Theory. Section 1.1. Amount and accumulation functions.

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▶ The effective rate of interest earned in the period [s, t] is

 $\frac{\text{final balance} - \text{invested amount}}{\text{invested amount}}$

Simon invests \$1000 in a bank account. Six months later, the amount in his bank account is \$1049.23.

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$$\frac{1049.23 - 1000}{1000} = 0.004923 = 0.4923\%.$$

Amount function

Suppose that an amount A(0) of money is invested at time 0. A(0) is the principal. Let A(t) denote the value at time t of the initial investment A(0). The function A(t), $t \ge 0$, is called the **amount function**. Usually, we assume that the amount function satisfies the following properties:

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Jessica invests \$5000 on March 1, 2008, in a fund which follows the accumulation function $A(t) = (5000) \left(1 + \frac{t}{40}\right)$, where t is the number of years after March 1, 2008.

(i) Find the balance in Jessica's account on October 1, 2008.

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$$\frac{A(7/12) - A(0)}{A(0)} = \frac{72.917}{5000} = 0.0145834 = 1.45834\%.$$

Cashflows

Often, we consider the case when several deposits/withdrawals are made into an account following certain amount function. A series of (deposits/withdrawals) payments made at different times is called a **cashflow**. The payments can be either made by the individual or to the individual. An **inflow** is payment to the individual. An **outflow** is a payment by the individual. We represent inflows by positive numbers and outflows by negative numbers. In a cashflow, we have a contribution of C_j at time t_j , for each $j = 1, \ldots, n$. C_j can be either positive or negative. We can represent a cashflow in a table:

Investments	C_1	C_2	•••	Cn
Time (in years)	t_1	t_2	•••	tn

Rule 1: Proportionality. If an investment strategy follows the amount function A(t), t > 0, an investment of k made at time 0 with the previous investment strategy, has a value of $\frac{kA(t)}{A(0)}$ at time t.

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- ▶ Investing 1 at time zero, we get $\frac{A(t)}{A(0)}$ at time t.
- Investing k at time zero, we get $\frac{kA(t)}{A(0)}$ at time t.

Present value

Definition 1

The **present value** at certain time of a cashflow is the amount of the money which need to invest at certain time in other to get the same balance as that obtained from a cashflow.

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Let x be the amount which need to invest at time zero to get a balance of k at time t. We have that $k = \frac{xA(t)}{A(0)}$. So, $x = \frac{kA(0)}{A(t)}$. Hence, the present value at time 0 of a balance of k had at time t is $\frac{kA(0)}{A(t)}$.

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- ► The present value at time t of a deposit of k made at time zero is $ka(t) (= \frac{kA(t)}{A(0)})$.
- The present value at time 0 of a balance of k had at time t is $\frac{k}{a(t)} \left(=\frac{kA(0)}{A(t)}\right)$.

The accumulation function of a fund is $a(t) = (1.03)^{2t}$, $t \ge 0$. (i) Amanda invests \$5000 at time zero in this fund. Find the balance into Amanda's fund at time 2.5 years. (ii) How much money does Kevin need to invest into the fund at time 0 to accumulate \$10000 at time 3?

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Solution: (i) The balance into Amanda's fund at time 2.5 years is

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Solution: (i) The balance into Amanda's fund at time 2.5 years is

$$ka(2.5) = (5000)(1.03)^{2(2.5)} = 5796.370371.$$

(ii) The amount which Kevin needs to invest at time 0 to accumulate 10000 at time 3 is

$$\frac{10000}{a(3)} = \frac{10000}{(1.03)^{2(3)}} = 8374.842567.$$

Rule 2. Grows–depends–on–balance rule. If an investment follows the amount function A(t), $t \ge 0$, the growth during certain period where no deposits/withdrawals are made depends on the balance on the account at the beginning of the period.

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In particular, the following two accounts have the same balance for times bigger than t:

- 1. An account where a unique deposit of k is made at time t.
- 2. An account where a unique deposit of $\frac{k}{A(t)}$ is made at time zero.

Theorem 1

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If an investment follows the amount function $A(\cdot)$, the present value at time t of a deposit of k made at time s is $\frac{kA(t)}{A(s)} = \frac{ka(t)}{a(s)}$. Proof. We need to invest $\frac{k}{A(s)}$ at time 0 to get a balance of k at time s. So, investing k at time s is equivalent to investing $\frac{k}{A(s)}$ at time 0. The future value at time t of an investment of $\frac{k}{A(s)}$ at time 0 is $\frac{kA(t)}{A(s)}$. Hence, investing k at time s is equivalent to investing $\frac{kA(t)}{A(s)}$ at time t. Another way to see the previous theorem is as follows. The following three accounts have the same balance at any time bigger than t:

1. An account where a unique deposit of A(0) is made at time zero.

- 2. An account where a unique deposit of A(s) is made at time s.
- 3. An account where a unique deposit of A(t) is made at time t.

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This means that:

- ► If t > s, an investment of A(s) made at time s has an accumulation value of A(t) at time t.
- If t < s, to get an accumulation of A(s) at time s, we need to invest A(t) at time t.

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1 at time s is equivalent to $\frac{A(t)}{A(s)}$ at time t.

► The present value at time t of an investment of k made at time s is \frac{kA(t)}{A(s)}, i.e.

k at time s is equivalent to $\frac{kA(t)}{A(s)}$ at time t.

The accumulation function of a fund follows the function $a(t) = 1 + \frac{t}{20}$, t > 0.

(i) Michael invests \$3500 into the fund at time 1. Find the value of Michael's fund account at time 4.

(ii) How much money needs Jason to invest at time 2 to accumulate \$700 at time 4.

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Solution: (i) The value of Michael's account at time 4 is $3500\frac{a(4)}{a(1)} = (3500)\frac{1+\frac{4}{20}}{1+\frac{1}{20}} = (3500)\frac{1.20}{1.05} = 4000.$

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$$3500\frac{a(4)}{a(1)} = (3500)\frac{1+\frac{4}{20}}{1+\frac{1}{20}} = (3500)\frac{1.20}{1.05} = 4000.$$

(ii) To accumulate \$700 at time 4, Jason needs to invest at time 2,
 $700\frac{a(2)}{a(4)} = 700\frac{1.1}{1.2} = 641.67.$

Theorem 3

Present value of a cashflow. If an investment account follows the amount function A(t), t > 0, the (equation of value) present value at time t of the cashflow

where $0 \le t_1 < t_2 < \cdots < t_n$, is

$$V(t) = \sum_{j=1}^{n} C_j \frac{A(t)}{A(t_j)}.$$

Proof. Let $s > t_n$.

Time	Balance before deposit	Balance after deposit
t_1	0	<i>C</i> ₁
<i>t</i> ₂	$C_1 \frac{a(t_2)}{a(t_1)} = \sum_{j=1}^1 C_j \frac{a(t_2)}{a(t_j)}$	$C_1 \frac{a(t_2)}{a(t_1)} + C_2 = \sum_{j=1}^2 C_j \frac{a(t_2)}{a(t_j)}$ $\sum_{j=1}^3 C_j \frac{a(t_3)}{a(t_j)}$
t ₃	$\sum_{j=1}^{2} C_{j} \frac{a(t_{3})}{a(t_{j})}$	$\sum_{j=1}^{3} C_j \frac{a(t_3)}{a(t_j)}$
t4	$\sum_{j=1}^{3} C_j \frac{a(t_4)}{a(t_j)}$	$\sum_{j=1}^{4} C_j \frac{a(t_4)}{a(t_j)}$
	•••	
tn	$\sum_{j=1}^{n-1} C_j \frac{a(t_n)}{a(t_j)}$	$\sum_{j=1}^{n} C_j \frac{a(t_n)}{a(t_j)}$

Since the balance after deposit at time t_1 is C_1 , the balance before deposit at time t_2 is $\frac{a(t_2)}{a(t_1)}C_1$. Since the balance after deposit at time t_2 is $\sum_{j=1}^2 C_j \frac{a(t_2)}{a(t_j)}$, the balance before deposit at time t_3 is $\frac{a(t_3)}{a(t_2)} \sum_{j=1}^2 C_j \frac{a(t_2)}{a(t_j)} = \sum_{j=1}^2 C_j \frac{a(t_3)}{a(t_j)}$. Hence, the balance at time s is

$$\frac{a(s)}{a(t_n)}\sum_{j=1}^n C_j \frac{a(t_n)}{a(t_j)} = \sum_{j=1}^n C_j \frac{a(s)}{a(t_j)}.$$

The future value of the cashflow at time t is

$$rac{a(t)}{a(s)}\sum_{j=1}^n C_j rac{a(s)}{a(t_j)} = \sum_{j=1}^n C_j rac{a(t)}{a(t_j)}.$$

end of proof.

Notice that the present value at time t of the cashflow

$$\begin{array}{c|cccc} \text{Deposits} & C_1 & C_2 & \cdots & C_n \\ \hline \hline \text{Time} & t_1 & t_2 & \cdots & t_n \end{array}$$

is the same as the sum of the present values at time t of n separated investment accounts each following the amount function A, with the *j*-the account having a unique deposit of C_i at time t_i .

The accumulation function of a fund follows the function $a(t) = 1 + \frac{t}{20}$, t > 0. Jared invests \$1000 into the fund at time 1 and he withdraws \$500 at time 3. Find the value of Jared's fund account at time 5.

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Solution: The cashflow is

deposit/withdrawal	1000	-500	
Time (in years)	1	3	

The value of Jared's account at time 5 is

$$1000\frac{a(5)}{a(1)} - 500\frac{a(5)}{a(3)} = 1000\frac{1 + \frac{5}{20}}{1 + \frac{1}{20}} - 500\frac{1 + \frac{5}{20}}{1 + \frac{3}{20}}$$
$$= (1000)\frac{1.25}{1.05} - (500)\frac{1.25}{1.15} = 1190.48 - 543.48 = 647.00$$