Manual for SOA Exam FM/CAS Exam 2.

Chapter 7. Derivatives markets.

Section 7.4. Call options.

Extract from:
"Arcones' Manual for the SOA Exam FM/CAS Exam 2, Financial Mathematics. Fall 2009 Edition",
available at http://www.actexmadriver.com/

Minimums and maximums

Definition 1

Given two real numbers a and b,
(i) $\min(a, b)$ *denotes the (minimum) smallest of the two numbers.*
(ii) $\max(a, b)$ *denotes the (maximum) biggest of the two numbers.*

Example 1

$\min(10, 5) = 5$, $\max(10, 5) = 10$, $\min(-1, 5) = -1$,
$\max(-1, 5) = 5$, $\min(-2, -100) = -100$, $\max(-2, -100) = -2$.

Definition 2

Given real numbers $a_1, \dots, a_n$,
(i) $\min(a_1, \dots, a_n)$ *denotes the (minimum) smallest of these numbers.*
(ii) $\max(a_1, \dots, a_n)$ *denotes the (maximum) biggest of these numbers.*

Example 2

$\min(-1, 5, 3, -6) = -6$, $\max(-1, 5, 3, -6) = 5$,
$\min(-2, -100, -50) = -100$ *and* $\max(-2, -100, -50) = -2$.

Theorem 1

For each $a, b, c \in \mathbb{R}$ *and each* $\lambda \geq 0$,

- $\min(a, b) = \min(b, a)$.
- $\max(a, b) = \max(b, a)$.
- $\min(\min(a, b), c) = \min(a, \min(b, c)) = \min(a, b, c)$.
- $\max(\max(a, b), c) = \max(a, \max(b, c)) = \max(a, b, c)$.
- $\min(a + c, b + c) = \min(a, b) + c$.
- $\max(a + c, b + c) = \max(a, b) + c$.
- $\min(\lambda a, \lambda b) = \lambda \min(a, b)$.
- $\max(\lambda a, \lambda b) = \lambda \max(a, b)$.
- $\min(-a, -b) = -\max(a, b)$.
- $\max(-a, -b) = -\min(a, b)$.

Definition 3

Given a real number a, $|a| = a$, if $a \geq 0$; and $|a| = -a$, if $a \leq 0$

Example 3

$|23| = 23$, $|-4| = 4$.

Theorem 2

For each $a, b \in \mathbb{R}$, $\min(a, b) + \max(a, b) = a + b$.

Proof.

$\min(a, b)$ and $\max(a, b)$ are a and b in some order. Hence, $\min(a, b) + \max(a, b) = a + b$. □

Theorem 3

For each $a \in \mathbb{R}$, $|a| = \max(a, 0) - \min(a, 0)$.

Proof.

If $a \geq 0$, then $\max(a, 0) = a$, $\min(a, 0) = 0$, and $\max(a, 0) - \min(a, 0) = a = |a|$. If $a \leq 0$, then $\max(a, 0) = 0$, $\min(a, 0) = a$, and $\max(a, 0) - \min(a, 0) = -a = |a|$. □

Call options

Definition 4

A **call option** *is a financial contract which gives the owner the right, but not the obligation, to buy a specified amount of a given asset at a specified price during a specified period of time.*

Call options

Definition 4

A **call option** *is a financial contract which gives the owner the right, but not the obligation, to buy a specified amount of a given asset at a specified price during a specified period of time.*

The call option owner exercises the option by buying the asset at the specified call price from the call writer. A call option is executed only if the call owner decides to do so. A call option owner executes a call option only when it benefits him, i.e. when the specified call price is smaller than the current (market value) spot price. Since the owner of a call option can make money if the option is exercised, call options are sold. The **owner of the call option** must pay to its counterpart for holding a call option. The price of a call option is called its **premium**.

- The (owner) buyer of a call option is called the **option call holder**. The holder of a call option is said to have a **long call position**.
- The seller of a call option is called the **option call writer**. The writer of a call is said to have a **short call position**.
- Assets used in call options are in commodities, currency exchange, stock shares and stock indices.
- A call option needs to specify the type and quality of the underlying.
- The asset used in the call option is called the **underlier** or **underlying asset**.
- The amount of the underlying asset to which the call option applies is called the **notional amount**.
- The specified price of an asset in a call option is called the **strike price**, or **exercise price**.
- A forward contract forces the buyer and seller to execute the sale. A call option is executed only if the call holder decides to do so.

- For an **European option**, the exercise of the option must occur at a certain time (the **expiration date**).
- For an **American option**, the exercise of the option must occur any time by the expiration date.
- For a **Bermudan option**, the buyer can exercise the call option during specified periods.

Unless say otherwise, we will assume that an option is an European option. European options are simpler and easier to study.

Example 4

Suppose that an investor buys a call option of 100 shares of XYZ stock with a strike price of \$76 per share. The exercise date is one year from now.
(i) If the spot price at expiration is \$70 per share, the call option holder does not exercise the option. The option is worthless. The call option holder can buy stock in the market for a price smaller than the call option price.
(ii) If the (the market price) spot price at expiration is \$80 per share, the call option holder exercises the call option, i.e. he buys 100 shares of XYZ stock for \$76 from the option seller. Since the call option holder can sell these shares for \$80 per share, the call option holder gets a payoff of $100(80 - 76) = \$400$.

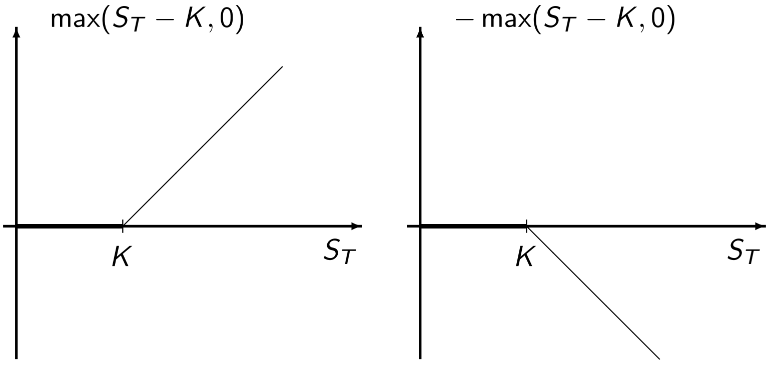
Let K be the strike price of a call option. Let S_T be the price of the asset at expiration.

- The call option holder's payoff is

$$\begin{cases} 0 & \text{if } S_T < K, \\ S_T - K & \text{if } S_T \geq K. \end{cases}$$

  We also can write this as $\max(0, S_T - K)$.
- The payoff for the call option writer is the opposite of the holder's payoff. The payoff for the call option writer is $-\max(0, S_T - K)$.
- A call–option is a zero–sum game. The sum of the two payoffs is zero.

Figure 1 shows a graph of the call option payoff as a function of S_T.



Payoff for the call option holder Payoff for the call option writer

Figure 1: Payoffs of a call option

Recall:

- The call option holder's payoff is

$$\max(0, S_T - K).$$

- The call option writer's payoff is

$$-\max(0, S_T - K).$$

We get from Figure 1 that:

- The minimum payoff for the call option holder is 0. The maximum payoff for the call option holder is ∞.
- The minimum payoff for the call option writer is $-\infty$. The maximum payoff for the call option writer is 0.

	minimum payoff	maximum payoff
call option holder	0	∞
call option writer	$-\infty$	0

Example 5

Andrew buys a 45–strike call option for XYZ stock with a nominal amount of 2000 shares. The expiration date is 6 months from now.
(i) Calculate Andrew's payoff for the following spot prices per share at expiration: 35, 40, 45, 55, 60.
(ii) Calculate Andrew's minimum and maximum payoffs.

Example 5

Andrew buys a 45–strike call option for XYZ stock with a nominal amount of 2000 shares. The expiration date is 6 months from now.
(i) Calculate Andrew's payoff for the following spot prices per share at expiration: 35, 40, 45, 55, 60.
(ii) Calculate Andrew's minimum and maximum payoffs.

Solution: (i) Andrew's payoff is $(2000)\max(S_T - 45, 0)$. The corresponding payoffs are:

$$\begin{aligned}
\text{if } S_T = 35,\ \text{payoff} &= (2000)\max(35-45,0) = 0,\\
\text{if } S_T = 40,\ \text{payoff} &= (2000)\max(40-45,0) = 0,\\
\text{if } S_T = 45,\ \text{payoff} &= (2000)\max(45-45,0) = 0,\\
\text{if } S_T = 50,\ \text{payoff} &= (2000)\max(50-45,0) = 10000,\\
\text{if } S_T = 55,\ \text{payoff} &= (2000)\max(55-45,0) = 20000.
\end{aligned}$$

(ii) Andrew's minimum payoff is 0. Andrew's maximum payoff is ∞.

Example 6

Madison sells a 45–strike call option for XYZ stock with a nominal amount of 2000 shares. The expiration date is 6 months from now.
(i) Calculate Madison's payoff for the following spot prices at expiration: 35, 40, 45, 55, 60.
(ii) Calculate Madison's minimum and maximum payoffs.

Example 6

Madison sells a 45–strike call option for XYZ stock with a nominal amount of 2000 shares. The expiration date is 6 months from now.
(i) Calculate Madison's payoff for the following spot prices at expiration: 35, 40, 45, 55, 60.
(ii) Calculate Madison's minimum and maximum payoffs.

Solution: (i) Madison's payoff is $-(2000)\max(S_T - 45, 0)$. The corresponding payoffs are:

$$\begin{aligned}
&\text{if } S_T = 35, \text{ payoff} = -(2000)\max(35-45,0) = 0,\\
&\text{if } S_T = 40, \text{ payoff} = -(2000)\max(40-45,0) = 0,\\
&\text{if } S_T = 45, \text{ payoff} = -(2000)\max(45-45,0) = 0,\\
&\text{if } S_T = 50, \text{ payoff} = -(2000)\max(50-45,0) = -10000,\\
&\text{if } S_T = 55, \text{ payoff} = -(2000)\max(55-45,0) = -20000.
\end{aligned}$$

(ii) Madison's payoff is $(2000)\max(S_T - 45, 0)$. Madison's minimum payoff is $-\infty$. Madison's maximum payoff is 0.

Let $\mathrm{Call}(K, T)$ be the premium per unit paid by the buyer of a call option with strike price K and expiration time T years. Notice that $\mathrm{Call}(K, T) > 0$. The premium of a call option for N units is $N\mathrm{Call}(K, T)$. Let i be the risk–free annual effective rate of interest.

- The call option holder's profit per unit is

$$\begin{aligned}&\max(S_T - K, 0) - \mathrm{Call}(K, T)(1 + i)^T\\ =&\begin{cases}-\mathrm{Call}(K, T)(1 + i)^T & \text{if } S_T < K,\\ S_T - K - \mathrm{Call}(K, T)(1 + i)^T & \text{if } S_T \geq K.\end{cases}\end{aligned}$$

- The call option seller's profit per unit is

$$\begin{aligned}&\mathrm{Call}(K, T)(1 + i)^T - \max(0, S_T - K)\\ =&\begin{cases}\mathrm{Call}(K, T)(1 + i)^T & \text{if } S_T < K,\\ \mathrm{Call}(K, T)(1 + i)^T - (S_T - K) & \text{if } S_T \geq K.\end{cases}\end{aligned}$$

The call option holder profit $\max(S_T - K, 0) - \mathrm{Call}(K, T)(1+i)^T$ as a function of S_T is nondecreasing. The call option holder benefits from the increase of the spot price.

- The minimum call option holder profit is

$$-\mathrm{Call}(K, T)(1+i)^T.$$

- The maximum call option holder profit is ∞.
- The profit for the call option holder is positive if

$$S_T > K + \mathrm{Call}(K, T)(1+i)^T.$$

- If $S_T < K + \mathrm{Call}(K, T)(1+i)^T$, the call option holder profit is negative.

The call option writer's profit $\mathrm{Call}(K, T)(1+i)^T - \max(0, S_T - K)$ as a function of S_T is nonincreasing. The call option writer benefits from the decrease of the spot price.

- The minimum call option writer profit is $-\infty$. The call option writer position is riskier than his counterpart. A call option writer can assumed unbounded loses.
- The maximum call option writer profit is $\mathrm{Call}(K, T)(1+i)^T$.
- The profit for the call option writer is positive if

$$S_T < K + \mathrm{Call}(K, T)(1+i)^T.$$

- The profit for the call option writer is negative if

$$S_T > K + \mathrm{Call}(K, T)(1+i)^T.$$

	profit
call option holder	$\max(S_T - K, 0) - \mathrm{Call}(K, T)(1+i)^T$
call option writer	$-\max(S_T - K, 0) + \mathrm{Call}(K, T)(1+i)^T$

	minimum profit	maximum profit
call option holder	$-\mathrm{Call}(K, T)(1+i)^T$	∞
call option writer	$-\infty$	$\mathrm{Call}(K, T)(1+i)^T$

Figure 2 shows the graph of the profit of a call option as a function of S_T.

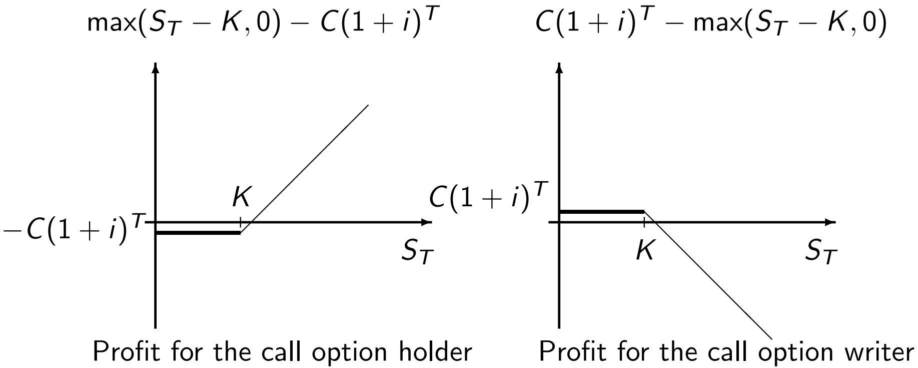


Figure 2: Profit of a call option

If r is the annual interest rate compounded continuously, then the profit for the call option holder is

$$\max(0, S_T - K) - \mathrm{Call}(K, T)e^{rT}$$

and the profit of the call option writer is

$$\mathrm{Call}(K, T)e^{rT} - \max(0, S_T - K).$$

Example 7

Ethan buys a 35–strike call option for XYZ stock for 4.337 *per share. The nominal amount of this call option is 2000 shares. The expiration date of this option is 18 months. The annual effective interest rate is* 5.5%.

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(i) Calculate Ethan's profit function.

Example 7

Ethan buys a 35–strike call option for XYZ stock for 4.337 *per share. The nominal amount of this call option is 2000 shares. The expiration date of this option is 18 months. The annual effective interest rate is* 5.5%.

(i) Calculate Ethan's profit function.

Solution: (i) Ethan's profit function is

$$
\begin{aligned}
&(2000)(\max(S_T - 35, 0) - 4.337(1.055)^{1.5}) \\
=&(2000)\max(S_T - 35, 0) - 9400.
\end{aligned}
$$

Example 7

Ethan buys a 35–strike call option for XYZ stock for 4.337 *per share. The nominal amount of this call option is 2000 shares. The expiration date of this option is 18 months. The annual effective interest rate is* 5.5%.

(ii) Calculate Ethan's profit for the following spot prices at expiration: 25, 30, 35, 40, 45.

Example 7

Ethan buys a 35–strike call option for XYZ stock for 4.337 *per share. The nominal amount of this call option is 2000 shares. The expiration date of this option is 18 months. The annual effective interest rate is* 5.5%.

(ii) Calculate Ethan's profit for the following spot prices at expiration: 25, 30, 35, 40, 45.

Solution: (ii) Since Ethan's profit is $(2000)\max(S_T - 35, 0) - 9400$, Ethan's profit for the considered spot prices is:

$$\begin{aligned}
&\text{if } S_T = 25, \text{ profit} = (2000)\max(25-35,0) - 9400 = -9400,\\
&\text{if } S_T = 30, \text{ profit} = (2000)\max(30-35,0) - 9400 = -9400,\\
&\text{if } S_T = 35, \text{ profit} = (2000)\max(35-35,0) - 9400 = -9400,\\
&\text{if } S_T = 40, \text{ profit} = (2000)\max(40-35,0) - 9400 = 600,\\
&\text{if } S_T = 45, \text{ profit} = (2000)\max(45-35,0) - 9400 = 10600.
\end{aligned}$$

Example 7

Ethan buys a 35–strike call option for XYZ stock for 4.337 *per share. The nominal amount of this call option is 2000 shares. The expiration date of this option is 18 months. The annual effective interest rate is* 5.5%.

(iii) Calculate Ethan's minimum and maximum profits.

Example 7

Ethan buys a 35–strike call option for XYZ stock for 4.337 *per share. The nominal amount of this call option is 2000 shares. The expiration date of this option is 18 months. The annual effective interest rate is* 5.5%.

(iii) Calculate Ethan's minimum and maximum profits.

Solution: (iii) Since Ethan's profit is $(2000)\max(S_T - 35, 0) - 9400$, Ethan's minimum profit is -9400 and Ethan's maximum profit is ∞.

Example 7

Ethan buys a 35–strike call option for XYZ stock for 4.337 *per share. The nominal amount of this call option is 2000 shares. The expiration date of this option is 18 months. The annual effective interest rate is* 5.5%.

(iv) Find the spot prices at which Ethan's profit is positive.

Example 7

Ethan buys a 35–strike call option for XYZ stock for 4.337 *per share. The nominal amount of this call option is 2000 shares. The expiration date of this option is 18 months. The annual effective interest rate is* 5.5%.

(iv) Find the spot prices at which Ethan's profit is positive.

Solution: (iv) Since Ethan's profit is $(2000)\max(S_T - 35, 0) - 9400$, Ethan's profit is positive if $(2000)\max(S_T - 35, 0) - 9400 > 0$, i.e. if $S_T > 35 + \frac{9400}{2000} = 39.7$.

Example 7

Ethan buys a 35–strike call option for XYZ stock for 4.337 *per share. The nominal amount of this call option is 2000 shares. The expiration date of this option is 18 months. The annual effective interest rate is* 5.5%*.*

(v) Calculate the spot price at expiration at which Ethan does not make or lose money on this contract.

Example 7

Ethan buys a 35–strike call option for XYZ stock for 4.337 *per share. The nominal amount of this call option is 2000 shares. The expiration date of this option is 18 months. The annual effective interest rate is* 5.5%*.*

(v) Calculate the spot price at expiration at which Ethan does not make or lose money on this contract.

Solution: (v) Since Ethan's profit is $(2000)\max(S_T - 35, 0) - 9400$, Ethan breaks even if $(2000)(S_T - 35) - 9400 = 0$, i.e. if $S_T = 35 + \frac{9400}{2000} = 39.7$.

Example 7

Ethan buys a 35–strike call option for XYZ stock for 4.337 *per share. The nominal amount of this call option is 2000 shares. The expiration date of this option is 18 months. The annual effective interest rate is* 5.5%.

(vi) Find the spot price at expiration at which Ethan makes an annual effective yield of 4.75%.

Example 7

Ethan buys a 35–strike call option for XYZ stock for 4.337 *per share. The nominal amount of this call option is 2000 shares. The expiration date of this option is 18 months. The annual effective interest rate is* 5.5%.

(vi) Find the spot price at expiration at which Ethan makes an annual effective yield of 4.75%.

Solution: (vi) Ethan invests $(2000)(4.337) = 8674$. If his yield is 4.75%, his payoff is

$$(8674)(1.0475)^{18/12} = 9300 = (2000)\max(S_T - 35, 0)$$

and

$$S_T = 35 + \frac{9300}{2000} = 39.65.$$

Example 7

Ethan buys a 35–strike call option for XYZ stock for 4.337 *per share. The nominal amount of this call option is 2000 shares. The expiration date of this option is 18 months. The annual effective interest rate is* 5.5%.

(vii) Find the annual effective rate of return earned by Ethan if the spot price at expiration is 38.

Example 7

Ethan buys a 35–strike call option for XYZ stock for 4.337 *per share. The nominal amount of this call option is 2000 shares. The expiration date of this option is 18 months. The annual effective interest rate is* 5.5%.

(vii) Find the annual effective rate of return earned by Ethan if the spot price at expiration is 38.

Solution: (vii) Let i be Ethan's annual effective rate of return. Ethan invests $(2000)(4.337) = 8674$. His payoff is $(2000)\max(38-35,0) = 6000$. Hence, $8674(1+i)^{1.5} = 6000$ and $i = -21.78538923\%$.

Example 8

Hannah sells a 35–strike call option for XYZ stock for 4.337 *per share. The nominal amount of this call option is 2000 shares. The expiration date of this option is 18 months. The annual effective interest rate is* 5.5%*. Hannah invests the proceeds of the sale in a zero–coupon bond.*

Example 8

Hannah sells a 35–strike call option for XYZ stock for 4.337 *per share. The nominal amount of this call option is 2000 shares. The expiration date of this option is 18 months. The annual effective interest rate is* 5.5%. *Hannah invests the proceeds of the sale in a zero–coupon bond.*

(i) Calculate Hannah's profit function.

Example 8

Hannah sells a 35–strike call option for XYZ stock for 4.337 *per share. The nominal amount of this call option is 2000 shares. The expiration date of this option is 18 months. The annual effective interest rate is* 5.5%. *Hannah invests the proceeds of the sale in a zero–coupon bond.*

(i) Calculate Hannah's profit function.

Solution: (i) Hannah's profit is

$$\begin{aligned}&-(2000)(\max(S_T-35,0)-4.337(1.055)^{1.5})\\=&9400-(2000)\max(S_T-35,0).\end{aligned}$$

Example 8

Hannah sells a 35–strike call option for XYZ stock for 4.337 *per share. The nominal amount of this call option is 2000 shares. The expiration date of this option is 18 months. The annual effective interest rate is* 5.5%. *Hannah invests the proceeds of the sale in a zero–coupon bond.*

(ii) Calculate Hannah's profit for the following spot prices at expiration: 25, 30, 35, 40, 45.

Example 8

Hannah sells a 35–strike call option for XYZ stock for 4.337 *per share. The nominal amount of this call option is 2000 shares. The expiration date of this option is 18 months. The annual effective interest rate is* 5.5%. *Hannah invests the proceeds of the sale in a zero–coupon bond.*

(ii) Calculate Hannah's profit for the following spot prices at expiration: 25, 30, 35, 40, 45.

Solution: (ii) Since Hannah's profit is $9400 - (2000)\max(S_T - 35, 0)$, Hannah's profit for the considered spot prices is:

$$\begin{aligned}
&\text{if } S_T = 25, \ \text{profit} = 9400 - (2000)\max(25 - 35, 0) = 9400,\\
&\text{if } S_T = 30, \ \text{profit} = 9400 - (2000)\max(30 - 35, 0) = 9400,\\
&\text{if } S_T = 35, \ \text{profit} = 9400 - (2000)\max(35 - 35, 0) = 9400,\\
&\text{if } S_T = 40, \ \text{profit} = 9400 - (2000)\max(40 - 35, 0) = -600,\\
&\text{if } S_T = 45, \ \text{profit} = 9400 - (2000)\max(45 - 35, 0) = -10600.
\end{aligned}$$

Example 8

Hannah sells a 35–strike call option for XYZ stock for 4.337 *per share. The nominal amount of this call option is 2000 shares. The expiration date of this option is 18 months. The annual effective interest rate is* 5.5%. *Hannah invests the proceeds of the sale in a zero–coupon bond.*

(iii) Calculate Hannah's minimum and maximum profits.

Example 8

Hannah sells a 35–strike call option for XYZ stock for 4.337 per share. The nominal amount of this call option is 2000 shares. The expiration date of this option is 18 months. The annual effective interest rate is 5.5%. Hannah invests the proceeds of the sale in a zero–coupon bond.

(iii) Calculate Hannah's minimum and maximum profits.

Solution: (iii) Since Hannah's profit is $9400 - (2000)\max(S_T - 35, 0)$, Hannah's minimum profit is $-\infty$ and Hannah's maximum profit is 9400.

Next we consider the pricing of a call option. The profit of a call option depends on S_T, which is random. In the case of uncertain scenarios, an arbitrage portfolio consists of a zero investment portfolio, which shows non–negative payoffs in all scenarios. This implies that if there exists no arbitrage, the profit function of a portfolio is either constantly zero, or its minimum is negative and its maximum positive.

Theorem 4

If there exist no arbitrage, then

$$\max(S_0 - (1+i)^{-T}K, 0) < \mathrm{Call}(K, T) < S_0.$$

Proof: Consider the portfolio consisting of selling a call option and buying the asset. The profit per unit at expiration is

$$\begin{aligned}&S_T - \max(S_T - K, 0) - (S_0 - \mathrm{Call}(K,T))(1+i)^T\\=&S_T + K - \max(S_T, K) - (S_0 - \mathrm{Call}(K,T))(1+i)^T\\=&\min(S_T, K) - (S_0 - \mathrm{Call}(K,T))(1+i)^T.\end{aligned}$$

The profit is nondecreasing on S_T. The minimum of this portfolio is $-(S_0 - \mathrm{Call}(K,T))(1+i)^T$. The maximum of this portfolio is $K - (S_0 - \mathrm{Call}(K,T))(1+i)^T$. If there exists no arbitrage and the profit function is not constant, the minimum profit is negative and the maximum profit is positive. Hence,

$$-(S_0 - \mathrm{Call}(K,T))(1+i)^T < 0 < K - (S_0 - \mathrm{Call}(K,T))(1+i)^T$$

which is equivalent to

$$S_0 - (1+i)^{-T}K < \mathrm{Call}(K,T) < S_0.$$

If the bounds in Theorem 4 do not hold, we can make arbitrage. For example, if the price of the call is bigger than the spot price, we can make money by buying the asset, selling the call and investing the proceeds in a zero–coupon bond. At redemption time, we have the asset which can use to satisfy the requirements of the call.

Example 9

Consider an European call option on a stock worth $S_0 = 32$*, with expiration date exactly one year from now, and with strike price \$30. The risk–free annual rate of interest compounded continuously is* $r = 5\%$.

Example 9

Consider an European call option on a stock worth $S_0 = 32$, with expiration date exactly one year from now, and with strike price \$30. The risk–free annual rate of interest compounded continuously is $r = 5\%$.

(i) If the call is worth \$3, find an arbitrage portfolio.

Example 9

Consider an European call option on a stock worth S_0 =32, with expiration date exactly one year from now, and with strike price \$30. The risk–free annual rate of interest compounded continuously is $r = 5\%$.

(i) If the call is worth \$3, find an arbitrage portfolio.

Solution: (i) We have that

$$\mathrm{Call}(K, T) - S_0 + (1+i)^{-T} K = 3 - 32 + e^{-0.05} 30 = -0.463117265 < 0.$$

We can do arbitrage by buying the call and shorting stock. If the spot price at expiration is more than 30, we buy the stock using the call option. If the spot price at expiration is less than 30, we buy the stock at market price. Any case, we buy stock for $\min(S_T, 30)$. Hence, the profit is

$$(32 - 3)e^{0.05} - \min(S_T, 30) \geq (32 - 3)e^{0.05} - 30 = 0.4868617949.$$

Example 9

Consider an European call option on a stock worth $S_0 = 32$*, with expiration date exactly one year from now, and with strike price \$30. The risk–free annual rate of interest compounded continuously is* $r = 5\%$.

(i) If the call is worth \$35, find an arbitrage portfolio.

Example 9

Consider an European call option on a stock worth $S_0 = 32$, with expiration date exactly one year from now, and with strike price \$30. The risk–free annual rate of interest compounded continuously is $r = 5\%$.

(i) If the call is worth \$35, find an arbitrage portfolio.

Solution: (ii) In this case $\mathrm{Call}(K, T) > S_0$. We can do arbitrage by selling the call and buying stock. If the spot price at expiration is more than 30, we sell the stock to the call option holder. If the spot price at expiration is less than 30, we sell the stock at the market price. In any case, we sell stock for $\min(S_T, 30)$. The profit is

$$(35 - 32)e^{0.05} + \min(S_T, 30) \geq (35 - 32)e^{0.05} = 3.153813289.$$

A call option is a way to buy stock in the future. A long forward is another way to buy stock in the future. Buying a call option, you are guaranteed that the price you pay is not bigger than the strike price. If you buy a call option, you can buy the asset at expiration for $\min(S_T, K)$. The baker in the example in Section 7.1, instead of buying a long forward for $F_{0,T}$, he can buy a call option to hedge against high wheat prices. Doing this we will be able to buy wheat at time T for $\min(S_T, K)$. The cost of this investment strategy is

$$\mathrm{Call}(K, T)e^{rT} + \min(S_T, K).$$

Recall that $F_{0,T}$ is the price of a forward contract with delivery in T years. The profit of a long forward is $S_T - F_{0,T}$. The minimum profit of a long forward is $-F_{0,T}$. The maximum profit of a long forward is ∞.

Example 10

Joseph buys a one–year long forward for 100 shares of a stock at \$74 per share. Samantha buys a call option of 100 shares of XYZ stock for \$76 per share. The exercise date is one year from now. The risk free effective annual interest rate is 6%. The premium of this call is \$6.4133 per share.

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(i) Make a table with Joseph's profit and Samantha's profit when the spot price at expiration is \$50, \$70, \$90 and \$110. Compare these profits.

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(i) Make a table with Joseph's profit and Samantha's profit when the spot price at expiration is \$50, \$70, \$90 and \$110. Compare these profits.

Solution: (i) Joseph's profit is given by the formula

$$(100)(S_T - 74).$$

Samantha's profit is

$$\begin{aligned} & 100 \max(0, S_T - K) - 100\mathrm{Call}(K, T)(1+i)^T \\ = \; & 100 \max(0, S_T - 76) - (100)(6.4133)(1.06) \\ = \; & 100 \max(0, S_T - 76) - 679.81. \end{aligned}$$

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(i) Make a table with Joseph's profit and Samantha's profit when the spot price at expiration is \$50, \$70, \$90 and \$110. Compare these profits.

Solution: (i) (continuation)

Joseph's profit	−2400	−400	1600	3600
Samantha's profit	−679.81	−679.81	720.19	2720.19
Spot Price	50	70	90	110

For high spot prices at expiration, Samantha's profits are smaller than John's profits. For low prices, Samantha's losses are smaller than Joseph's losses.

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(ii) Calculate Joseph's profit and Samantha's minimum and maximum payoffs.

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(ii) Calculate Joseph's profit and Samantha's minimum and maximum payoffs.

Solution: (ii) Joseph's minimum profit is -7400. Joseph's maximum profit is ∞. Samantha's minimum profit is -679.81. Samantha's maximum profit is ∞.

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(iii) Which is the minimum spot price at expiration at which Joseph makes a profit? Which is the minimum spot price at expiration at which Samantha makes a profit?

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(iii) Which is the minimum spot price at expiration at which Joseph makes a profit? Which is the minimum spot price at expiration at which Samantha makes a profit?

Solution: (iii) Joseph is even if $S_T = 74$. Samantha is even if $100(S_T - 76) - 679.81 = 0$, i.e. $S_T = 76 + (679.81/100) = 82.7981$.

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Joseph buys a one–year long forward for 100 shares of a stock at \$74 per share. Samantha buys a call option of 100 shares of XYZ stock for \$76 per share. The exercise date is one year from now. The risk free effective annual interest rate is 6%. The premium of this call is \$6.4133 per share.

(iv) Draw the graph of the profit versus the spot price at expiration for Joseph and Samantha.

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(iv) Draw the graph of the profit versus the spot price at expiration for Joseph and Samantha.

Solution: (iv) The graphs of (long forward) Joseph's profit and (purchased call) Samantha's profit are in Figure 3.

Example 10

Joseph buys a one–year long forward for 100 shares of a stock at \$74 per share. Samantha buys a call option of 100 shares of XYZ stock for \$76 per share. The exercise date is one year from now. The risk free effective annual interest rate is 6%. The premium of this call is \$6.4133 per share.

(v) Find the spot price at redemption at which both profits are equal.

Example 10

Joseph buys a one–year long forward for 100 shares of a stock at \$74 per share. Samantha buys a call option of 100 shares of XYZ stock for \$76 per share. The exercise date is one year from now. The risk free effective annual interest rate is 6%. The premium of this call is \$6.4133 per share.

(v) Find the spot price at redemption at which both profits are equal.

Solution: (v) We solve $(100)(S_T - 74) = 100\max(0, S_T - 76) - 679.81$ for S_T. There is not solution with $S_T \geq 76$. If $S_T < 76$ we have the equation $(100)(S_T - 74) = -679.81$, or $S_T = 74 - 6.7981 = 67.2019$.

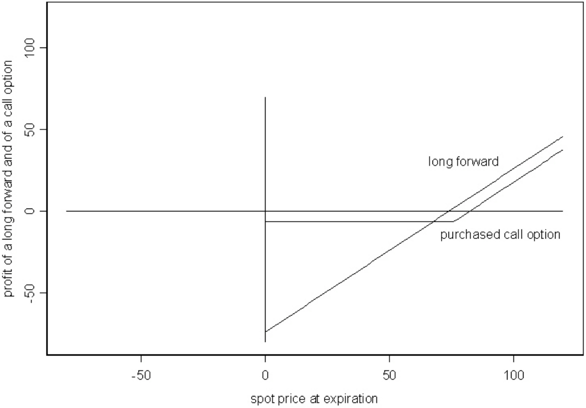

Figure 3: Example 10. Profit for long forward and purchased call.

A purchased call option reduces losses over a long forward. Notice that in Figure 3 the losses for a long forward holder can be big if the spot price at redemption is small. A call option is an insured long position in an asset. In return for not having large losses, the possible profits in a call option are smaller. The spot price needed to make money is bigger for a purchased call than for a long forward. The profit for the call option holder is positive if

$$S_T > K + \mathrm{Call}(K, T)(1+i)^T.$$

The profit for the long forward is positive if $S_T > F_{0,T}$. By Theorem 5,

$$K + \mathrm{Call}(K, T)(1+i)^T > F_{0,T}.$$

To make a positive profit, a call option holder needs a bigger increase on the spot price than a long forward holder.

Theorem 5

If there exists no arbitrage, then

$$(1+i)^{-T}\max(F_{0,T}-K,0) < \mathrm{Call}(K,T) < (1+i)^{-T}F_{0,T}.$$

Proof: Suppose that you enter into a short forward contract and you buy a call option. Both contracts have the same expiration time and nominal amount. At expiration, the profit of this strategy is

$$\begin{aligned}&F_{0,T} - S_T + \max(S_T - K, 0) - (1+i)^T \mathrm{Call}(K, T)\\ =&F_{0,T} + \max(-K, -S_T) - (1+i)^T \mathrm{Call}(K, T)\\ =&F_{0,T} - \min(K, S_T) - (1+i)^T \mathrm{Call}(K, T).\end{aligned}$$

This profit function is increasing on S_T and it not constant. The minimum profit of this portfolio is

$$F_{0,T} - K - (1+i)^T \mathrm{Call}(K, T).$$

The maximum profit of this portfolio is

$$F_{0,T} - (1+i)^T \mathrm{Call}(K, T).$$

If there is no arbitrage,

$$F_{0,T} - K - (1+i)^T \mathrm{Call}(K, T) < 0 < F_{0,T} - (1+i)^T \mathrm{Call}(K, T).$$

Example 11

The current price of a forward contract for 1000 *units of an asset with expiration date two years from now is* \$120000*. The risk–free annual rate of interest compounded continuously is* 5%*. The price of a two–year* 100*–strike European call option for 1000 units of the asset is* \$15000*. Find an arbitrage portfolio and its minimum profit.*

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Solution: Since

$$\begin{aligned} & e^{-rT}(F_{0,T} - K) = e^{-(2)(0.05)}(120000 - (100)(1000)) \\ = & 18096.74836 > 15000, \end{aligned}$$

the call option is under priced. Consider the portfolio consisting of buying the call and entering into a short forward. The profit is

$$\begin{aligned} & 1000 \max(S_T - 100, 0) - 15000e^{(2)(0.05)} + 120000 - 1000S_T \\ = & 1000 \max(-100, -S_T) + 103422.4362 \\ = & 103422.4362 - 1000 \min(100, S_T). \end{aligned}$$

The minimum profit is $103422.4362 - 1000(100) = 3422.4362$.

Another motive to buy call options is to speculate. Call options allow betting in the increase of the price of a particular asset for a small cash outlay. Buying a call option, a speculator achieves leverage. Call options provide price exposure without having to pay, hold and warehouse the underlying asset. If a speculator believes that an asset price is going to increase and it is right, he can get a much higher yield of return buying a call option than buying the asset.

Example 12

Rachel is a speculator. She anticipates XYZ stock to appreciate from its current level of $130 per share in four months. Rachel buys a four–month 1000–share call option with a strike price of $150 per share and a premium of $1.8074 per share. Luke is also a speculator. He also expects XYZ stock to appreciate and buys XYZ stock at the current market price.

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(i) Find Rachel's annual effective rate of return in her investment for the following spot prices at expiration 130, 150, 160 and 170.

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(i) Find Rachel's annual effective rate of return in her investment for the following spot prices at expiration 130, 150, 160 and 170.

Solution: (i) Rachel invests $(1000)(1.8074) = 1807.4$. Four months later, she receives $(1000)\max(S_T - 150, 0)$.

If $S_T \leq 150$, Rachel loses all her money and her yield of return is -100%. If $S_T = 160$, Rachel receives $(1000)(160 - 150) = 10000$ at expiration. Rachel's annual rate of return is $\left(\frac{10000}{1807.4}\right)^3 - 1 = 168.3702647 = 16837.02647\%$. If $S_T = 170$, Rachel receives $(1000)(170 - 150) = 20000$ at expiration. Rachel's annual rate of return is $\left(\frac{20000}{1807.4}\right)^3 - 1 = 1353.962117 = 135396.2117\%$.

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(ii) Luke sells his stock at the end of four months. Find Luke's annual effective rate of return in his investment for the spot prices in (i).

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(ii) Luke sells his stock at the end of four months. Find Luke's annual effective rate of return in his investment for the spot prices in (i).

Solution: (ii) Luke invests 130 per share. His annual rate of return j satisfies $S_T = 130(1+j)^{1/3}$. So, $j = \left(\frac{S_T}{130}\right)^3 - 1$.

If $S_T = 130$, $j = 0\%$.

If $S_T = 150$, $j = 53.61857078\%$.

If $S_T = 160$, $j = 86.43604916\%$.

If $S_T = 170$, $j = 123.6231224\%$.

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(iii) Compare the rates in (i) and (ii).

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(iii) Compare the rates in (i) and (ii).

Solution: (iii) In the case that XYZ stock does not appreciate, Rachel loses all her money. But in the cases where XYZ stock appreciates, Rachel makes a much higher yield than Luke.

Let j_C be the rate of return which an investor makes buying a call option. Since the payoff per share is $\max(S_T - K, 0)$, we have that

$$\frac{\max(S_T - K, 0)}{\mathrm{Call(K,T)}} = (1 + j_C)^T.$$

Let j_B be the rate of return which an investor makes buying an asset and holding it for T years. Since the payoff per share is S_T, we have that

$$S_T = (1 + j_B)^T S_0.$$

We have that $j_C > j_B$ if

$$\frac{\max(S_T - K, 0)}{\mathrm{Call(K,T)}} > \frac{S_T}{S_0},$$

which is equivalent to $S_0 > \mathrm{Call(K,T)}$ and

$$S_T > \frac{\frac{K}{\mathrm{Call(K,T)}}}{\frac{1}{\mathrm{Call(K,T)}} - \frac{1}{S_0}} = \frac{KS_0}{S_0 - \mathrm{Call(K,T)}}.$$

We conclude that if S_T is large enough, investing in an option call gives a larger yield than buying an asset.

Next we consider call options with different strike prices. If $0 < K_1 < K_2$, then

$$\max(S_T - K_2, 0) \leq \max(S_T - K_1, 0),$$

i.e. the payoff of a K_1–strike call option is higher than the payoff of a K_2–strike call option (see Figure 4). Hence, the price of the call is bigger for the call with smaller strike price (see Theorem 6).

Example 13

The current price of XYZ stock is \$75 per share. The annual effective rate of interest is 5%. The redemption time is one year from now. Draw the payoff and profit diagrams for the buyer of:

Example 13

The current price of XYZ stock is \$75 per share. The annual effective rate of interest is 5%. The redemption time is one year from now. Draw the payoff and profit diagrams for the buyer of:

(i) a \$70 strike call option with a premium of \$10.755.

Example 13

The current price of XYZ stock is \$75 per share. The annual effective rate of interest is 5%. The redemption time is one year from now. Draw the payoff and profit diagrams for the buyer of:
(i) a \$70 strike call option with a premium of \$10.755.
Solution: (i) The payoff is $\max(S_T - 70, 0)$. The diagram of this payoff is in Figure 4. The profit is

$$\max(S_T - 70, 0) - (10.755)(1.05) = \max(S_T - 70, 0) - 11.29275.$$

The diagram of this profit is in Figure 5.

Example 13

The current price of XYZ stock is \$75 per share. The annual effective rate of interest is 5%. The redemption time is one year from now. Draw the payoff and profit diagrams for the buyer of:
(ii) a \$80 strike call option with a premium of \$5.445.

Example 13

The current price of XYZ stock is \$75 per share. The annual effective rate of interest is 5%. The redemption time is one year from now. Draw the payoff and profit diagrams for the buyer of:
(ii) a \$80 strike call option with a premium of \$5.445.
Solution: (ii) The payoff is $\max(S_T - 80, 0)$. The diagram of this payoff is in Figure 4. The profit is

$$\max(S_T - 80, 0) - (5.445)(1.05) = \max(S_T - 80, 0) - 5.71725.$$

The diagram of this profit is in Figure 5.

Example 13

The current price of XYZ stock is \$75 per share. The annual effective rate of interest is 5%. The redemption time is one year from now. Draw the payoff and profit diagrams for the buyer of:

(iii) Find the spot price at redemption at which both profits are equal.

Example 13

The current price of XYZ stock is $75 per share. The annual effective rate of interest is 5%. The redemption time is one year from now. Draw the payoff and profit diagrams for the buyer of:
(iii) Find the spot price at redemption at which both profits are equal.

Solution: (iii) The profit amounts are equal for some $S_T \in (70, 80)$. So,

$$S_T - 70 - 11.29275 = \max(S_T - 70, 0) - 11.29275$$
$$= \max(S_T - 80, 0) - 5.71725 = -5.71725$$

and $S_T = 70 + 11.29275 - 5.71725 = 75.5755$.

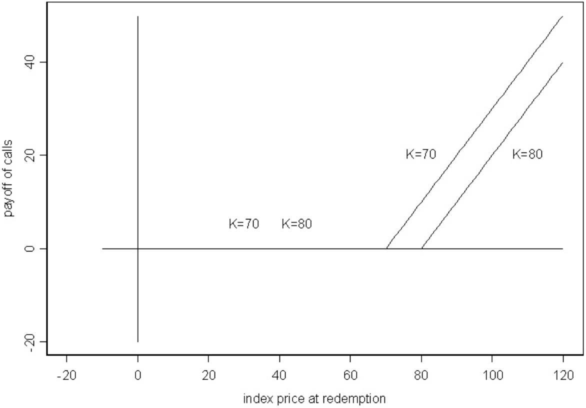

Figure 4: Example 13. Payoff for two calls with different strikes.

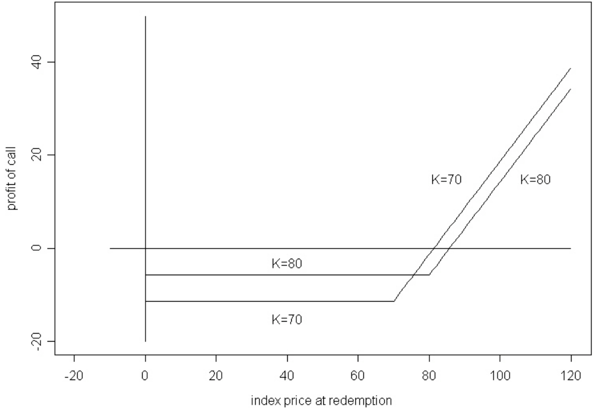

Figure 5: Example 13. Profit for two calls with different strikes.

Theorem 6

If $0 < K_1 < K_2$, *then*

$$\mathrm{Call}(K_2, T) \le \mathrm{Call}(K_1, T) \le \mathrm{Call}(K_2, T) + (K_2 - K_1)e^{-rT}.$$

Proof.

We have that

$$\begin{aligned}
&\max(S_T - K_2, 0) \\
\leq &\max(S_T - K_1, 0) = K_2 - K_1 + \max(S_T - K_2, K_1 - K_2) \\
\leq &K_2 - K_1 + \max(S_T - K_2, 0).
\end{aligned}$$

In other words,
(i) The payoff for a K_2–strike call is smaller than the payoff for a K_1–strike call.
(ii) The payoff for a K_1–strike call is smaller than $K_2 - K_1$ plus the payoff for a K_2–strike call.
Hence, if there exist no arbitrage, then

$$\mathrm{Call}(K_2, T) \leq \mathrm{Call}(K_1, T) \leq \mathrm{Call}(K_2, T) + (K_2 - K_1)e^{-rT}.$$

$\square$

Example 14

Consider two European call options on a stock worth $S_0 = 32$, both with expiration date exactly two years from now and the same nominal amount. The risk–free annual rate of interest compounded continuously is 5%. One call option has strike price \$30 and the other one \$35. The price of the 30–strike call is 7.

Example 14

Consider two European call options on a stock worth $S_0 = 32$, both with expiration date exactly two years from now and the same nominal amount. The risk–free annual rate of interest compounded continuously is 5%. One call option has strike price \$30 and the other one \$35. The price of the 30–strike call is 7.

(i) Suppose that the price of the 35–strike call option is 8, find an arbitrage portfolio.

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(i) Suppose that the price of the 35–strike call option is 8, find an arbitrage portfolio.

Solution: (i) Here, $\mathrm{Call}(35, T) \le \mathrm{Call}(30, T)$ does not hold. We can do arbitrage by a buying a 30–strike call option and selling a 35–strike call option, both for the same nominal amount. The profit per share is

$$\begin{aligned} &\max(S_T - 30, 0) - \max(S_T - 35, 0) + (8 - 7)e^{0.05} \\ \geq&(8 - 7)e^{0.05} = 1.051271096. \end{aligned}$$

Example 14

Consider two European call options on a stock worth $S_0 = 32$*, both with expiration date exactly two years from now and the same nominal amount. The risk–free annual rate of interest compounded continuously is* 5%*. One call option has strike price \$30 and the other one \$35. The price of the 30–strike call is 7.*
(ii) Suppose that the price of the 35–strike call option is 1, find an arbitrage portfolio.

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Consider two European call options on a stock worth $S_0 = 32$, both with expiration date exactly two years from now and the same nominal amount. The risk–free annual rate of interest compounded continuously is 5%. One call option has strike price \$30 and the other one \$35. The price of the 30–strike call is 7.
(ii) Suppose that the price of the 35–strike call option is 1, find an arbitrage portfolio.
Solution: (ii) We have that

$$\begin{aligned}&\mathrm{Call}(K_2, T) - \mathrm{Call}(K_1, T) + (K_2 - K_1)e^{-rT}\\ =&1 - 7 + (35 - 30)e^{-0.05} = -1.243852877 < 0.\end{aligned}$$

We can do arbitrage by buying a 35–strike call option and selling a 30–strike call option.

Example 14

Consider two European call options on a stock worth S_0 =32, both with expiration date exactly two years from now and the same nominal amount. The risk–free annual rate of interest compounded continuously is 5%. One call option has strike price \$30 and the other one \$35. The price of the 30–strike call is 7.

(ii) Suppose that the price of the 35–strike call option is 1, find an arbitrage portfolio.

Solution: (ii) (continuation) We can do arbitrage by buying a 35–strike call option and selling a 30–strike call option. The profit per share is

$$\begin{aligned} &\max(S_T - 35, 0) - \max(S_T - 30, 0) + (7-1)e^{0.05} \\ =& -5 + \max(S_T - 30, 5) - \max(S_T - 30, 0) + (7-1)e^{0.05} \\ \geq& -5 + (7-1)e^{0.05} = -5 + 6.307626578 = 1.307626578. \end{aligned}$$

The premium of a call option of an asset depends on several factors, like asset price, interest rate, expiration time, strike price, and asset price variability. We have the following rules of thump for the price of a call:

- Higher asset prices lead to higher call option prices.
- Higher strike prices lead to lower call option prices.
- Higher interest rates lead to higher call option prices.
- Higher expiration time leads to higher call option prices.
- Higher variation of an asset price leads to higher call option prices.

Since the call option buyer's payoff decreases as the strike increases, the (price) premium of a call option decrease as the strikes increases. Hence, between two call options with different strike prices:
(i) The call option with smaller strike price has a bigger premium.
(ii) If the spot price is low enough, both call options substain a loss. The loss is bigger for the call option with the smaller strike price.
(iii) If the spot price is high enough, both call options have a positive profit. The profit is bigger for the call option with the smaller strike price.
We can check the previous assertions analytically using Theorem 6.

The strike price is paid at the expiration time, as higher the interest rate is as higher the call option premium is. As higher the expiration time as higher the call option premium is. The greater the past variability of the price of an asset is as more likely is that the option will be exercised. So, higher variation of an asset price leads to higher call option prices.

The common method to find the price of a call option of a stock is to use the **Black–Scholes formula**[1]

$$\mathrm{Call}(K, T) = S_0 e^{-\delta T}\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

where

$$d_1 = \frac{\log(S_0/K) + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}};$$

$$d_2 = d_1 - \sigma\sqrt{T};$$

S_0 is the current price of the stock; K is the strike price; r is the risk free continuously compounded annual interest rate; δ is the continuous rate of dividend payments; T is the expiration time in years of the option; σ is the implied volatility for the underlying asset and Φ the cumulative distribution function of a standard normal distribution.

[1] In 1973, Fischer Black and Myron Scholes published a paper presenting the pricing formula for call and put options.

Table 1 shows the premium of a call option for different strike prices. We have used $S_0 = 75$, $T = 1$, $\sigma = 0.20$, $\delta = 0$, $r = \ln(1.05)$.

Table 1:

K	65	70	75	80	85
$\mathrm{Call}(K, T)$	14.31722	10.75552	7.78971	5.444947	3.680736

Example 15

The current price of XYZ stock is \$75 per share. The annual effective rate of interest is 5%. The redemption time is one year from now. The price of stock one year from now is \$73.5. Calculate the profit per share at expiration for the holder of each one of the call options in Table 1.

Solution: The profit is

$$\begin{aligned}&\max(S_T - K, 0) - (1.05)\mathrm{Call}(K, T)\\ =&\max(73.5 - K, 0) - (1.05)\mathrm{Call}(K, T).\end{aligned}$$

The corresponding profits are:

$$\begin{aligned}
&\text{if } K = 65, \max(73.5 - 65, 0) - (1.05)(14.31722) = -6.533081,\\
&\text{if } K = 70, \max(73.5 - 70, 0) - (1.05)(10.75552) = -7.793296,\\
&\text{if } K = 75, \max(73.5 - 75, 0) - (1.05)(7.78971) = -8.1791955,\\
&\text{if } K = 80, \max(73.5 - 80, 0) - (1.05)(5.444947) = -5.71719435,\\
&\text{if } K = 85, \max(73.5 - 85, 0) - (1.05)(3.680736) = -3.8647728.
\end{aligned}$$

If K is very small, the call option will almost certainly be executed. Hence, if K is very small, $\mathrm{Call}(K, T) = F_{0,T}$, i.e. $\lim_{K \to 0+} \mathrm{Call}(K, T) = F_{0,T}$. If K is very large, the call option will almost certainly not be executed. Hence, $\lim_{K \to \infty} \mathrm{Call}(K, T) = 0$. As a function on K, $\mathrm{Call}(K, T)$ is a decreasing function with $\lim_{K \to 0+} \mathrm{Call}(K, T) = F_{0,T}$ and $\lim_{K \to \infty} \mathrm{Call}(K, T) = 0$. Figure 6 shows the graph of $\mathrm{Call}(K, T)$ as a function of T.

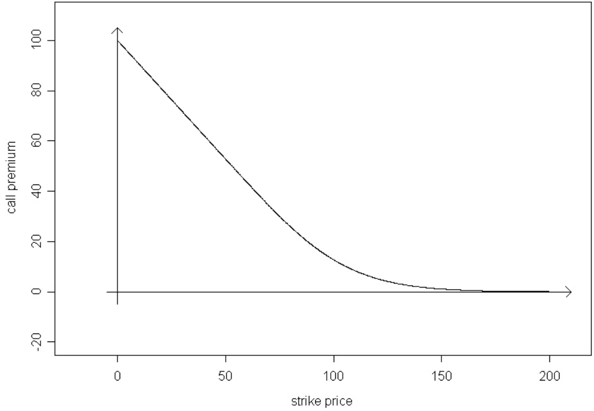

Figure 6: Example 16. Graph of $\mathrm{Call}(K, T)$ as a function of K.

Example 16

Using the Black–Scholes formula with $T = 1$, $S_0 = 100$, $T = 1$, $\sigma = 0.25$, $r = \ln(1.06)$ and $\delta = 0.0$, the following table of call option premiums was obtained:

$\mathrm{Call}(K, T)$	76.4150	52.8366	30.0399	12.7562	4.1341	0.8417	0.2672	0.0605
K	25	50	75	100	125	150	175	200

Figure 6 shows the graph of this function.

When we consider $\mathrm{Call}(K, T)$ as function of T. If T is small enough, then the option will be exercised if $S_0 > K$ with a profit of $S_0 - K$. Hence, if $S_0 > K$, $\lim_{T \to 0+} \mathrm{Call}(K, T) = S_0 - K$. Notice that by buying the call option for $\mathrm{Call}(K, T)$, we buy an asset worth S_0 for K. If T is small enough and $S_0 < K$, the option is not exercised and his value is zero, i.e. $\lim_{T \to 0+} \mathrm{Call}(K, T) = 0$.

An option is **in–the–money option** if it would have a positive payoff if exercised immediately. An option is **out–the–money** option if it would have a negative payoff if exercised immediately. An option is **at–the–money** option if it would have a zero payoff if exercised immediately. The previous definition hold for both call and put options. Put options will considered shortly. For a purchased call option, we have

- The purchased call option is in–the–money, if $S_0 > K$.
- The purchased call option is out–the–money, if $S_0 < K$.
- The purchased call option is at–the–money, if $S_0 = K$.

