Chapter 1

Introduction

In this chapter we introduce the notions of a **P**artial **D**ifferential **E**quation (PDE) and its solution. We attempt classifying all partial differential equations in at least three different ways.

1.1 Preliminaries

Partial differential equation is an equation involving an unknown function (possibly a vector-valued) of two or more variables and a finite number of its partial derivatives.
In the sequel we reserve the following terminology and notations:

- **Independent variables:** denoted by $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \Omega \subseteq \mathbb{R}^n$ $(n \geq 2)$
- **Dependent variables:** denoted by $\mathbf{u} = (u_1, u_2, \cdots, u_p) \in \mathbb{R}^p$ also called unknown function.
- Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Then $D^\alpha u$ denotes

$$D^\alpha u = \frac{\partial^\alpha u}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2}\, \cdots \partial x_n^{\alpha_n}} \tag{1.1}$$

- For $l \in \mathbb{N}$, $D^l \mathbf{u}$ denotes the tensor of all partial derivatives of order l. That is, collection of all partial derivatives $D^\alpha u$ such that $|\alpha| = l$ of the vector function $\mathbf{u}$.

We now define a PDE more formally.

Definition 1.1 (PDE) *Let $\Omega \subseteq \mathbb{R}^n$, $m \in \mathbb{N}$ and $\mathbf{F} : \Omega \times \mathbb{R}^p \times \mathbb{R}^{np} \times \mathbb{R}^{n^2 p} \times \cdots \times \mathbb{R}^{n^m p} \longrightarrow \mathbb{R}^q$ be a function. A system of Partial differential equations of* **order m** *is defined by the equation*

$$\mathbf{F}\left(\mathbf{x},\, \mathbf{u},\, D\mathbf{u},\, D^2\mathbf{u}, \cdots, D^m\mathbf{u}\right) = 0, \tag{1.2}$$

where some m^{th} order partial derivative of the vector function $\mathbf{u}$ appears in the system of equations (1.2).

Remark 1.2 *The equation* (1.2) *consists of q equations. Note that the unknown vector function $\mathbf{u}$ has p components. If $p = q$, the system of PDE is called determined. If $p < q$, then the system of PDE is called over-determined system. If $p > q$, then the system of PDE is called under-determined system. I feel that we should not attach too much of importance to this terminology.*

Definition 1.3 (Solution of a PDE) *Let U be an open subset of $\mathbb{R}^n$ and $\mathbf{\Phi} : U \longrightarrow \mathbb{R}^p$ be a function which is m times differentiable. Then $\mathbf{\Phi}$ is said to be a solution of the PDE* (1.2) *if it satisfies*

$$\mathbf{F}\left(\mathbf{x},\, \mathbf{\Phi}(\mathbf{x}),\, D\mathbf{\Phi}(\mathbf{x}),\, D^2\mathbf{\Phi}(\mathbf{x}), \cdots, D^m\mathbf{\Phi}(\mathbf{x})\right) = 0 \text{ for all } \mathbf{x} \in \Omega.$$

What equations we study? In our course we restrict our studies to equations where $(m, n, p, q) = (1, 2, 1, 1)$ or $(m, n, p, q) = (2, 2, 1, 1)$. We are going to study non-linear first order PDE and linear second order PDE.

Remark 1.4

1. *There is no guarantee that an equation such as* (1.2) *will have a solution. In fact, the PDE* $(u_x)^2 + 1 = 0$ *has no solution. Thus we cannot hope to have a very general existience theorem for equations of type* (1.2)*.*

2. *To convince ourselves that we do not expect every PDE to have a solution, let us recall the situation with other types of equations involving Polynomials, Systems of linear equations, Implicit functions. In each of these cases, existence of solutions was proved under some conditions. Some of those results also characterised equations that have solution(s), for example, for systems of linear equations the characterisation was in terms of ranks of matrix defining the linear system and the corresponding augmented matrix.*

3. *In the context of ODE, there are two basic theorems that hold for equations of a special form called* **normal** *form. They are Peano's existence theorem and Cauchy-Lipschitz-Picard's existence and uniqueness theorem. We refer the reader to any good book on ODE or my lecture notes on ODE (available on my homepage). Recall that these theorems address the existence of solutions to initial value problems for a first order system of Ordinary differential equations, and this was enough to study any ODE in normal form (of any order). Can we do the same for PDE as well? Unfortunately, it is not possible. We will see this at a later time. One problem is that same kind of problems are not interesting for all PDEs.*

1.1.1 Examples of PDE

1. **Laplace Equation**
$$\Delta u \equiv \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0. \tag{1.3}$$

2. **Heat Equation**
$$\frac{\partial u}{\partial t} - \Delta u = 0. \tag{1.4}$$

3. **Wave Equation**
$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0. \tag{1.5}$$

4. **Schrödinger Equation**
$$i\frac{\partial u}{\partial t} = -\frac{\hbar}{2m}\Delta u + V(x)u(t,x) = 0, \quad t > 0,\, x \in \mathbb{R}. \tag{1.6}$$

5. **Burgers' Equation**
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2}, \quad t > 0,\, x \in \mathbb{R}, \tag{1.7}$$
where $\mu \geq 0$.

6. **Korteweg-de Vries (KdV) Equation**
$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u\frac{\partial u}{\partial x} = 0 \quad t > 0,\, x \in \mathbb{R}. \tag{1.8}$$

7. **Benjamin-Bona-Mahony (BBM) Equation**

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial t\, \partial x^2} + u\frac{\partial u}{\partial x} = 0 \quad t > 0,\, x \in \mathbb{R}. \tag{1.9}$$

8. **Vlasov-Poisson (VP) Equation**

$$\begin{aligned} &\frac{\partial f}{\partial t} + \mathbf{v} \bullet \nabla_{\mathbf{x}} f + \mathbf{E} \bullet \nabla_{\mathbf{v}} f = 0 \quad t > 0,\, \mathbf{x} \in \mathbb{R}^n,\, \mathbf{v} \in \mathbb{R}^n, \\ &\mathbf{E} = \nabla_{\mathbf{x}} V, \quad \Delta V = \int_{\mathbb{R}^n} f(t, \mathbf{x}, \mathbf{v})\, d\mathbf{v},\; V = V(\mathbf{x}),\; f \geq 0. \end{aligned} \tag{1.10}$$

9. **Maxwell's Equations**

$$\begin{aligned} \frac{\partial \mathbf{E}}{\partial t} &= \mathrm{curl}\mathbf{B} \\ \frac{\partial \mathbf{B}}{\partial t} &= -\mathrm{curl}\mathbf{E} \\ \mathrm{div}\mathbf{B} &= \mathrm{div}\mathbf{E} = 0. \end{aligned} \tag{1.11}$$

10. **Euler's Equations for incompressible, inviscid flow**

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \bullet D\mathbf{u} &= -Dp \\ \mathrm{div}\mathbf{u} &= 0. \end{aligned} \tag{1.12}$$

11. **Navier-Stokes Equations for incompressible, viscous flow**

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \bullet D\mathbf{u} - \boldsymbol{\Delta}\mathbf{u} &= -Dp \\ \mathrm{div}\mathbf{u} &= 0. \end{aligned} \tag{1.13}$$

12. **Minimal Surface Equation**

$$\left(1 + |\nabla u|^2\right) \Delta u - \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i\, \partial x_j} = 0. \tag{1.14}$$

Differences between ODE and PDE

1. A general solution of an ODE involves arbitrary constants. Obtaining a general solution for PDEs is difficult and a general solution would involve arbitrary functions (See § 2.1.2). Let us look at a simple example now. Consider the PDE $u_x = 0$. Any arbitrary function of y solves this PDE. This is the simplest possible linear equation of first order and it has an infinite dimensional space of solutions. Compare this situation with that of a linear first order ODE $\frac{d\mathbf{y}}{dt} = 0$ where $\mathbf{y} = (y_1, \cdots, y_p)$, whose solution space is $\mathbb{R}^p$ which is finite dimensional.

2. In differential equations the unknown function has the interpretation of the state of a system when the equations decribe evolution of a physical system in time. For ODEs the independent variable is time and for PDEs one of the independent variables has the interpretation of time. Now the initial state (state of the system at time $t = 0$) for ODEs is prescribed as an element of $\mathbb{R}^n$ (n is the length of the unknown vector $\mathbf{y}$); while for PDEs the initial state varies in a function space. Thus solving a PDE means finding the states of the system at different times and each of these states vary in an infinite dimensional space of function while solving ODE means finding the states of the system but are in a finite dimensional

space.[1]

For those of you who know some Functional analysis, you understand the difference between finite and infinite dimensional spaces. One important theorem which is lacking in infinite dimensional spaces is a Heine-Borel theorem concerning compactness. Thus the topologies more suited to infinite dimensional spaces are some kind of weak topologies which we do not address here.

3. Linear ODEs have global solutions. Linear PDEs posed on $\mathbb{R}^2$ do not necessarily have solutions defined on $\mathbb{R}^2$ (See § 2.2.5).

1.2 Classification

Partial differential equations can be classified in at least three ways. They are

1. Order of PDE.
2. Linear, Semi-linear, Quasi-linear, and fully non-linear.
3. Scalar equation, System of equations.

Classification based on the number of unknowns and number of equations in the PDE

If a PDE consists of more than one unknown function or more than one equation, it is called a System of PDEs. Otherwise it is called a single PDE or a scalar PDE. These kinds of definitions will have some problems. Think why!

Exercise 1.5 *Classify the examples in § 1.1.1 into systems and scalar PDEs.*

Classification based on highest order derivative appearing in the PDE

Exercise 1.6 *Find the orders of each of the PDEs appearing in § 1.1.1.*

Classification via Algebra

This classification is somewhat different from the previous ones. This is analogous to the **Postal Index Number (PIN)**. In PIN code, the first digit stands for a region (usually consists of more than one State); the second and third digits correspond to a district and the last three digits determine the location of the Post Office within the district. Similarly, PDEs may be classified just like ODEs are classified, namely by categorising them into two: Linear and Nonlinear equations. But in the case of PDEs it turns out that there can be another way of catergorising into two: Quasi-linear and Non-Quasilinear. Quasi-linear PDEs are further categorised into two: Semi-linear, Non-semilinear. Semi-linear PDEs are further categorised into two: Linear and Nonlinear. We have the following picture.

$$\textbf{Linear PDE} \subsetneq \textbf{Semi-linear PDE} \subsetneq \textbf{Quasi-linear PDE} \subsetneq \textbf{PDE}.$$

We now define each of the terminology used above for scalar PDE, and one can extend these concepts to systems of PDE easily. We choose to define them in english and then see explicitly what it means for first order PDEs.

Definition 1.7

[1] I learnt this from my teacher.

1. *A PDE of order m is called Quasi-linear if it is linear in the derivatives of order m with coefficients that depend on the independent variables and derivatives of the unknown function or order strictly less than m.*

2. *A Quasi-linear PDE where the coefficients of derivatives of order m are functions of the independent variables alone is called a Semi-linear PDE.*

3. *A PDE which is linear in the unknown function and all its derivatives with coefficients depending on the independent variables alone is called a Linear PDE.*

4. *A PDE which is not Quasi-linear is called a Fully nonlinear PDE.*

Remark 1.8

1. *A single first order Quasi-linear PDE must be of the form*

$$a(x, y, u)\, u_x + b(x, y, u)\, u_y = c(x, y, u) \tag{1.15}$$

2. *A single Quasi-linear PDE where a, b are functions of x and y alone is a Semi-linear PDE.*

3. *A single Semi-linear PDE where $c(x, y, u) = c_0(x, y)u + c_1(x, y)$ is a Linear PDE.*

Examples of Linear PDEs Linear PDEs can further be classified into two: Homogeneous and Nonhomogeneous. Every linear PDE can be written in the form

$$\mathcal{L}[\mathbf{u}] = \mathbf{f}, \tag{1.16}$$

where $\mathbf{u} \longmapsto \mathcal{L}[\mathbf{u}]$ is a linear map, and $\mathbf{f}$ is a function of independent variables only. The linear PDE (1.16) is said to be homogeneous if $\mathbf{f}$ is the zero function; otherwise it is called a nonhomogeneous linear PDE.

Why to Classify?

1. Classification process does not achieve anything. One could have avoided doing it. However since everyone does it, we also do it.

2. Some of the classifications are just branding a PDE. It is just immaterial what the branding is.

3. Some of the classifications help people identify or guess or anticipate the properties of solutions of PDEs in that class. For example, there could be one existence theorem that covers all equations which fall under a particular classification.

1.3 Problems to be studied

In general it is difficult to find general solutions to PDEs. Recall that for first order ODEs we studied Initial value problems and boundary value problems for the second order ODE. For PDEs, we are going to study similar problems. It turns out that results for PDEs are somewhat different from the "corresponding problems" for ODEs.

1.4 Properly posed problems in the sense of Hadamard

A mathematical problem is said to be properly-posed or well-posed in the sense of Hadamard if the following three conditions are satisfied:

1. The problem should admit at least one solution.

2. The problem should admit at most one solution.

3. The solution should depend continuously on the data in the problem.

1.5 Exercises

(1) We know the following classification of partial differential equations

$$\textbf{Linear PDE} \subsetneqq \textbf{Semi-linear PDE} \subsetneqq \textbf{Quasi-linear PDE} \subsetneqq \textbf{PDE}.$$

Each of the above inclusions is a strict inclusion. Justify this statement by giving examples.

(2) Give at least three examples of fifth order PDE belonging to each of the above classes.

(3) Classify the following equations by all the three ways of classification.

(i) $\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial^3 u}{\partial x^3} = 1.$

(ii) $\sin\left(1 + \frac{\partial u}{\partial x}\right)^2 + u^3 = \sin x.$

(iii) $\Delta u = 0.$

(iv) $e^{\Delta u} = 1.$

(v) $u_{tt} - \Delta u = \sin u.$

(vi) $\sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{|\alpha|=k} a_\alpha(\mathbf{x}, u) D^\alpha u = \sin \|\mathbf{x}\|.$

(vii) $\sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{|\alpha|=k} D^\alpha u = \sin \|\mathbf{x}\|.$