Methods of Analysis and Selected Topics (dc)

8.9 NODAL ANALYSIS (GENERAL APPROACH)

For mesh analysis we used Kirchhoff's voltage law around each closed loop.

Now we employ Kirchhoff's current law to develop the method: **nodal analysis.**

A **node** is defined as a junction of two or more branches.

Define one node as a reference ⟹ remaining nodes will have a potential relative to this reference.

For a network of ***N*** nodes, there is ***(N-1)*** nodes with a fixed voltage relative to the reference node.

Equations relating these nodal voltages are written by applying KCL to each of the ***(N-1)*** nodes.

The equations are then solved for these nodal voltages.

1. ***Determine the number of nodes within the network.***
2. ***Pick a reference node, and label each remaining node with a subscripted value of voltage:*** V_1, V_2, ***and so on.***
3. ***Apply Kirchhoff's current law at each node except the reference. Assume that all unknown currents leave the node for each application of Kirchhoff's current law. In other words, for each node, don't be influenced by the direction that an unknown current for another node may have had. Each node is to be treated as a separate entity, independent of the application of Kirchhoff's current law to the other nodes.***
4. ***Solve the resulting equations for the nodal voltages.***

EXAMPLE 8.19 Apply nodal analysis to the network of Fig. 8.40.

Solution:

Steps 1 and 2: The network has two nodes, as shown in Fig. 8.41. The lower node is defined as the reference node at ground potential (zero volts), and the other node as V_1, the voltage from node 1 to ground.

Step 3: I_1 and I_2 are defined as leaving the node in Fig. 8.42, and Kirchhoff's current law is applied as follows:

$$I = I_1 + I_2$$

The current I_2 is related to the nodal voltage V_1 by Ohm's law:

$$I_2 = \frac{V_{R_2}}{R_2} = \frac{V_1}{R_2}$$

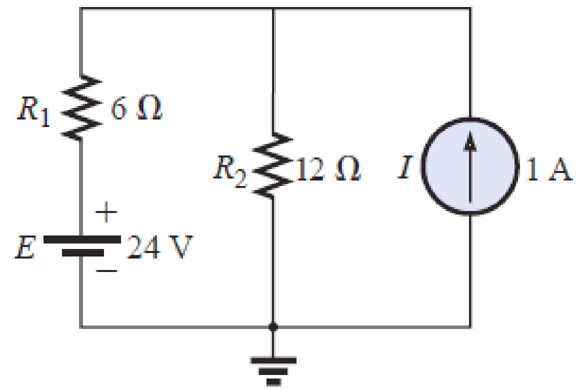
The current I_1 is also determined by Ohm's law as follows:

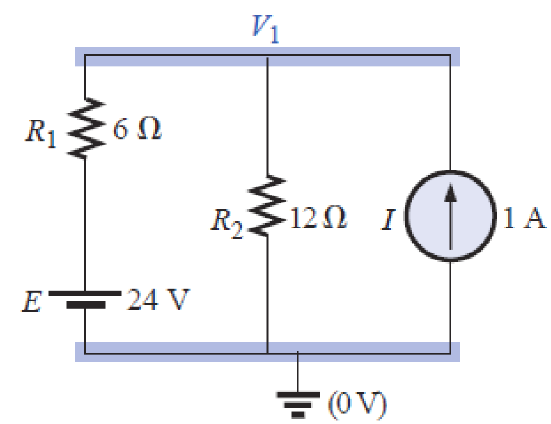
$$I_1 = \frac{V_{R_1}}{R_1}$$

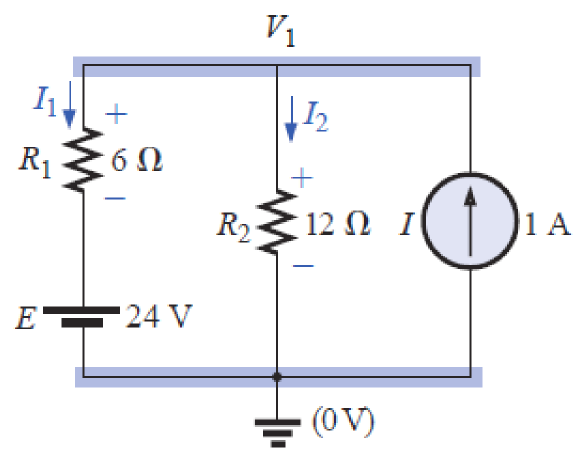
with $$V_{R_1} = V_1 - E$$

Substituting into the Kirchhoff's current law equation:

$$I = \frac{V_1 - E}{R_1} + \frac{V_1}{R_2}$$







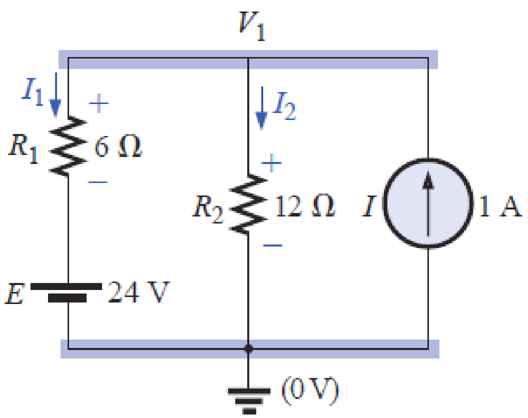
and rearranging, we have

$$I = \frac{V_1}{R_1} - \frac{E}{R_1} + \frac{V_1}{R_2} = V_1\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - \frac{E}{R_1}$$

or

$$V_1\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{E}{R_1} + I$$

Substituting numerical values, we obtain



$$V_1\left(\frac{1}{6\ \Omega} + \frac{1}{12\ \Omega}\right) = \frac{24\ \text{V}}{6\ \Omega} + 1\ \text{A} = 4\ \text{A} + 1\ \text{A}$$

$$V_1\left(\frac{1}{4\ \Omega}\right) = 5\ \text{A}$$

$$V_1 = \mathbf{20\ V}$$

The currents I_1 and I_2 can then be determined using the preceding equations:

$$I_1 = \frac{V_1 - E}{R_1} = \frac{20\ \text{V} - 24\ \text{V}}{6\ \Omega} = \frac{-4\ \text{V}}{6\ \Omega}$$

$$= \mathbf{-0.667\ A}$$

The minus sign indicates simply that the current I_1 has a direction opposite to that appearing in Fig. 8.42.

$$I_2 = \frac{V_1}{R_2} = \frac{20\ \text{V}}{12\ \Omega} = \mathbf{1.667\ A}$$

EXAMPLE 8.20 Apply nodal analysis to the network of Fig. 8.43.

Solution 1:

Steps 1 and 2: The network has three nodes, as defined in Fig. 8.44, with the bottom node again defined as the reference node (at ground potential, or zero volts), and the other nodes as V_1 and V_2.

Step 3: For node V_1 the currents are defined as shown in Fig. 8.45, and Kirchhoff's current law is applied:

$$0 = I_1 + I_2 + I$$

with

$$I_1 = \frac{V_1 - E}{R_1}$$

and

$$I_2 = \frac{V_{R_2}}{R_2} = \frac{V_1 - V_2}{R_2}$$

so that

$$\frac{V_1 - E}{R_1} + \frac{V_1 - V_2}{R_2} + I = 0$$

or

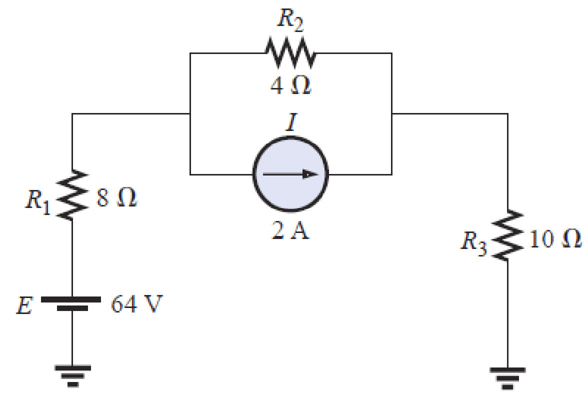
$$\frac{V_1}{R_1} - \frac{E}{R_1} + \frac{V_1}{R_2} - \frac{V_2}{R_2} + I = 0$$

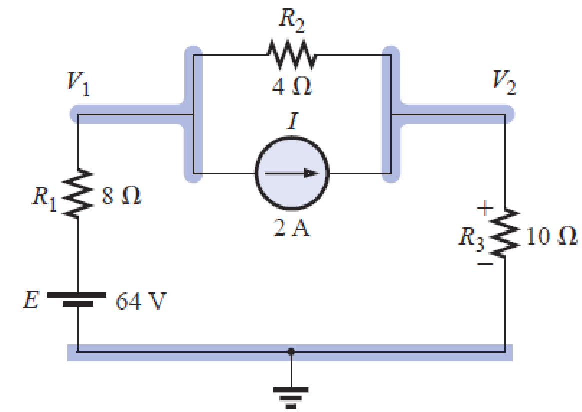
and

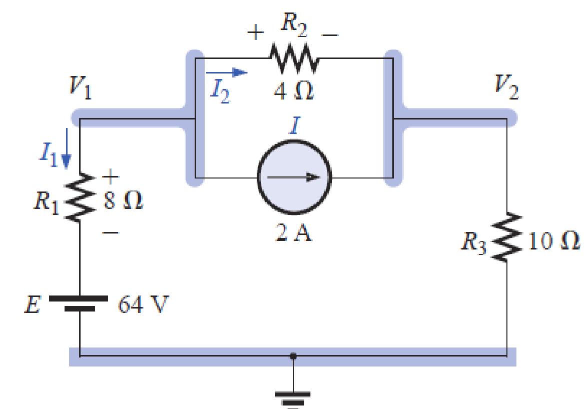
$$V_1\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - V_2\left(\frac{1}{R_2}\right) = -I + \frac{E}{R_1}$$

Substituting values:

$$V_1\left(\frac{1}{8\ \Omega} + \frac{1}{4\ \Omega}\right) - V_2\left(\frac{1}{4\ \Omega}\right) = -2\ \text{A} + \frac{64\ \text{V}}{8\ \Omega} = 6\ \text{A}$$







For node V_2 the currents are defined as shown in Fig. 8.46, and Kirchhoff's current law is applied:

$$I = I_2 + I_3$$

with

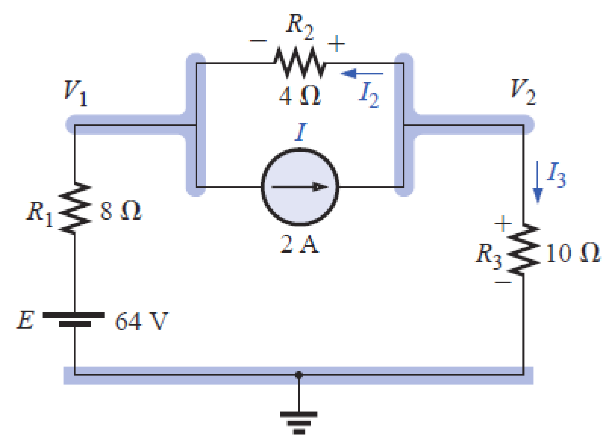
$$I = \frac{V_2 - V_1}{R_2} + \frac{V_2}{R_3}$$

or

$$I = \frac{V_2}{R_2} - \frac{V_1}{R_2} + \frac{V_2}{R_3}$$

and

$$V_2\left(\frac{1}{R_2} + \frac{1}{R_3}\right) - V_1\left(\frac{1}{R_2}\right) = I$$



Substituting values:

$$V_2\left(\frac{1}{4\ \Omega} + \frac{1}{10\ \Omega}\right) - V_1\left(\frac{1}{4\ \Omega}\right) = 2\ \text{A}$$

Step 4: The result is two equations and two unknowns:

$$V_1\left(\frac{1}{8\ \Omega} + \frac{1}{4\ \Omega}\right) - V_2\left(\frac{1}{4\ \Omega}\right) = 6\ \text{A}$$

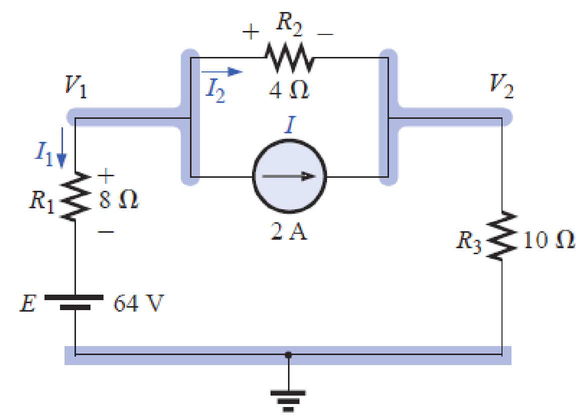
$$-V_1\left(\frac{1}{4\ \Omega}\right) + V_2\left(\frac{1}{4\ \Omega} + \frac{1}{10\ \Omega}\right) = 2\ \text{A}$$

which become

$$0.375V_1 - 0.25V_2 = 6$$
$$-0.25V_1 + 0.35V_2 = 2$$

Using determinants,

$$V_1 = \mathbf{37.818\ V}$$
$$V_2 = \mathbf{32.727\ V}$$



Since E is greater than V_1, the current I_1 flows from ground to V_1 and is equal to

$$I_{R_1} = \frac{E - V_1}{R_1} = \frac{64\ \text{V} - 37.818\ \text{V}}{8\ \Omega} = \mathbf{3.273\ A}$$

The positive value for V_2 results in a current I_{R_3} from node V_2 to ground equal to

$$I_{R_3} = \frac{V_{R_3}}{R_3} = \frac{V_2}{R_3} = \frac{32.727\ \text{V}}{10\ \Omega} = \mathbf{3.273\ A}$$

Since V_1 is greater than V_2, the current I_{R_2} flows from V_1 to V_2 and is equal to

$$I_{R_2} = \frac{V_1 - V_2}{R_2} = \frac{37.818\ \text{V} - 32.727\ \text{V}}{4\ \Omega} = \mathbf{1.273\ A}$$

EXAMPLE 8.21 Determine the nodal voltages for the network of Fig. 8.48.

Solution:

Steps 1 and 2: As indicated in Fig. 8.49.

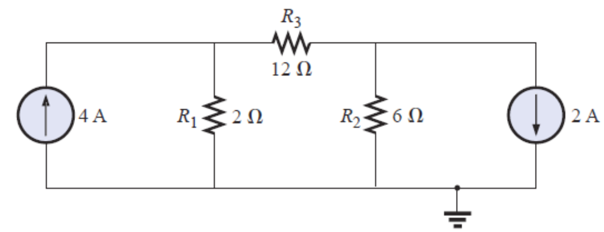
Step 3: Included in Fig. 8.49 for the node V_1. Applying Kirchhoff's current law:

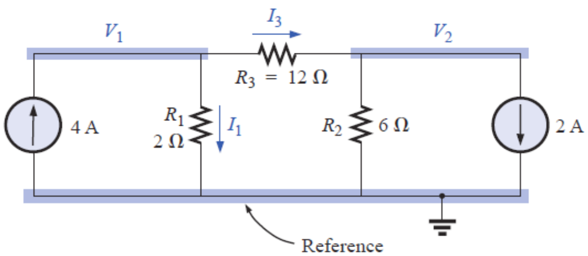
$$4\text{ A} = I_1 + I_3$$

and
$$4\text{ A} = \frac{V_1}{R_1} + \frac{V_1 - V_2}{R_3} = \frac{V_1}{2\ \Omega} + \frac{V_1 - V_2}{12\ \Omega}$$

Expanding and rearranging:

$$V_1\left(\frac{1}{2\ \Omega} + \frac{1}{12\ \Omega}\right) - V_2\left(\frac{1}{12\ \Omega}\right) = 4\text{ A}$$





For node V_2 the currents are defined as in Fig. 8.50.

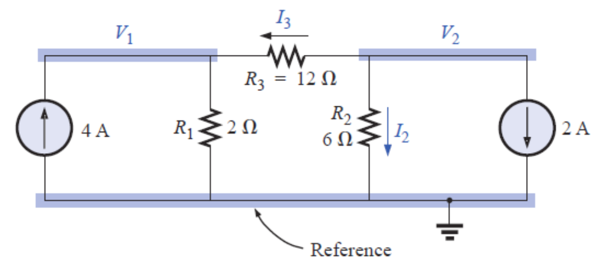
Applying Kirchhoff's current law:

$$0 = I_3 + I_2 + 2\text{ A}$$

and $$\frac{V_2 - V_1}{R_3} + \frac{V_2}{R_2} + 2\text{ A} = 0 \longrightarrow \frac{V_2 - V_1}{12\ \Omega} + \frac{V_2}{6\ \Omega} + 2\text{ A} = 0$$

Expanding and rearranging:

$$V_2\left(\frac{1}{12\ \Omega} + \frac{1}{6\ \Omega}\right) - V_1\left(\frac{1}{12\ \Omega}\right) = -2\text{ A}$$



resulting in two equations and two unknowns (numbered for later reference):

$$\left.\begin{aligned} V_1\left(\frac{1}{2\ \Omega} + \frac{1}{12\ \Omega}\right) - V_2\left(\frac{1}{12\ \Omega}\right) &= +4\text{ A} \\ V_2\left(\frac{1}{12\ \Omega} + \frac{1}{6\ \Omega}\right) - V_1\left(\frac{1}{12\ \Omega}\right) &= -2\text{ A} \end{aligned}\right\} \tag{8.3}$$

producing

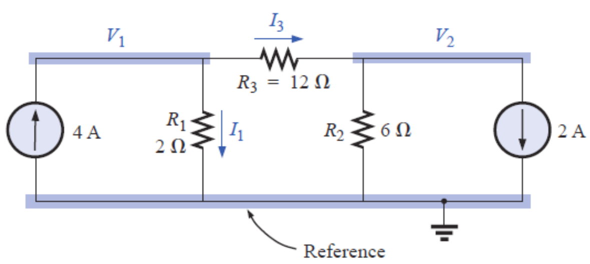
$$\left.\begin{aligned} \frac{7}{12}V_1 - \frac{1}{12}V_2 &= +4 \\ -\frac{1}{12}V_1 + \frac{3}{12}V_2 &= -2 \end{aligned}\right\} \begin{aligned} 7V_1 - V_2 &= 48 \\ -1V_1 + 3V_2 &= -24 \end{aligned}$$

and $$V_1 = \frac{\begin{vmatrix} 48 & -1 \\ -24 & 3 \end{vmatrix}}{\begin{vmatrix} 7 & -1 \\ -1 & 3 \end{vmatrix}} = \frac{120}{20} = \mathbf{+6\ V}$$

$$V_2 = \frac{\begin{vmatrix} 7 & 48 \\ -1 & -24 \end{vmatrix}}{20} = \frac{-120}{20} = \mathbf{-6\ V}$$

Since V_1 is greater than V_2, the current through R_3 passes from V_1 to V_2. Its value is

$$I_{R_3} = \frac{V_1 - V_2}{R_3} = \frac{6\text{ V} - (-6\text{ V})}{12\ \Omega} = \frac{12\text{ V}}{12\ \Omega} = \mathbf{1\ A}$$



The fact that V_1 is positive results in a current I_{R_1} from V_1 to ground equal to

$$I_{R_1} = \frac{V_{R_1}}{R_1} = \frac{V_1}{R_1} = \frac{6\text{ V}}{2\ \Omega} = \mathbf{3\ A}$$

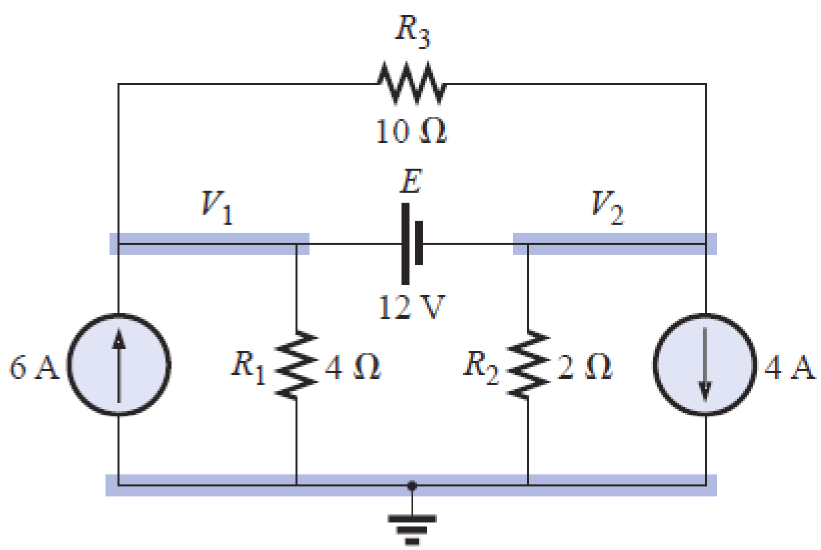
Finally, since V_2 is negative, the current I_{R_2} flows from ground to V_2 and is equal to

$$I_{R_2} = \frac{V_{R_2}}{R_2} = \frac{V_2}{R_2} = \frac{6\text{ V}}{6\ \Omega} = \mathbf{1\ A}$$

Supernode

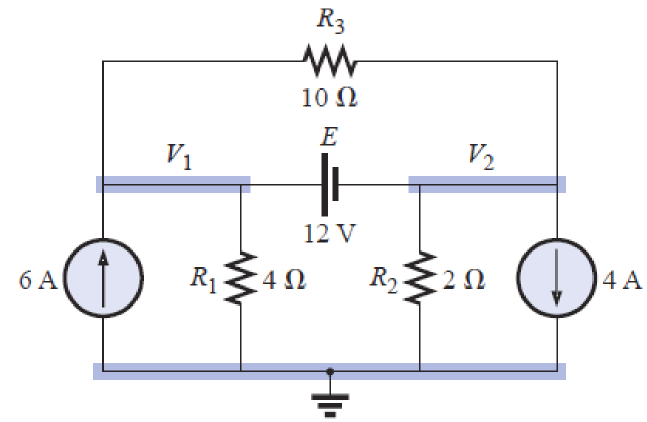
When there is a voltage source in the network:

- Start as before step 1 and 2, (voltage source is treated as resistor)
- Mentally replace the voltage source with a short circuit and apply KCL to the defined nodes
- Any node including the effect of other nodes is said **supernode**.
- Finally relate the defined nodes voltages to the voltage source



EXAMPLE 8.22 Determine the nodal voltages V_1 and V_2 of Fig. 8.51 using the concept of a supernode.

Solution: Replacing the independent voltage source of 12 V with a short-circuit equivalent will result in the network of Fig. 8.52. Even though the mental application of a short-circuit equivalent is discussed above, it would be wise in the early stage of development to redraw the network as shown in Fig. 8.52. The result is a single supernode for which Kirchhoff's current law must be applied. Be sure to leave the other defined nodes in place and use them to define the currents from that region of the network. In particular, note that the current I_3 will leave the supernode at V_1 and then enter the same supernode at V_2. It must therefore appear twice when applying Kirchhoff's current law, as shown below:



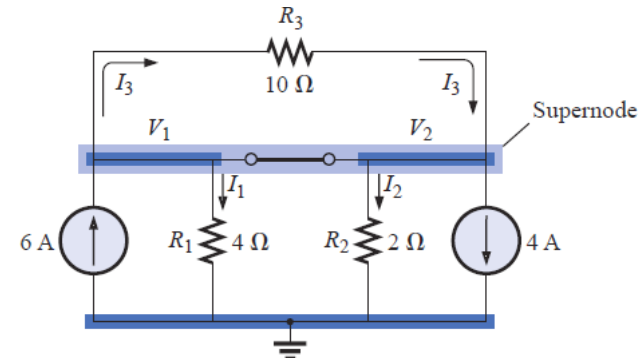


FIG. 8.52
Defining the supernode for the network of Fig. 8.51.

$$\Sigma I_i = \Sigma I_o$$

$$6\text{ A} + I_3 = I_1 + I_2 + 4\text{ A} + I_3$$

or $$I_1 + I_2 = 6\text{ A} - 4\text{ A} = 2\text{ A}$$

Then $$\frac{V_1}{R_1} + \frac{V_2}{R_2} = 2\text{ A}$$

and $$\frac{V_1}{4\ \Omega} + \frac{V_2}{2\ \Omega} = 2\text{ A}$$

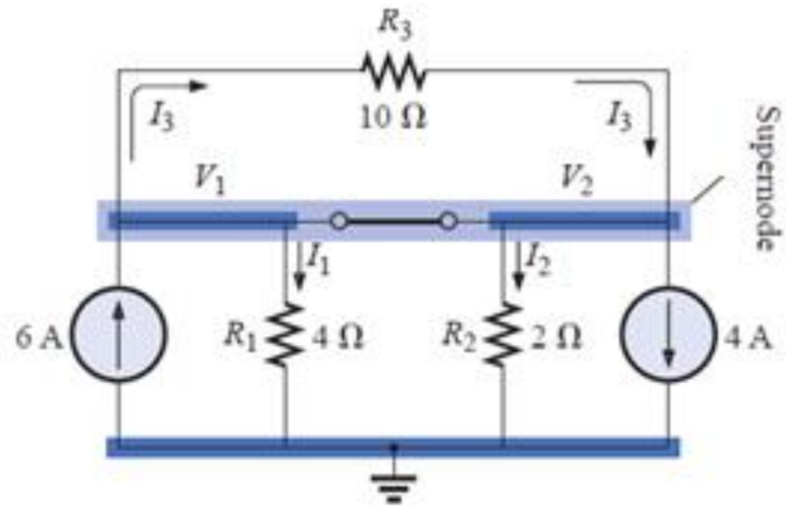


FIG. 8.52
Defining the supernode for the network of Fig. 8.51

Relating the defined nodal voltages to the independent voltage source, we have

$$V_1 - V_2 = E = 12\text{ V}$$

which results in two equations and two unknowns:

$$\begin{aligned} 0.25V_1 + 0.5V_2 &= 2 \\ V_1 - \quad 1V_2 &= 12 \end{aligned}$$

Substituting:

$$V_1 = V_2 + 12$$

$$0.25(V_2 + 12) + 0.5V_2 = 2$$

and
$$0.75V_2 = 2 - 3 = -1$$

so that
$$V_2 = \frac{-1}{0.75} = \mathbf{-1.333\text{ V}}$$

and
$$V_1 = V_2 + 12\text{ V} = -1.333\text{ V} + 12\text{ V} = \mathbf{+10.667\text{ V}}$$

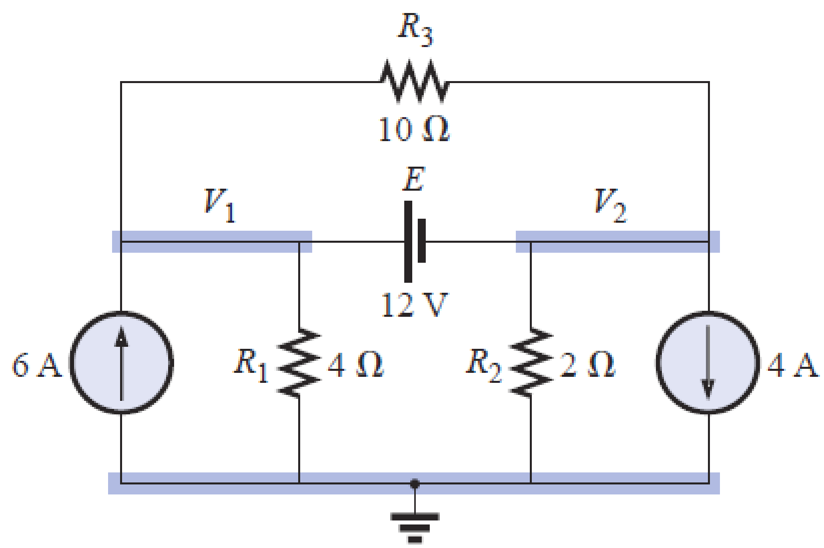
The current of the network can then be determined as follows:

$$I_1\downarrow = \frac{V}{R_1} = \frac{10.667\text{ V}}{4\ \Omega} = \mathbf{2.667\text{ A}}$$

$$I_2\uparrow = \frac{V_2}{R_2} = \frac{1.333\text{ V}}{2\ \Omega} = \mathbf{0.667\text{ A}}$$

$$\underset{\rightarrow}{I_3} = \frac{V_1 - V_2}{10\ \Omega} = \frac{10.667\text{ V} - (-1.333\text{ V})}{10\ \Omega} = \frac{12\text{ V}}{10\ \Omega} = \mathbf{1.2\text{ A}}$$

A careful examination of the network at the beginning of the analysis would have revealed that the voltage across the resistor R_3 must be 12 V and I_3 must be equal to 1.2 A.



8.10 NODAL ANALYSIS (FORMAT APPROACH)

1. *Choose a reference node and assign a subscripted voltage label to the (N -1) remaining nodes of the network.*

2. *The number of equations required for a complete solution is equal to the number of subscripted voltages (N-1).*

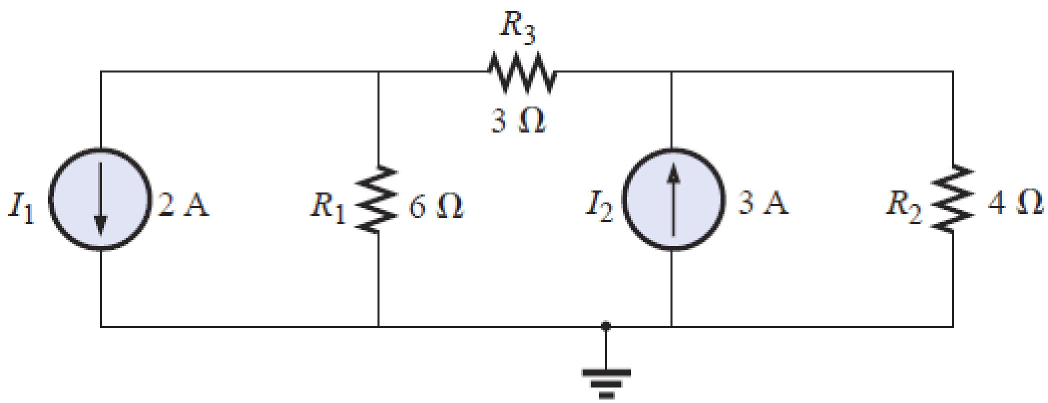
   a. *Column 1 of each equation is formed by summing the conductances tied to the node of interest and multiplying the result by that subscripted nodal voltage.*

3. *the mutual terms, (tying two nodes), are subtracted from the first column. It is possible to have more than one mutual term. Each mutual term is the product of the mutual conductance and the other nodal voltage tied to that conductance.*

4. *The column to the right of the equality sign is the algebraic sum of the current sources tied to the node of interest. A current source is assigned a positive sign if it supplies current to a node and a negative sign if it draws current from the node.*

5. *Solve the resulting simultaneous equations for the desired voltages.*

EXAMPLE 8.23 Write the nodal equations for the network of Fig. 8.53.



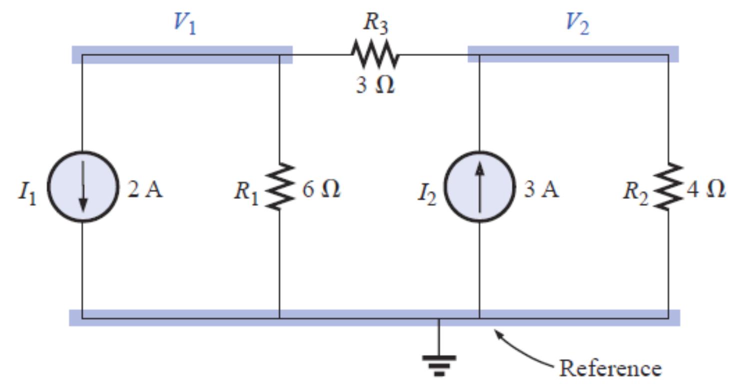
Solution:

Step 1: The figure is redrawn with assigned subscripted voltages in Fig. 8.54.

Steps 2 to 4:

$$V_1:\quad \underbrace{\left(\frac{1}{6\ \Omega}+\frac{1}{3\ \Omega}\right)}_{\substack{\text{Sum of}\\ \text{conductances}\\ \text{connected}\\ \text{to node 1}}} V_1 - \underbrace{\left(\frac{1}{3\ \Omega}\right)}_{\substack{\text{Mutual}\\ \text{conductance}}} V_2 = \overset{\substack{\text{Drawing current}\\ \text{from node 1}\\ \downarrow}}{-2\ \text{A}}$$

$$V_2:\quad \underbrace{\left(\frac{1}{4\ \Omega}+\frac{1}{3\ \Omega}\right)}_{\substack{\text{Sum of}\\ \text{conductances}\\ \text{connected}\\ \text{to node 2}}} V_2 - \underbrace{\left(\frac{1}{3\ \Omega}\right)}_{\substack{\text{Mutual}\\ \text{conductance}}} V_1 = \overset{\substack{\text{Supplying current}\\ \text{to node 2}\\ \downarrow}}{+3\ \text{A}}$$

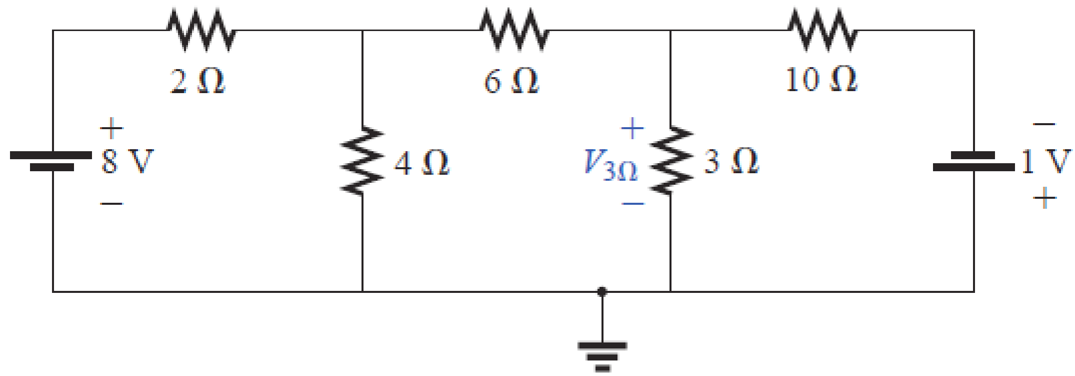


and

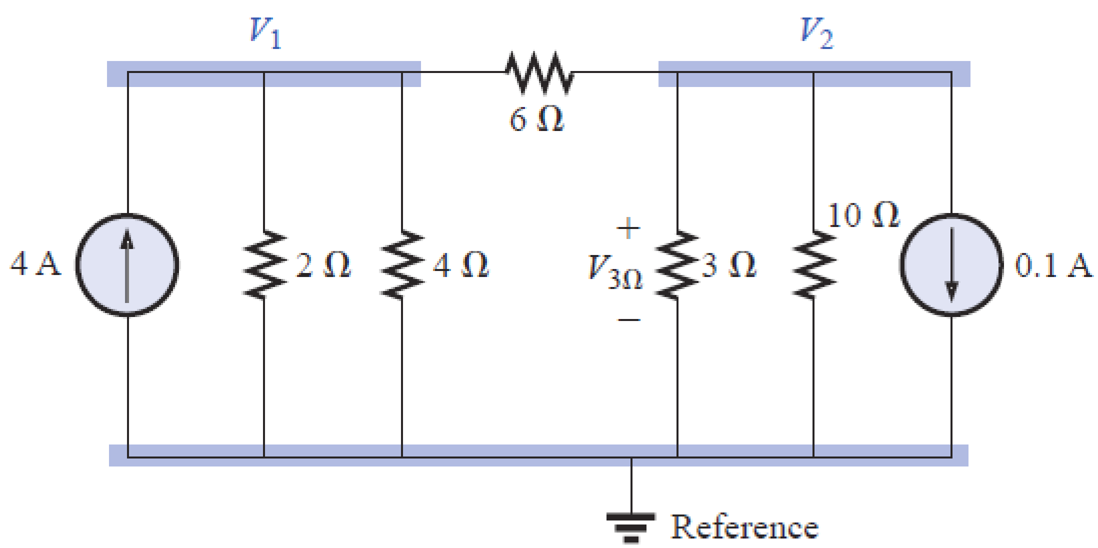
$$\frac{1}{2}V_1 - \frac{1}{3}V_2 = -2$$

$$-\frac{1}{3}V_1 + \frac{7}{12}V_2 = 3$$

EXAMPLE 8.24 Find the voltage across the 3-Ω resistor of Fig. 8.55 by nodal analysis.



Solution: Converting sources and choosing nodes (Fig. 8.56), we have



$$\left.\begin{aligned}\left(\frac{1}{2\,\Omega}+\frac{1}{4\,\Omega}+\frac{1}{6\,\Omega}\right)V_1-\left(\frac{1}{6\,\Omega}\right)V_2&=+4\text{ A}\\ \left(\frac{1}{10\,\Omega}+\frac{1}{3\,\Omega}+\frac{1}{6\,\Omega}\right)V_2-\left(\frac{1}{6\,\Omega}\right)V_1&=-0.1\text{ A}\end{aligned}\right\}$$

$$\frac{11}{12}V_1-\frac{1}{6}V_2=4$$

$$-\frac{1}{6}V_1+\frac{3}{5}V_2=-0.1$$

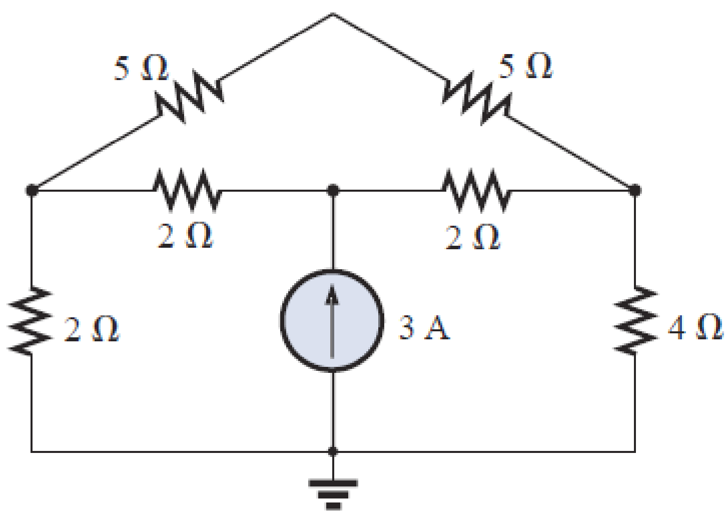
resulting in

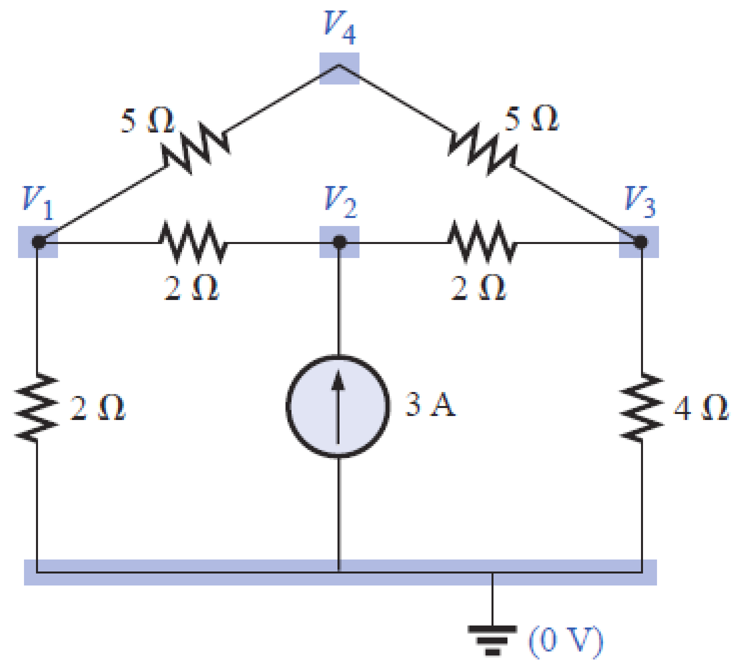
$$\begin{aligned}11V_1-\ \ 2V_2&=+48\\ -5V_1+18V_2&=-3\end{aligned}$$

and

$$V_2=V_{3\Omega}=\frac{\begin{vmatrix}11 & 48\\ -5 & -3\end{vmatrix}}{\begin{vmatrix}11 & -2\\ -5 & 18\end{vmatrix}}=\frac{-33+240}{198-10}=\frac{207}{188}=\mathbf{1.101\ V}$$

EXAMPLE 8.25 Using nodal analysis, determine the potential across the 4-Ω resistor in Fig. 8.57.

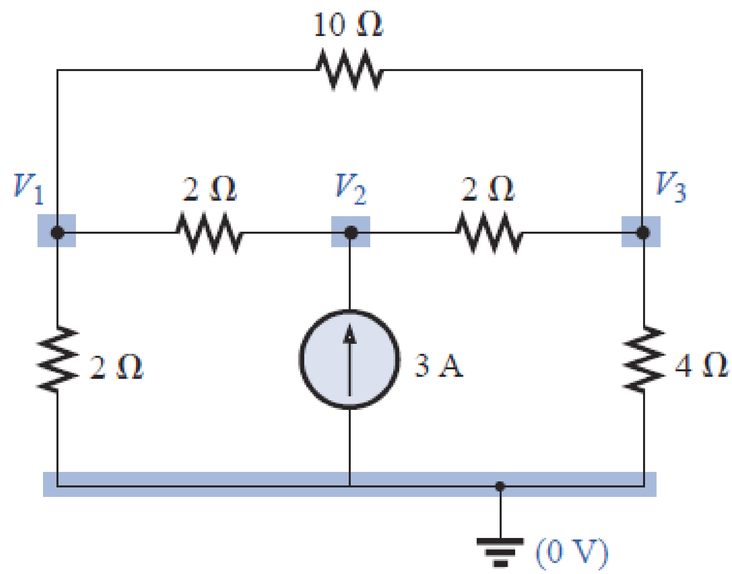




$$V_1: \quad \left(\frac{1}{2\ \Omega} + \frac{1}{2\ \Omega} + \frac{1}{10\ \Omega}\right)V_1 - \left(\frac{1}{2\ \Omega}\right)V_2 - \left(\frac{1}{10\ \Omega}\right)V_3 = 0$$

$$V_2: \quad \left(\frac{1}{2\ \Omega} + \frac{1}{2\ \Omega}\right)V_2 - \left(\frac{1}{2\ \Omega}\right)V_1 - \left(\frac{1}{2\ \Omega}\right)V_3 = 3\ \text{A}$$

$$V_3: \quad \left(\frac{1}{10\ \Omega} + \frac{1}{2\ \Omega} + \frac{1}{4\ \Omega}\right)V_3 - \left(\frac{1}{2\ \Omega}\right)V_2 - \left(\frac{1}{10\ \Omega}\right)V_1 = 0$$



$$\begin{aligned} 1.1V_1 - 0.5V_2 - 0.1V_3 &= 0 \\ V_2 - 0.5V_1 - 0.5V_3 &= 3 \\ 0.85V_3 - 0.5V_2 - 0.1V_1 &= 0 \end{aligned}$$

For determinants,

$$\begin{aligned} 1.1V_1 - 0.5V_2 - 0.1V_3 &= 0 \\ -0.5V_1 + 1V_2 - 0.5V_3 &= 3 \\ -0.1V_1 - 0.5V_2 + 0.85V_3 &= 0 \end{aligned}$$

$$V_3 = V_{4\Omega} = \frac{\begin{vmatrix} 1.1 & -0.5 & 0 \\ -0.5 & +1 & 3 \\ -0.1 & -0.5 & 0 \end{vmatrix}}{\begin{vmatrix} 1.1 & -0.5 & -0.1 \\ -0.5 & +1 & -0.5 \\ -0.1 & -0.5 & +0.85 \end{vmatrix}} = \mathbf{4.645\ V}$$