Network Theorems

9.1 INTRODUCTION

This chapter will introduce the important **fundamental theorems** of network analysis. Included are the **superposition**, **Thévenin's**, **Norton's**, and **maximum power transfer theorems.** We will consider a number of areas of application for each. A thorough understanding of each theorem is important because a number of these theorems will be applied repeatedly in the material to follow.

9.2 SUPERPOSITION THEOREM

The **superposition theorem** can be used to find solution to networks with many sources.
The theorem states that:

The current through, or voltage across, an element in a linear bilateral network is equal to the algebraic sum of the currents or voltages produced independently by each source.

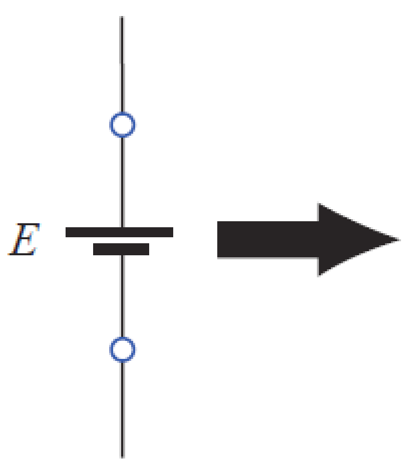
· **Linear** ≡ elements of the network are independent of the voltage applied across or the current through them,
· **Bilateral** ≡ no change in the behavior if current or voltage is reversed.

$$\begin{array}{c}\text{Number of networks}\\ \text{to be analyzed}\end{array} = \begin{array}{c}\text{Number of}\\ \text{independent sources}\end{array}$$

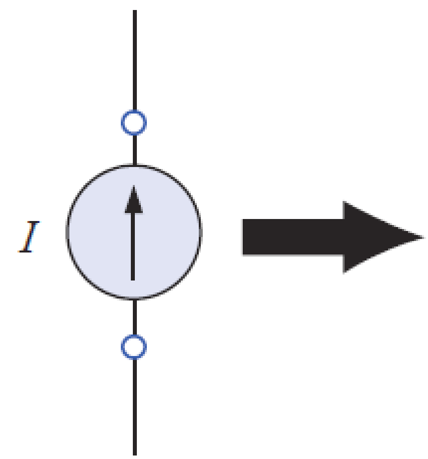
To reduce the number of network to be analyzed we **can consider the effect of more than one source at a time**.

To consider the effects of each source independently requires that other **sources be removed** and **replaced without affecting the final result**.

Removing the effects of ideal sources.

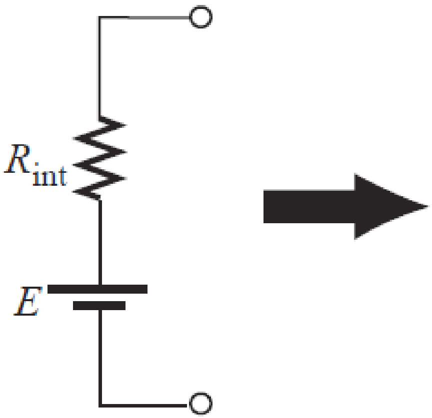


Voltage source are replaced by a short

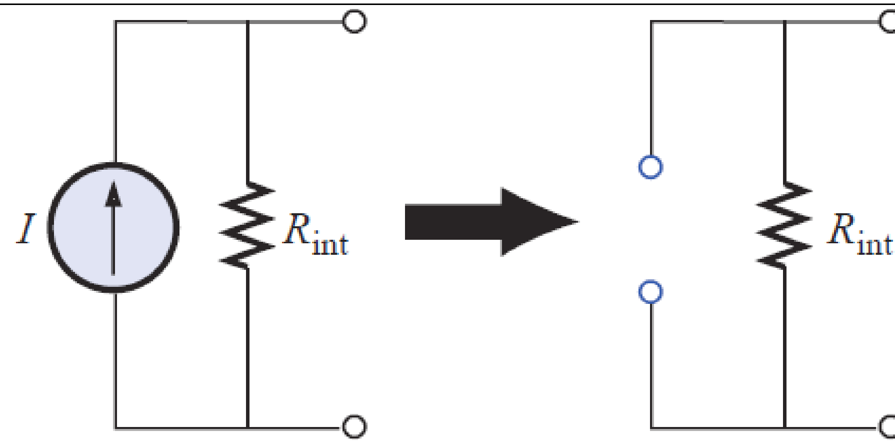


Current source are replaced by an open

Removing the effects of practical sources.



Voltage source are replaced by a short



Current source are replaced by an open

I_1 and I_2 in the same direction: $I_T = I_1 + I_2$

Power from Source 1 is: $P_1 = I_1^2 \cdot R$

Power from Source 2 is: $P_2 = I_2^2 \cdot R$

If we assume total power is just the sum of powers delivered by each source then:

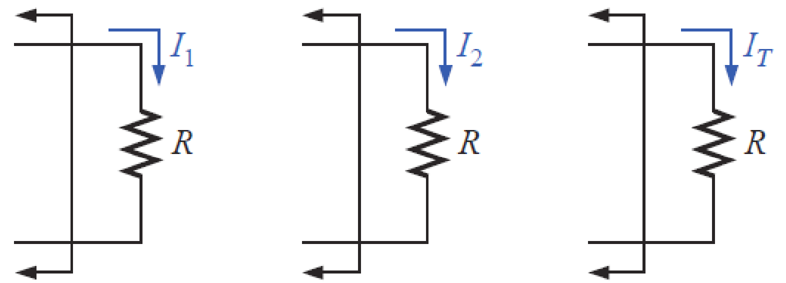
$$P_T = P_1 + P_2 = R \cdot \left(I_1^2 + I_2^2\right)$$

$$\Longrightarrow I_T^2 = I_1^2 + I_2^2$$

This is not equal to the value calculated from the theorem that is:

$$I_T^2 = (I_1 + I_2)^2 = I_1^2 + 2 \cdot I_1 \cdot I_2 + I_2^2$$

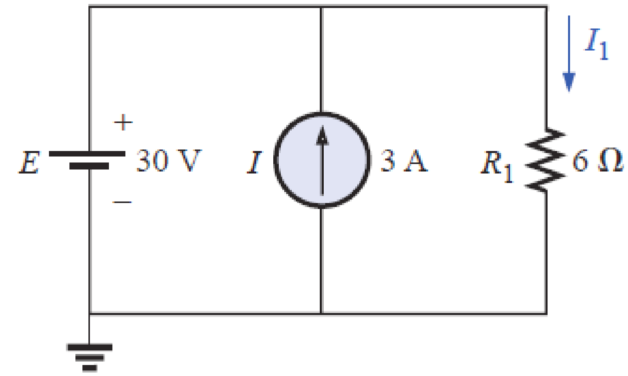
Which is different.



$P_1 = I_1^2R$ (a) $P_2 = I_2^2R$ (b) $P_T = I_T^2R$ (c)

the total power delivered to a resistive element must be determined using the total current through or the total voltage across the element and cannot be determined by a simple sum of the power levels established by each source.

EXAMPLE 9.1 Determine I_1 for the network of Fig. 9.4.



Solution: Setting $E = 0$ V for the network of Fig. 9.4 results in the network of Fig. 9.5(a), where a short-circuit equivalent has replaced the 30-V source.

As shown in Fig. 9.5(a), the source current will choose the short-circuit path, and $I'_1 = 0$ A. If we applied the current divider rule,

$$I'_1 = \frac{R_{sc}I}{R_{sc} + R_1} = \frac{(0\ \Omega)I}{0\ \Omega + 6\ \Omega} = 0\text{ A}$$

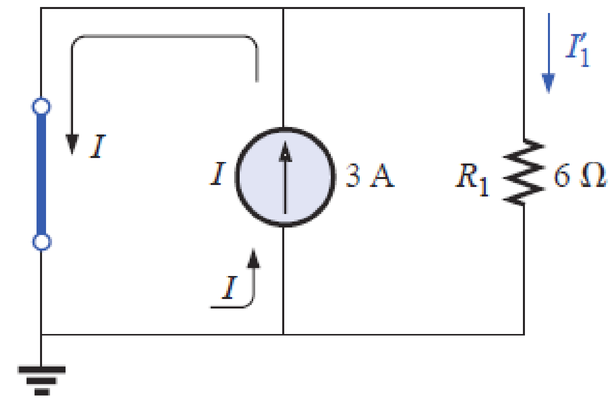
Setting I to zero amperes will result in the network of Fig. 9.5(b), with the current source replaced by an open circuit. Applying Ohm's law,

$$I''_1 = \frac{E}{R_1} = \frac{30\text{ V}}{6\ \Omega} = 5\text{ A}$$

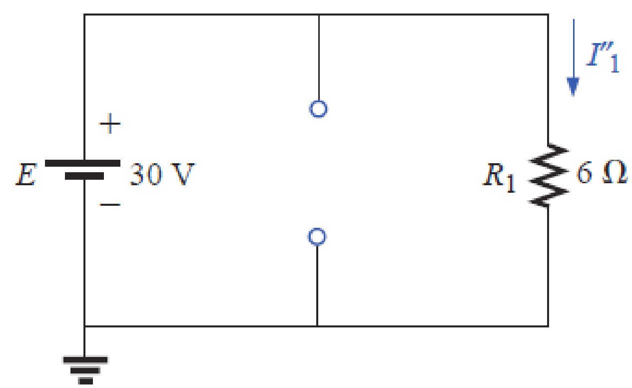
Since I'_1 and I''_1 have the same defined direction in Fig. 9.5(a) and (b), the current I_1 is the sum of the two, and

$$I_1 = I'_1 + I''_1 = 0\text{ A} + 5\text{ A} = \mathbf{5\ A}$$

Note in this case that the current source has no effect on the current through the 6-Ω resistor. The voltage across the resistor must be fixed at 30 V because they are parallel elements.

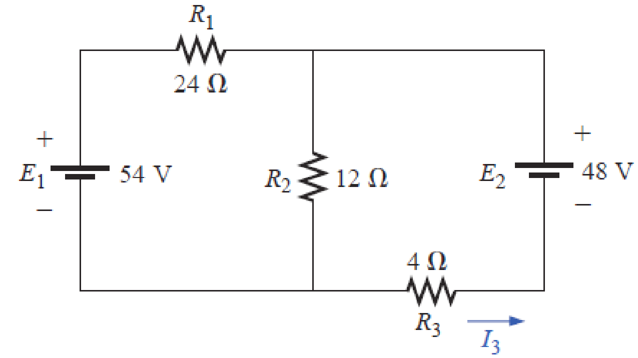


(a)



(b)

EXAMPLE 9.2 Using superposition, determine the current through the 4-Ω resistor of Fig. 9.6. Note that this is a two-source network of the type considered in Chapter 8.



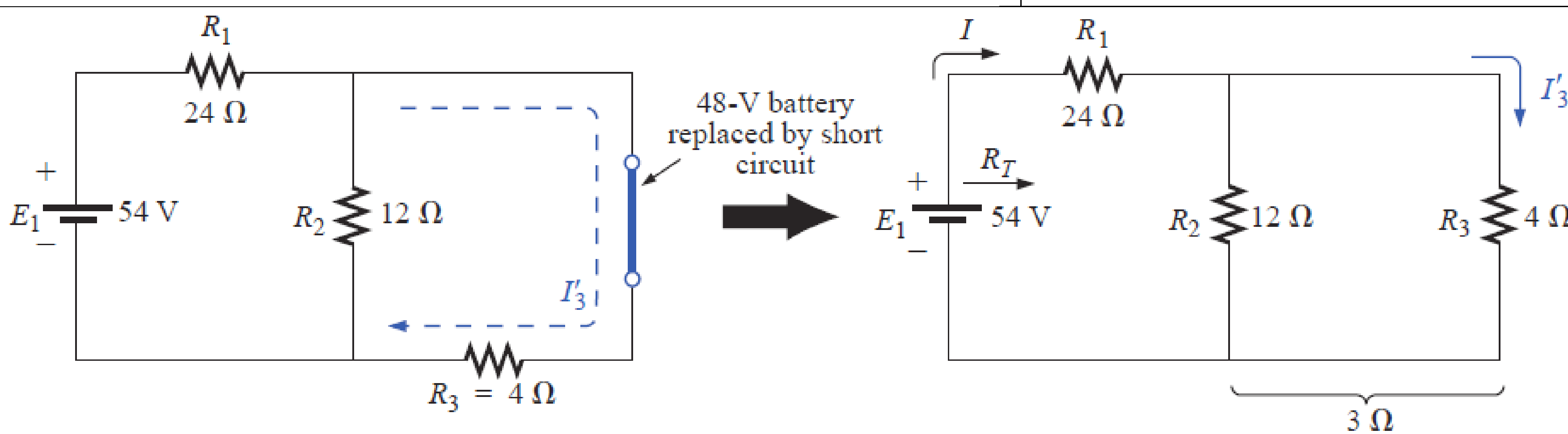
Solution: *Considering the effects of a 54-V source (Fig. 9.7):*

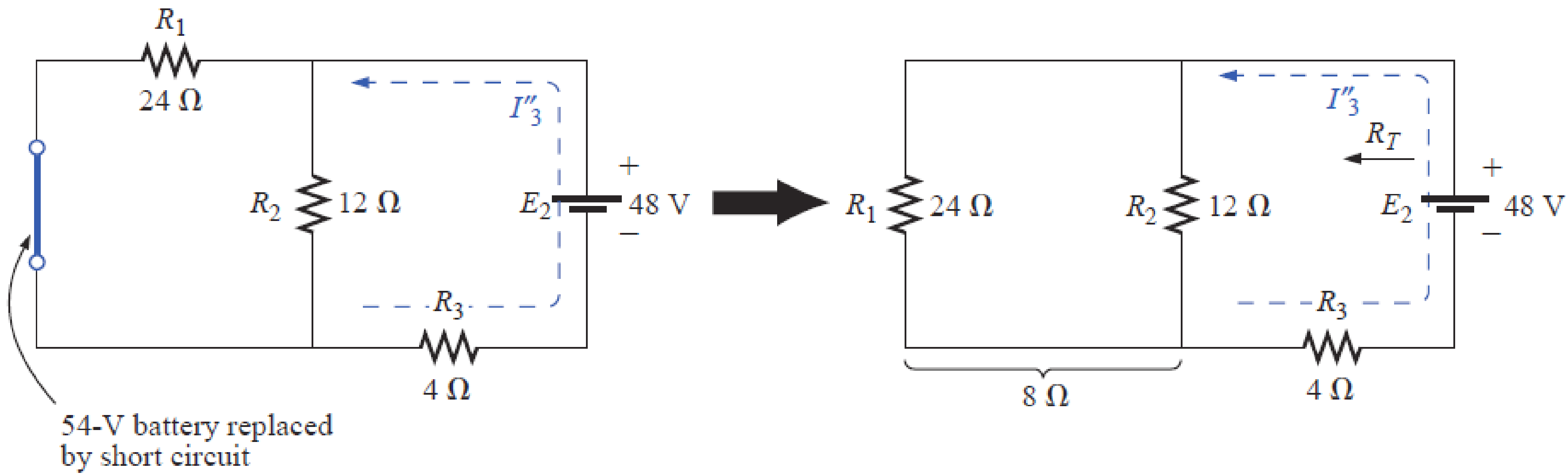
$$R_T = R_1 + R_2 \parallel R_3 = 24\ \Omega + 12\ \Omega \parallel 4\ \Omega = 24\ \Omega + 3\ \Omega = 27\ \Omega$$

$$I = \frac{E_1}{R_T} = \frac{54\text{ V}}{27\ \Omega} = 2\text{ A}$$

Using the current divider rule,

$$I'_3 = \frac{R_2 I}{R_2 + R_3} = \frac{(12\ \Omega)(2\text{ A})}{12\ \Omega + 4\ \Omega} = \frac{24\text{ A}}{16} = 1.5\text{ A}$$





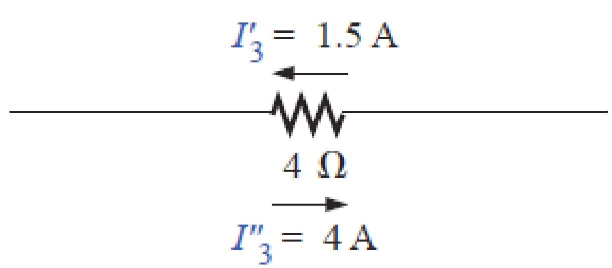
Considering the effects of the 48-V source (Fig. 9.8):

$$R_T = R_3 + R_1 \parallel R_2 = 4\ \Omega + 24\ \Omega \parallel 12\ \Omega = 4\ \Omega + 8\ \Omega = 12\ \Omega$$

$$I''_3 = \frac{E_2}{R_T} = \frac{48\ \text{V}}{12\ \Omega} = 4\ \text{A}$$

The total current through the 4-Ω resistor (Fig. 9.9) is

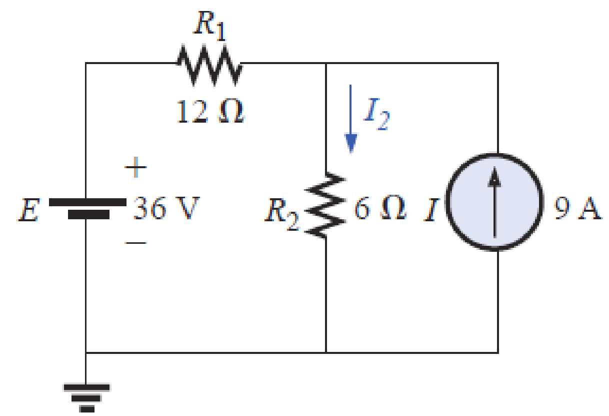
$$I_3 = I''_3 - I'_3 = 4\ \text{A} - 1.5\ \text{A} = \mathbf{2.5\ A} \qquad (\text{direction of } I''_3)$$



EXAMPLE 9.3

a. Using superposition, find the current through the 6-Ω resistor of the network of Fig. 9.10.

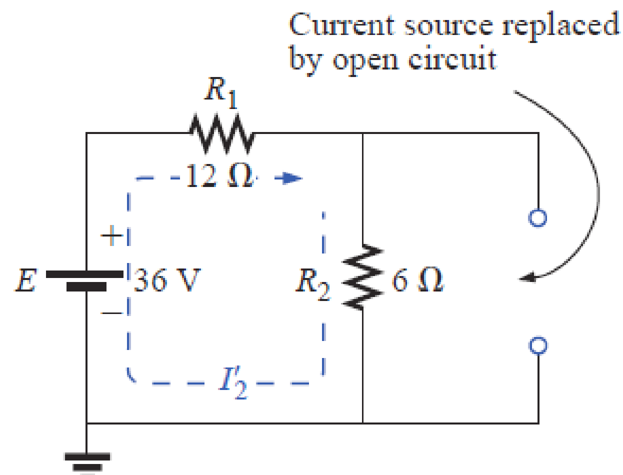
b. Demonstrate that superposition is not applicable to power levels.



Solutions:

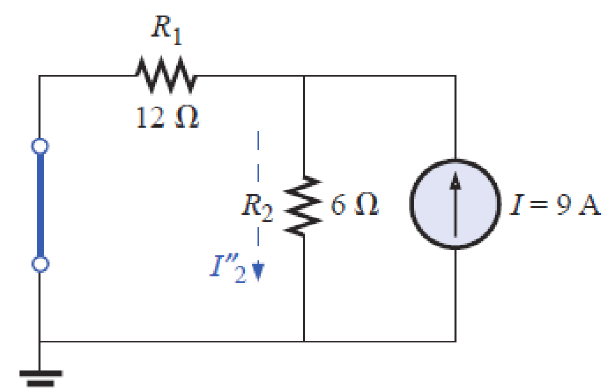
a. *Considering the effect of the 36-V source (Fig. 9.11):*

$$I'_2 = \frac{E}{R_T} = \frac{E}{R_1 + R_2} = \frac{36\text{ V}}{12\ \Omega + 6\ \Omega} = 2\text{ A}$$



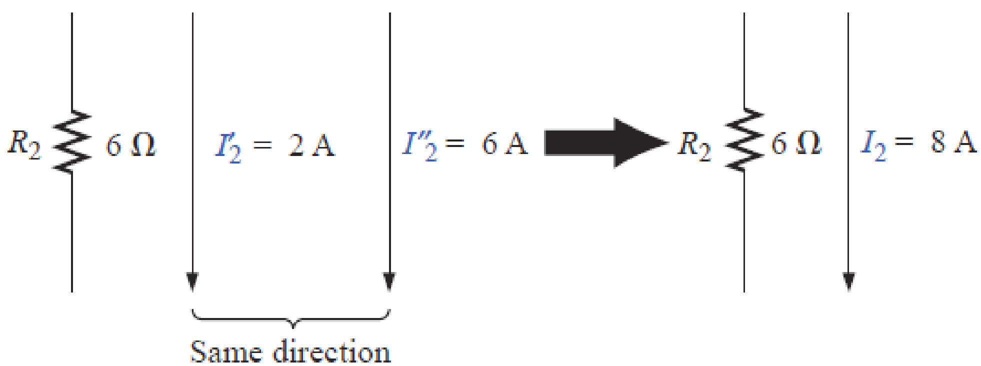
Considering the effect of the 9-A source (Fig. 9.12):
Applying the current divider rule,

$$I''_2 = \frac{R_1 I}{R_1 + R_2} = \frac{(12\ \Omega)(9\text{ A})}{12\ \Omega + 6\ \Omega} = \frac{108\text{ A}}{18} = 6\text{ A}$$



The total current through the 6-Ω resistor (Fig. 9.13) is

$$I_2 = I'_2 + I''_2 = 2\text{ A} + 6\text{ A} = \mathbf{8\text{ A}}$$



b. The power to the 6-Ω resistor is

$$\text{Power} = I^2R = (8\ \text{A})^2(6\ \Omega) = \mathbf{384\ W}$$

The calculated power to the 6-Ω resistor due to each source, *misusing* the principle of superposition, is

$$P_1 = (I'_2)^2R = (2\ \text{A})^2(6\ \Omega) = 24\ \text{W}$$

$$P_2 = (I''_2)^2R = (6\ \text{A})^2(6\ \Omega) = 216\ \text{W}$$

$$P_1 + P_2 = 240\ \text{W} \neq 384\ \text{W}$$

This results because 2 A + 6 A = 8 A, but

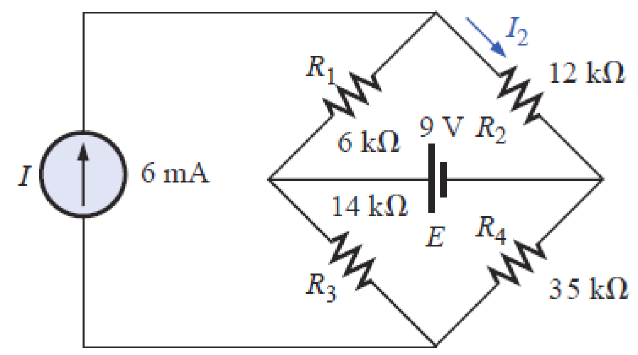
$$(2\ \text{A})^2 + (6\ \text{A})^2 \neq (8\ \text{A})^2$$

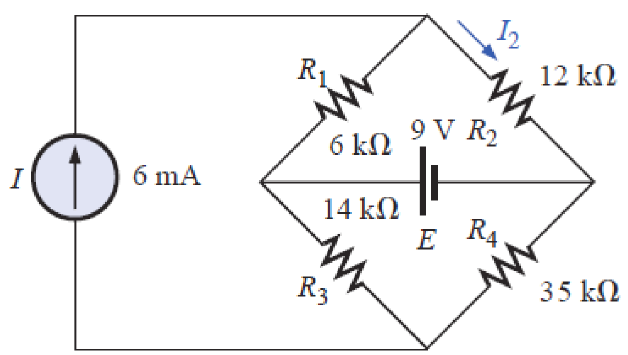
EXAMPLE 9.4 Using the principle of superposition, find the current I_2 through the 12-kΩ resistor of Fig. 9.16.

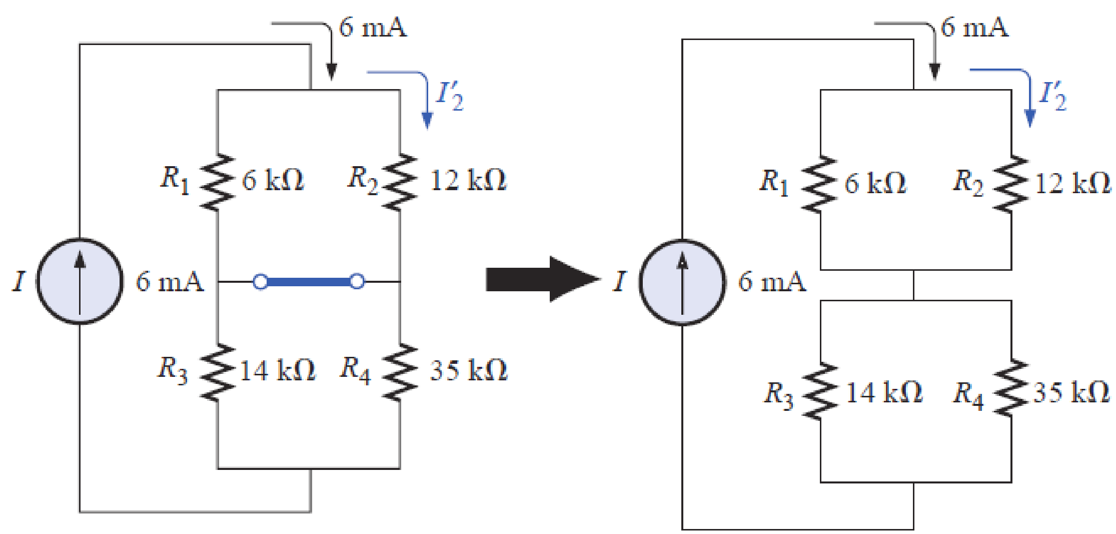
Solution: *Considering the effect of the 6-mA current source (Fig. 9.17):*

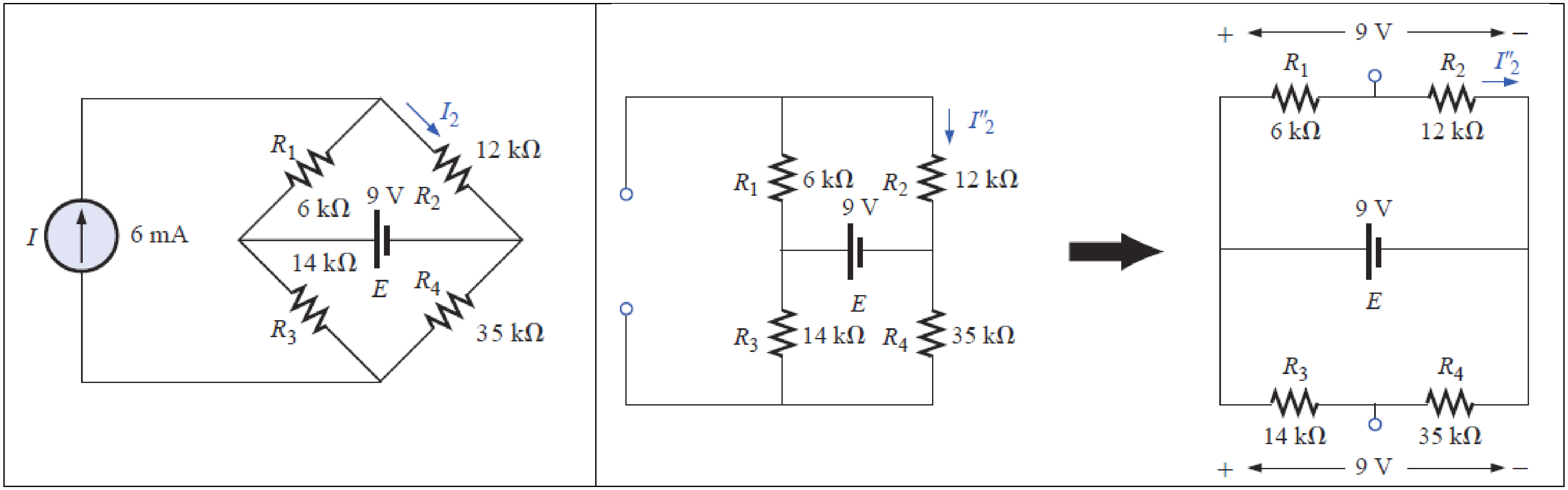
Current divider rule:

$$I'_2 = \frac{R_1 I}{R_1 + R_2} = \frac{(6\ \text{k}\Omega)(6\ \text{mA})}{6\ \text{k}\Omega + 12\ \text{k}\Omega} = 2\ \text{mA}$$









Considering the effect of the 9-V voltage source (Fig. 9.18):

$$I''_2 = \frac{E}{R_1 + R_2} = \frac{9\ \text{V}}{6\ \text{k}\Omega + 12\ \text{k}\Omega} = 0.5\ \text{mA}$$

Since I'_2 and I''_2 have the same direction through R_2, the desired current is the sum of the two:

$$\begin{aligned} I_2 &= I'_2 + I''_2 \\ &= 2\ \text{mA} + 0.5\ \text{mA} \\ &= \mathbf{2.5\ mA} \end{aligned}$$

EXAMPLE 9.5 Find the current through the 2-Ω resistor of the network of Fig. 9.19. The presence of three sources will result in three different networks to be analyzed.

Solution: *Considering the effect of the 12-V source (Fig. 9.20):*

$$I'_1 = \frac{E_1}{R_1 + R_2} = \frac{12\text{ V}}{2\ \Omega + 4\ \Omega} = \frac{12\text{ V}}{6\ \Omega} = 2\text{ A}$$

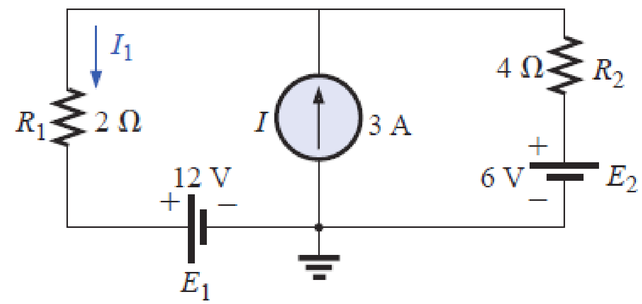
Considering the effect of the 6-V source (Fig. 9.21):

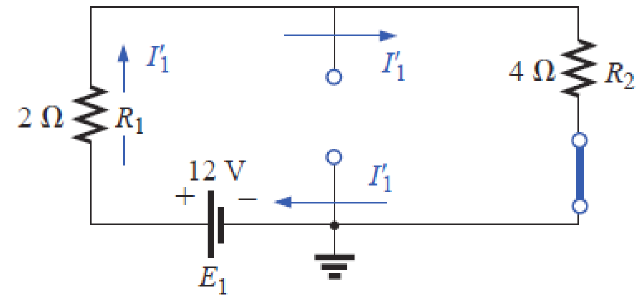
$$I''_1 = \frac{E_2}{R_1 + R_2} = \frac{6\text{ V}}{2\ \Omega + 4\ \Omega} = \frac{6\text{ V}}{6\ \Omega} = 1\text{ A}$$

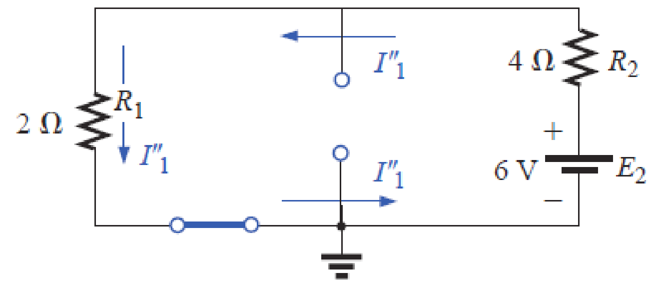
Considering the effect of the 3-A source (Fig. 9.22):

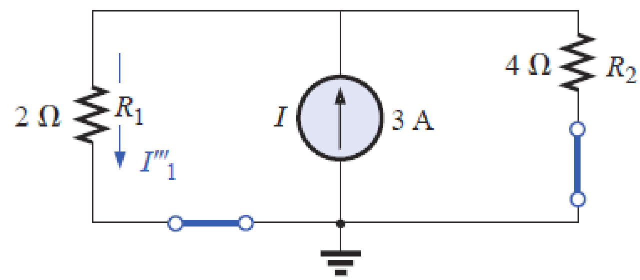
Applying the current divider rule,

$$I'''_1 = \frac{R_2 I}{R_1 + R_2} = \frac{(4\ \Omega)(3\text{ A})}{2\ \Omega + 4\ \Omega} = \frac{12\text{ A}}{6} = 2\text{ A}$$





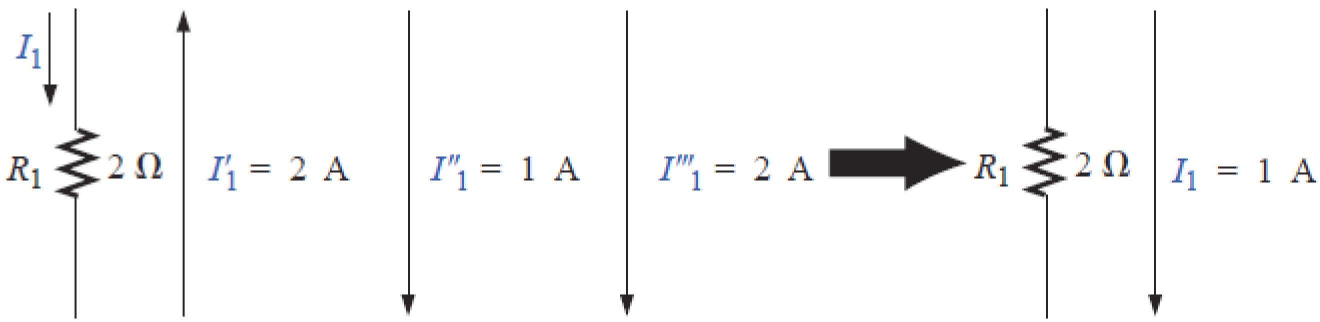




The total current through the 2-Ω resistor appears in Fig. 9.23, and

$$I_1 = \overbrace{I''_1 + I'''_1}^{\text{Same direction as } I_1 \text{ in Fig. 9.19}} - \underset{\text{Opposite direction to } I_1 \text{ in Fig. 9.19}}{I'_1}$$

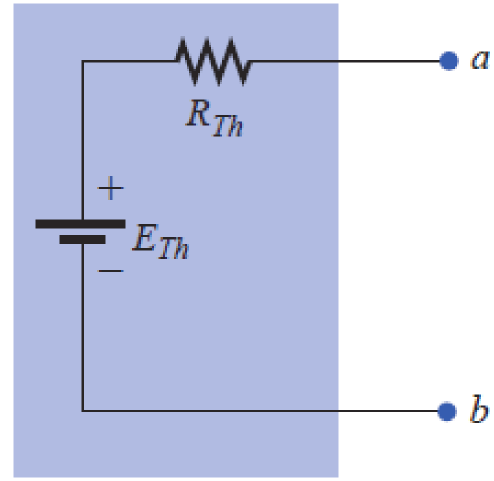
$$= 1\text{A} + 2\text{A} - 2\text{A} = 1\textbf{A}$$



9.3 THÉVENIN'S THEOREM

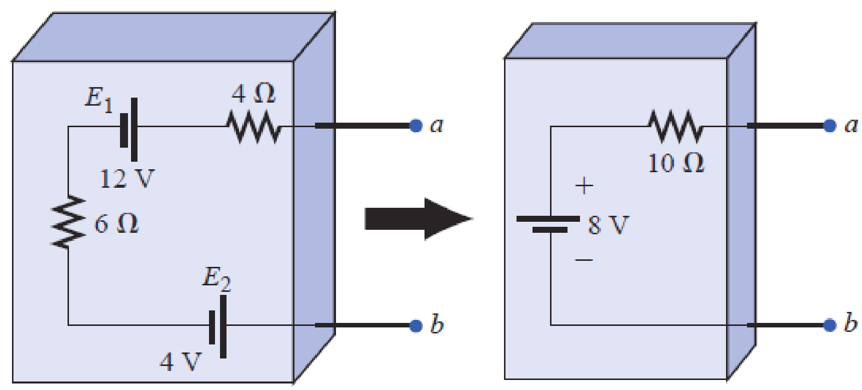
Any two-terminal, linear bilateral dc network can be replaced by an equivalent circuit consisting of a voltage source and a series resistor, as shown.

the Thévenin equivalent circuit provides an equivalence at the terminals only—the internal construction and characteristics of the original network and the Thévenin equivalent are usually quite different.



The Thévenin equivalent circuit can be found quite directly by simply combining the series batteries and resistors.

To apply the theorem, the network to be reduced to the Thévenin equivalent form must be isolated as shown in the figure, and the two "holding" terminals identified.



(a) (b)

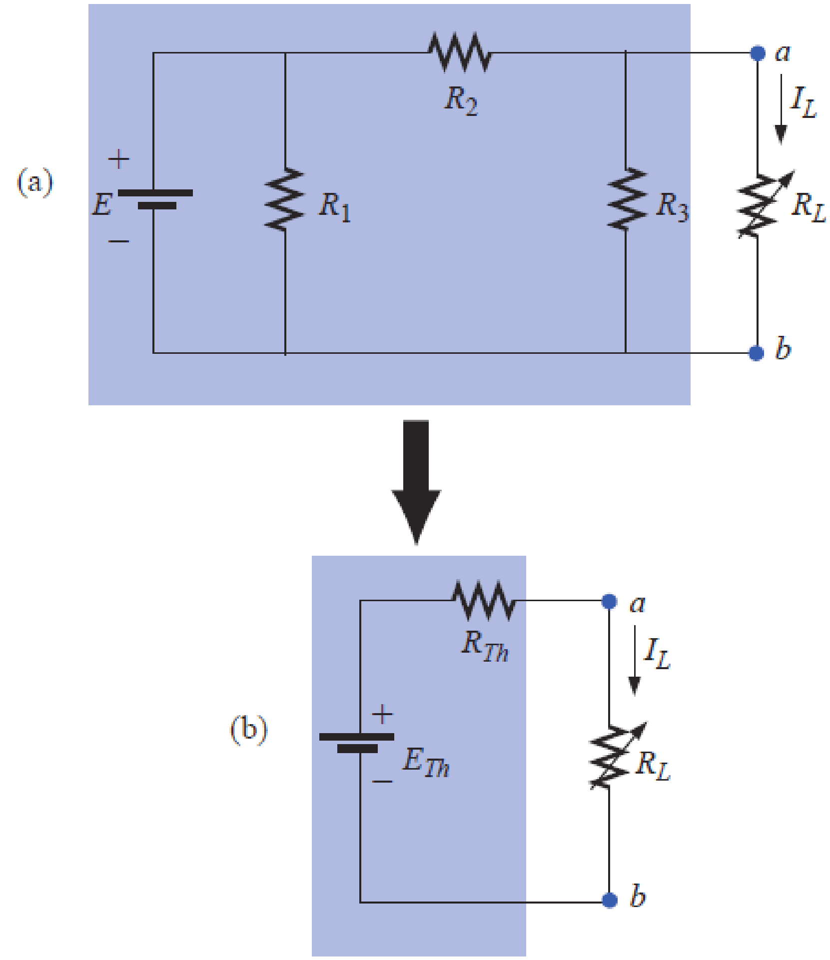
The voltage, current, or resistance readings between the two “holding” terminals is the same whether the original or the Thévenin equivalent circuit is connected to the left of terminals ***a*** and ***b***.

The theorem achieves two important objectives:

1- It allows us to find any particular voltage or current in a linear network as the previous methods.

2- We can concentrate on a specific portion of the network by replacing the remaining with an equivalent circuit by finding the Thévenin equivalent circuit for the network in the shaded area, we can quickly calculate the change in current through or voltage across the variable resistor R_L for the various values that it may assume.

the current through or voltage across R_L must be the same for either network for any value of R_L.



Preliminary:

1. ***Remove** the portion of the network across which the Thévenin equivalent circuit is to be found.*
2. ***Mark the terminals** of the remaining two-terminal network.*

R_{Th}:

3. *Calculate R_{Th} by first setting **all sources to zero** (voltage sources are replaced by short circuits, and current sources by open circuits) and then finding the **resultant resistance between the two marked terminals**. (If the internal resistance of the voltage and/or current sources is included in the original network, it must remain when the sources are set to zero.)*

E_{Th}:

4. *Calculate E_{Th} by first returning all sources to their original position and finding the **open-circuit voltage between the marked terminals**.*

Conclusion:

5. *Draw the Thévenin equivalent circuit with the portion of the circuit previously removed **replaced between the terminals of the equivalent circuit**.*

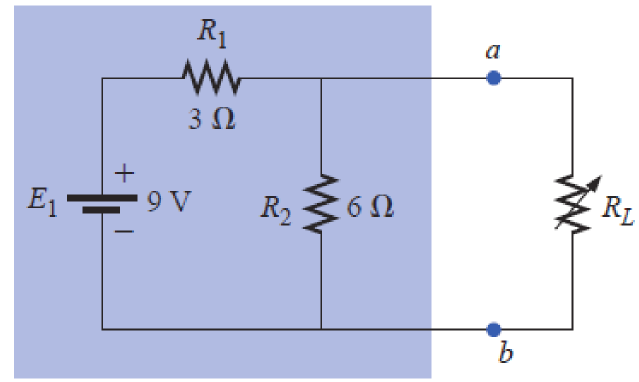
EXAMPLE 9.6 Find the Thévenin equivalent circuit for the network in the shaded area of the network of Fig. 9.27. Then find the current through R_L for values of 2 Ω, 10 Ω, and 100 Ω.

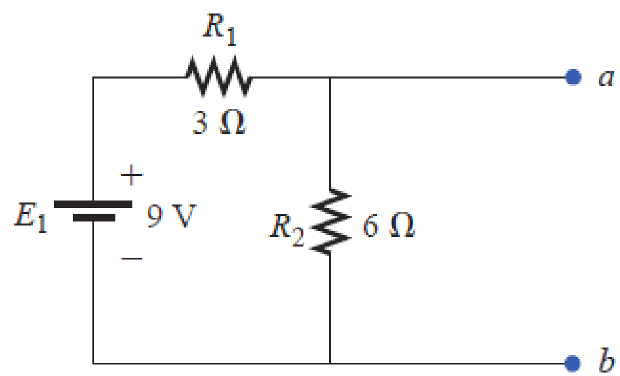
Solution:

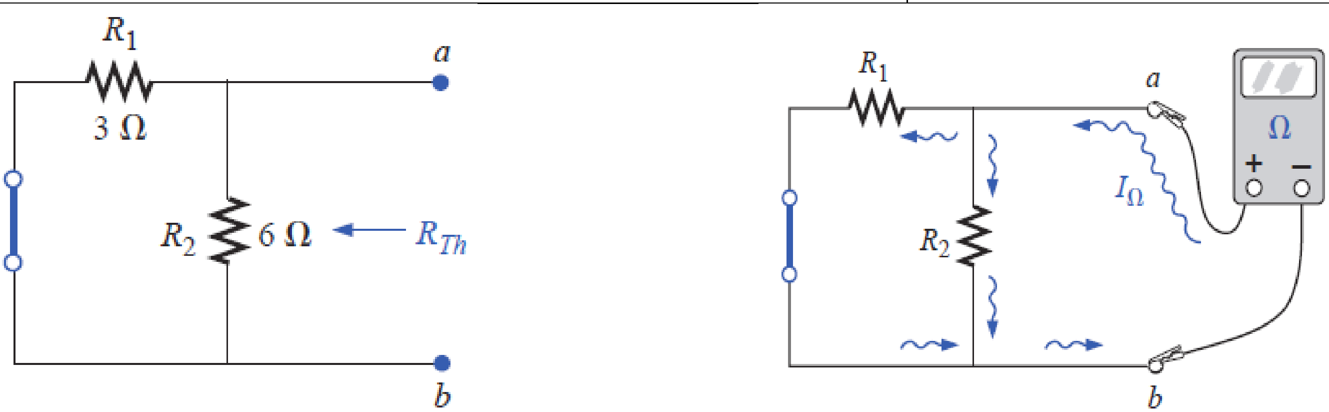
Steps 1 and 2 produce the network of Fig. 9.28. Note that the load resistor R_L has been removed and the two "holding" terminals have been defined as *a* and *b*.

Step 3: Replacing the voltage source E_1 with a short-circuit equivalent yields the network of Fig. 9.29(a), where

$$R_{Th} = R_1 \parallel R_2 = \frac{(3\ \Omega)(6\ \Omega)}{3\ \Omega + 6\ \Omega} = \mathbf{2\ \Omega}$$



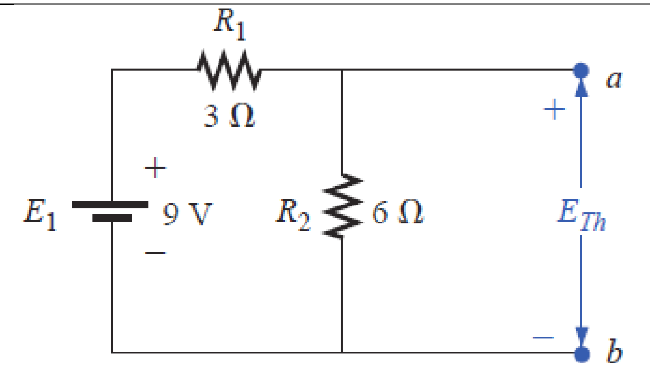


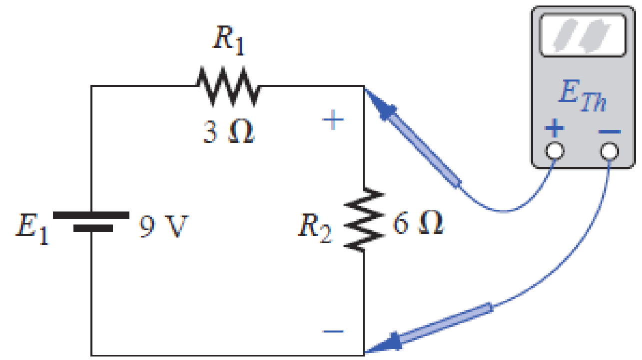


(a) (b)

Step 4: Replace the voltage source (Fig. 9.30). For this case, the open-circuit voltage E_{Th} is the same as the voltage drop across the 6-Ω resistor. Applying the voltage divider rule,

$$E_{Th} = \frac{R_2 E_1}{R_2 + R_1} = \frac{(6\ \Omega)(9\ \text{V})}{6\ \Omega + 3\ \Omega} = \frac{54\ \text{V}}{9} = \mathbf{6\ V}$$





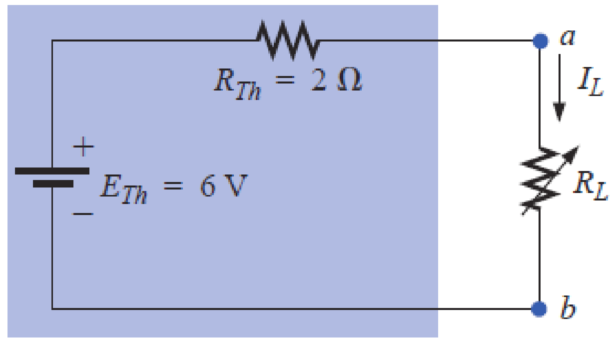
Step 5 (Fig. 9.32):

$$I_L = \frac{E_{Th}}{R_{Th} + R_L}$$

$$R_L = 2\ \Omega: \quad I_L = \frac{6\ \text{V}}{2\ \Omega + 2\ \Omega} = \mathbf{1.5\ A}$$

$$R_L = 10\ \Omega: \quad I_L = \frac{6\ \text{V}}{2\ \Omega + 10\ \Omega} = \mathbf{0.5\ A}$$

$$R_L = 100\ \Omega: \quad I_L = \frac{6\ \text{V}}{2\ \Omega + 100\ \Omega} = \mathbf{0.059\ A}$$



EXAMPLE 9.7 Find the Thévenin equivalent circuit for the network in the shaded area of the network of Fig. 9.33.

Solution:

Steps 1 and 2 are shown in Fig. 9.34.

Step 3 is shown in Fig. 9.35. The current source has been replaced with an open-circuit equivalent, and the resistance determined between terminals *a* and *b*.

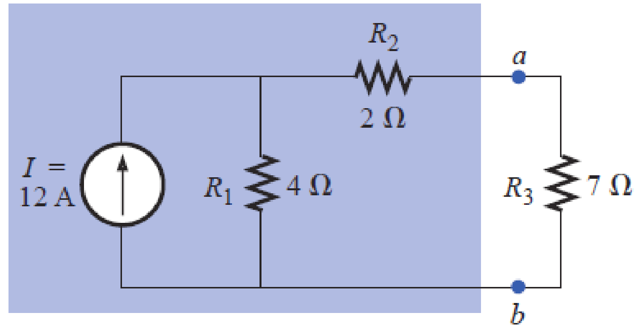
In this case an ohmmeter connected between terminals *a* and *b* would send out a sensing current that would flow directly through R_1 and R_2 (at the same level). The result is that R_1 and R_2 are in series and the Thévenin resistance is the sum of the two.

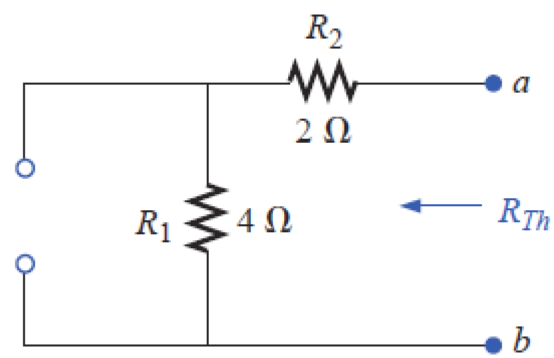
$$R_{Th} = R_1 + R_2 = 4\ \Omega + 2\ \Omega = \mathbf{6\ \Omega}$$

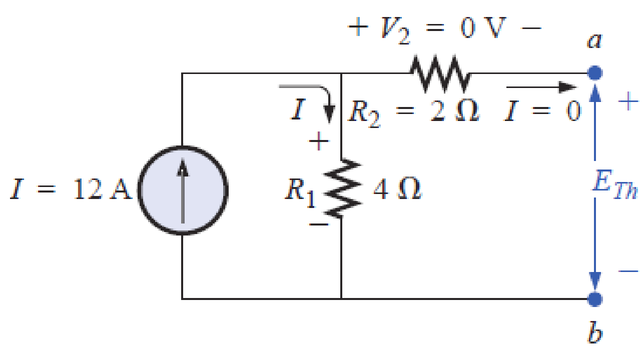
Step 4 (Fig. 9.36): In this case, since an open circuit exists between the two marked terminals, the current is zero between these terminals and through the 2-Ω resistor. The voltage drop across R_2 is, therefore,

$$V_2 = I_2R_2 = (0)R_2 = 0\ \text{V}$$

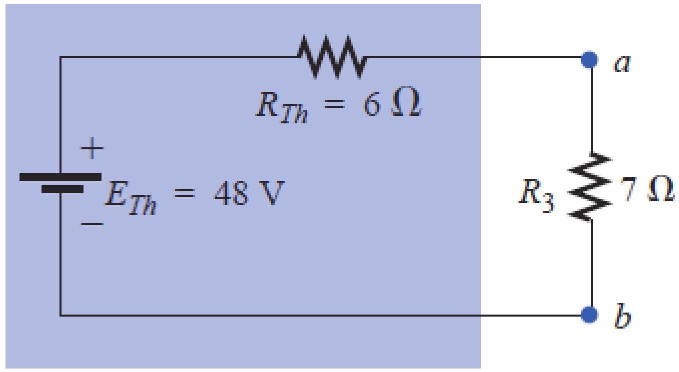
and $$E_{Th} = V_1 = I_1R_1 = IR_1 = (12\ \text{A})(4\ \Omega) = \mathbf{48\ V}$$





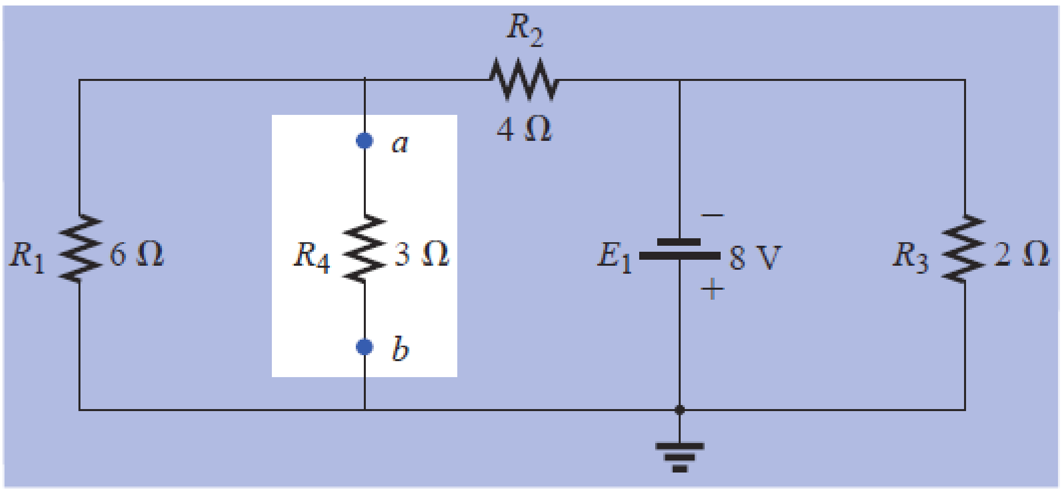


Step 5 is shown in Fig. 9.37.



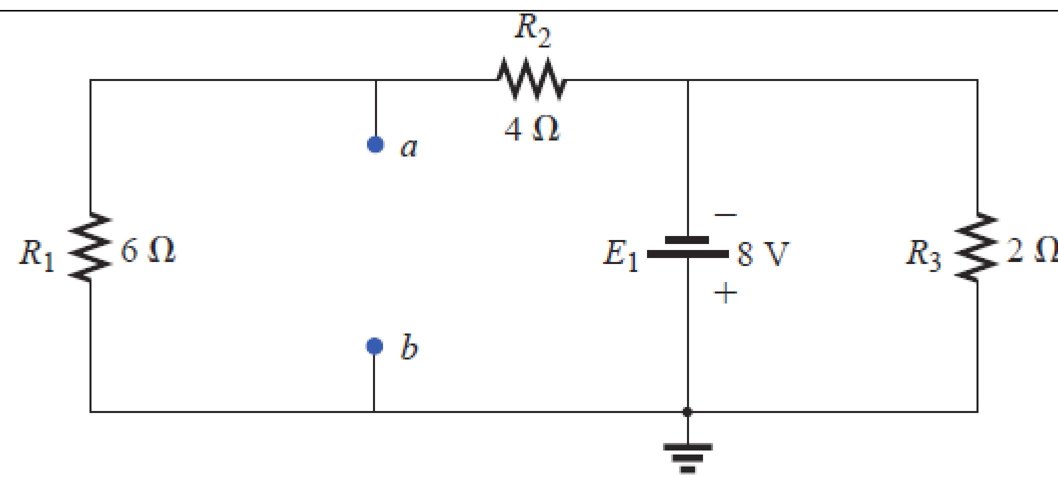
EXAMPLE 9.8 Find the Thévenin equivalent circuit for the network in the shaded area of the network of Fig. 9.38. Note in this example that

there is no need for the section of the network to be preserved to be at the "end" of the configuration.



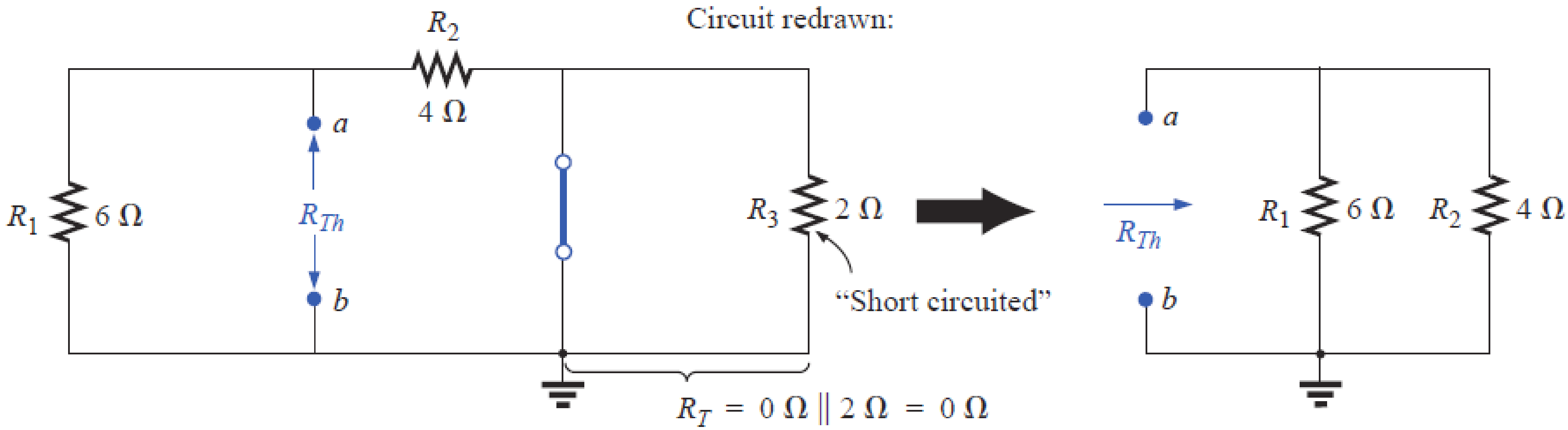
Solution:

Steps 1 and 2: See Fig. 9.39.



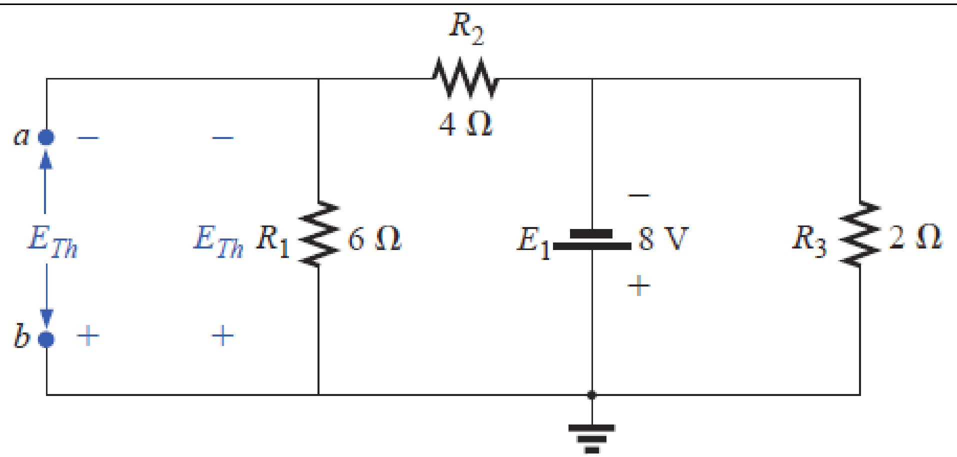
Step 3: See Fig. 9.40. Steps 1 and 2 are relatively easy to apply, but now we must be careful to "hold" onto the terminals *a* and *b* as the Thévenin resistance and voltage are determined. In Fig. 9.40, all the remaining elements turn out to be in parallel, and the network can be redrawn as shown.

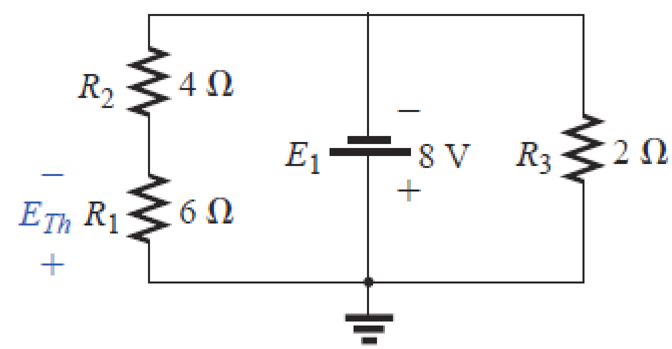
$$R_{Th} = R_1 \parallel R_2 = \frac{(6\ \Omega)(4\ \Omega)}{6\ \Omega + 4\ \Omega} = \frac{24\ \Omega}{10} = \mathbf{2.4\ \Omega}$$



Step 4: See Fig. 9.41. In this case, the network can be redrawn as shown in Fig. 9.42, and since the voltage is the same across parallel elements, the voltage across the series resistors R_1 and R_2 is E_1, or 8 V. Applying the voltage divider rule,

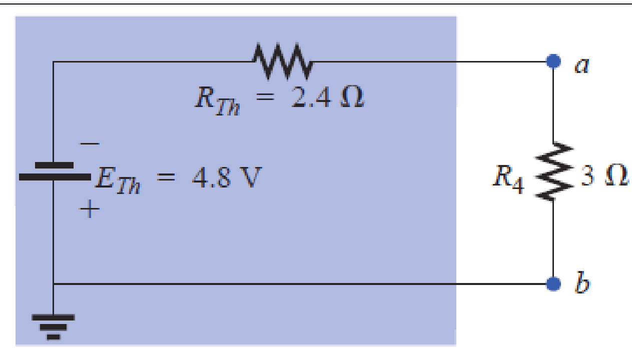
$$E_{Th} = \frac{R_1 E_1}{R_1 + R_2} = \frac{(6\ \Omega)(8\ \text{V})}{6\ \Omega + 4\ \Omega} = \frac{48\ \text{V}}{10} = \mathbf{4.8\ V}$$





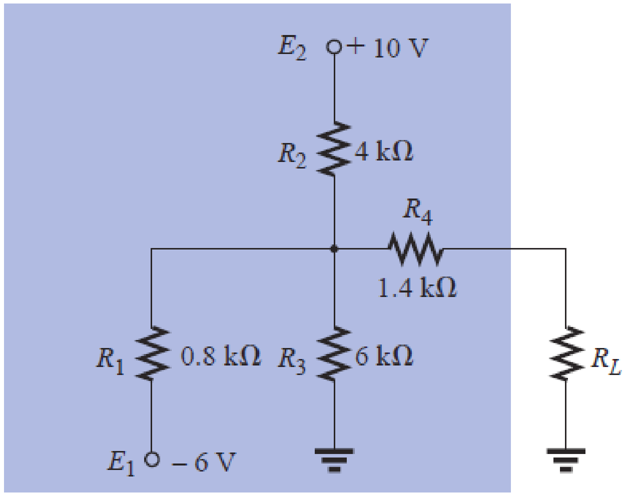
Step 5: See Fig. 9.43.

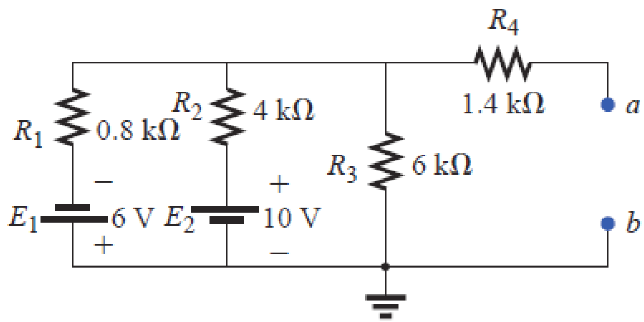
The importance of marking the terminals should be obvious from Example 9.8. Note that there is no requirement that the Thévenin voltage have the same polarity as the equivalent circuit originally introduced.



EXAMPLE 9.10 (Two sources) Find the Thévenin circuit for the network within the shaded area of Fig. 9.49.

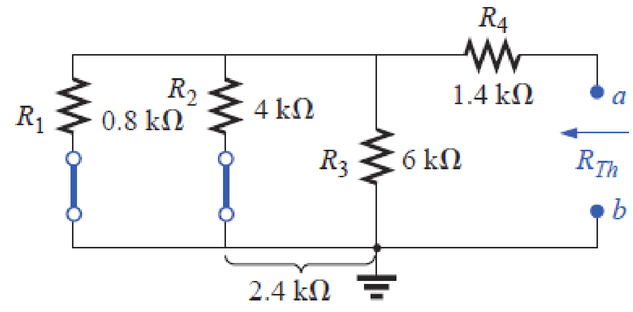
Solution: The network is redrawn and *steps 1 and 2* are applied as shown in Fig. 9.50.





Step 3: See Fig. 9.51.

$$
\begin{aligned}
R_{Th} &= R_4 + R_1 \parallel R_2 \parallel R_3 \\
&= 1.4\ \text{k}\Omega + 0.8\ \text{k}\Omega \parallel 4\ \text{k}\Omega \parallel 6\ \text{k}\Omega \\
&= 1.4\ \text{k}\Omega + 0.8\ \text{k}\Omega \parallel 2.4\ \text{k}\Omega \\
&= 1.4\ \text{k}\Omega + 0.6\ \text{k}\Omega \\
&= \mathbf{2\ k\Omega}
\end{aligned}
$$



Step 4: Applying superposition, we will consider the effects of the voltage source E_1 first. Note Fig. 9.52. The open circuit requires that $V_4 = I_4R_4 = (0)R_4 = 0$ V, and

$$E'_{Th} = V_3$$
$$R'_T = R_2 \parallel R_3 = 4\text{ k}\Omega \parallel 6\text{ k}\Omega = 2.4\text{ k}\Omega$$

Applying the voltage divider rule,

$$V_3 = \frac{R'_T E_1}{R'_T} + R_1 \quad \text{(Error)} \quad V_3 = \frac{R'_T E_1}{R'_T + R_1}$$

$$= \frac{(2.4\text{ k}\Omega)(6\text{ V})}{2.4\text{ k}\Omega + 0.8\text{ k}\Omega} = \frac{14.4\text{ V}}{3.2} = 4.5\text{ V}$$

$$E'_{Th} = V_3 = 4.5\text{ V}$$

For the source E_2, the network of Fig. 9.53 will result. Again, $V_4 = I_4R_4 = (0)R_4 = 0$ V, and

$$E''_{Th} = V_3$$
$$R'_T = R_1 \parallel R_3 = 0.8\text{ k}\Omega \parallel 6\text{ k}\Omega = 0.706\text{ k}\Omega$$

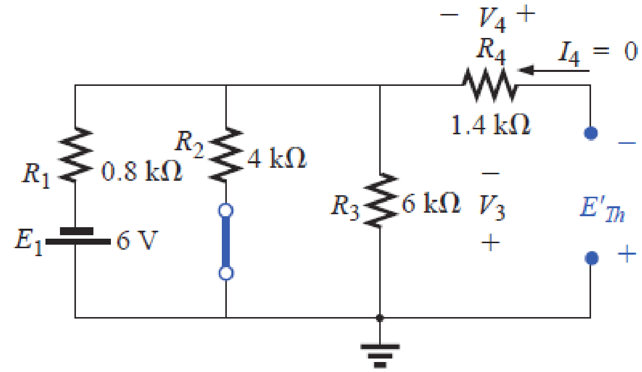
and $$V_3 = \frac{R'_T E_2}{R'_T + R_2} = \frac{(0.706\text{ k}\Omega)(10\text{ V})}{0.706\text{ k}\Omega + 4\text{ k}\Omega} = \frac{7.06\text{ V}}{4.706} = 1.5\text{ V}$$

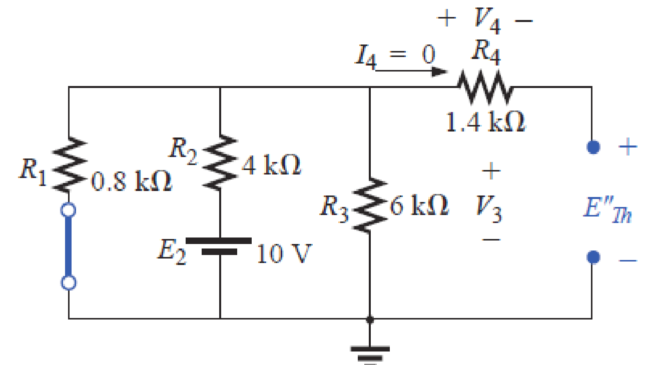
$$E''_{Th} = V_3 = 1.5\text{ V}$$

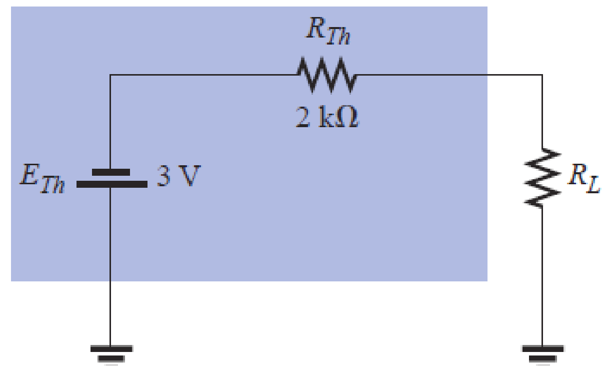
Since E'_{Th} and E''_{Th} have opposite polarities,

$$\begin{aligned} E_{Th} &= E'_{Th} - E''_{Th} \\ &= 4.5\text{ V} - 1.5\text{ V} \\ &= \mathbf{3\text{ V}} \qquad \text{(polarity of } E'_{Th}\text{)} \end{aligned}$$

Step 5: See Fig. 9.54.







9.4 NORTON'S THEOREM

Any two-terminal linear bilateral dc network can be replaced by an equivalent circuit consisting of a current source and a parallel resistor, as shown in Fig.

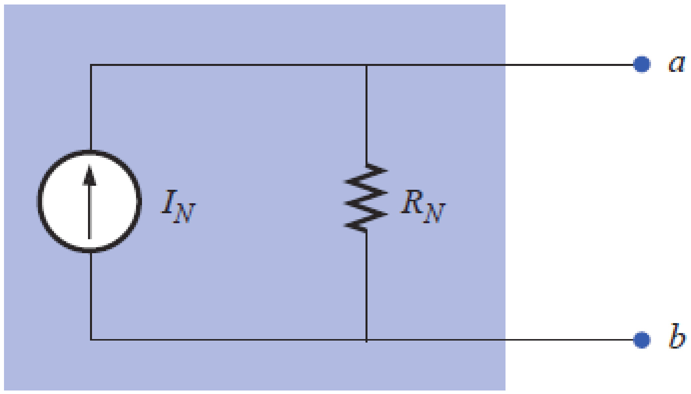


FIG. 9.58
Norton equivalent circuit.

Preliminary:

1. *Remove the portion of the network across which the Norton equivalent circuit is to be found.*
2. *Mark the terminals of the remaining two-terminal network.*

R_N:

3. *Calculate R_N by first setting all sources to zero (voltage sources are replaced by short circuits, and current sources by open circuits) and then finding the resultant resistance between the two marked terminals. (If the internal resistance of the voltage and/or current sources is included in the original network, it must remain when the sources are set to zero.) Since $R_N = R_{Th}$, the procedure and value obtained using the approach described for Thévenin's theorem will determine the proper value of R_N.*

I_N:

4. *Calculate I_N by first returning all sources to their original position and finding the short-circuit current between the marked terminals.*

Conclusion:

5. *Draw the Norton equivalent circuit with the portion of the circuit previously removed replaced between the terminals of the equivalent circuit.*

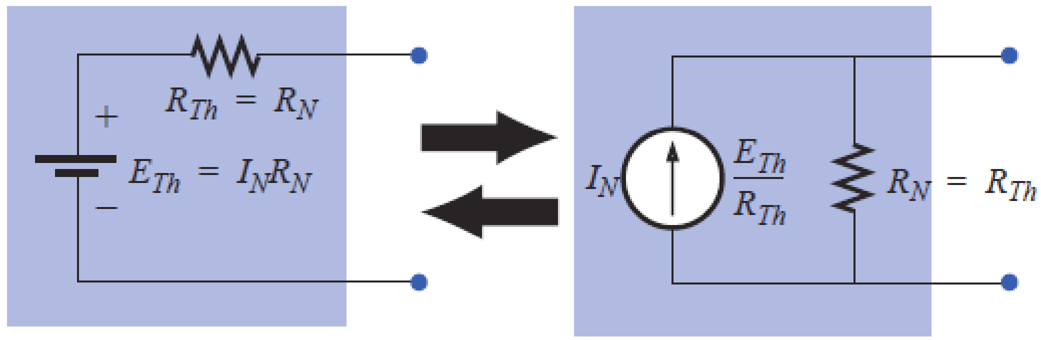


FIG. 9.59
Converting between Thévenin and Norton equivalent circuits.

EXAMPLE 9.13 (Two sources) Find the Norton equivalent circuit for the portion of the network to the left of *a-b* in Fig. 9.71.

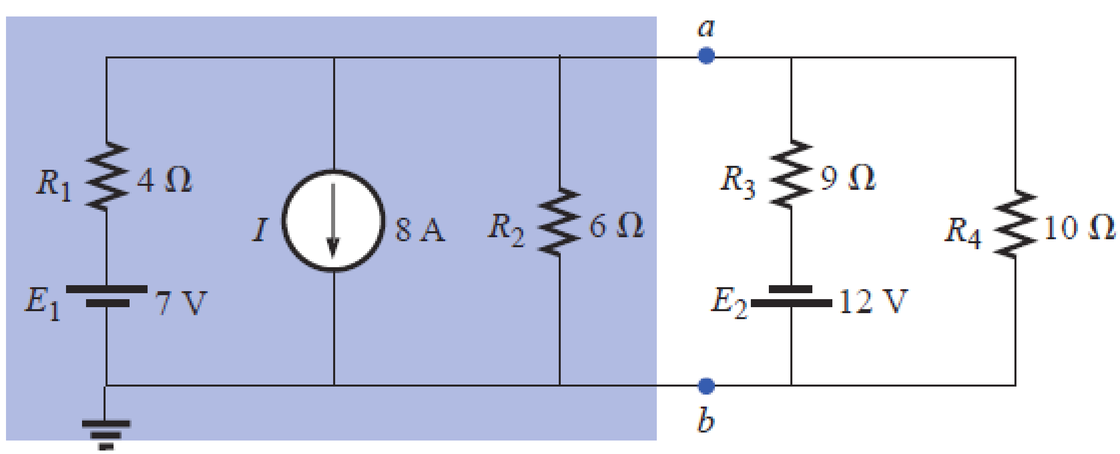


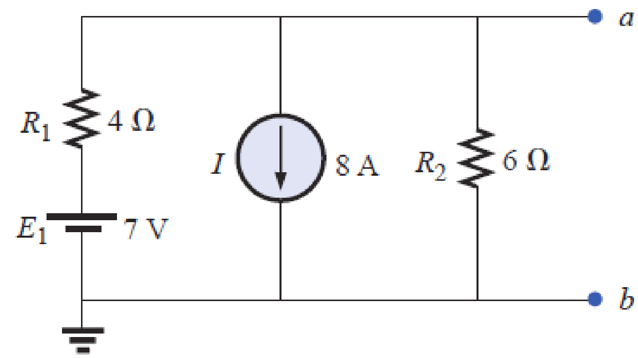
FIG. 9.71
Example 9.13.

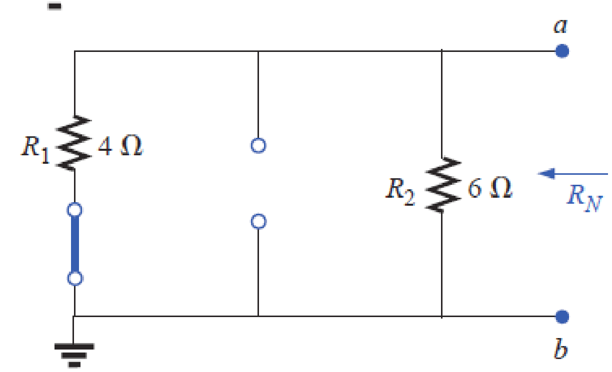
Solution:

Steps 1 and 2: See Fig. 9.72.

Step 3 is shown in Fig. 9.73, and

$$R_N = R_1 \parallel R_2 = 4\ \Omega \parallel 6\ \Omega = \frac{(4\ \Omega)(6\ \Omega)}{4\ \Omega + 6\ \Omega} = \frac{24\ \Omega}{10} = \mathbf{2.4\ \Omega}$$





Step 4: (Using superposition) For the 7-V battery (Fig. 9.74),

$$I'_N = \frac{E_1}{R_1} = \frac{7\text{ V}}{4\ \Omega} = 1.75\text{ A}$$

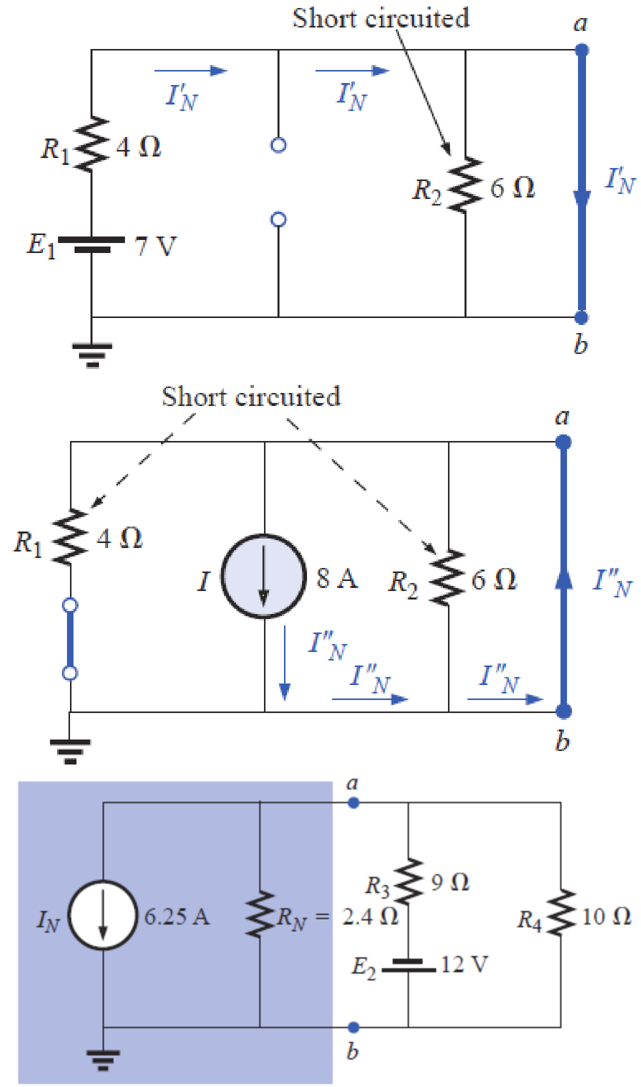
For the 8-A source (Fig. 9.75), we find that both R_1 and R_2 have been "short circuited" by the direct connection between a and b, and

$$I''_N = I = 8\text{ A}$$

The result is

$$I_N = I''_N - I'_N = 8\text{ A} - 1.75\text{ A} = \mathbf{6.25\text{ A}}$$

Step 5: See Fig. 9.76.



9.5 MAXIMUM POWER TRANSFER THEOREM

A load will receive maximum power from a linear bilateral dc network when its total resistive value is exactly equal to the Thévenin resistance of the network as "seen" by the load.

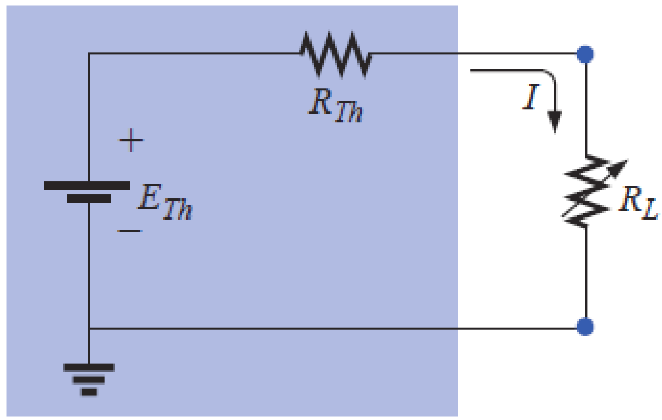
$$\boxed{R_L = R_{Th}}$$

$$\boxed{R_L = R_N}$$

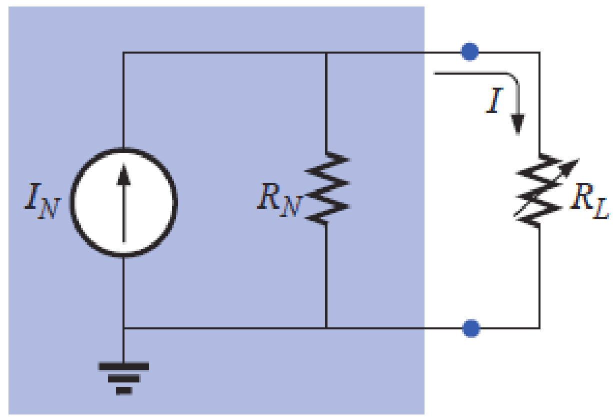
$$I = \frac{E_{Th}}{R_{Th} + R_L}$$

and $$P_L = I^2 R_L = \left(\frac{E_{Th}}{R_{Th} + R_L}\right)^2 R_L$$

so that $$P_L = \frac{E_{Th}^2 R_L}{(R_{Th} + R_L)^2}$$



conditions for maximum power to a load using the Thévenin circuit.



conditions for maximum power to a load using the Norton circuit.

$$I_L = \frac{E_{Th}}{R_{Th}+R_L} \Rightarrow \boxed{I_{max} = \frac{E_{Th}}{R_{Th}}} \;;\; V_L = \frac{R_L}{R_{Th}+R_L} E_{Th} \Rightarrow \boxed{V_{max} = E_{Th}}$$

9.5 MAXIMUM POWER TRANSFER THEOREM

When designing a circuit, it is often important to be able to answer one of the following questions:

What load should be applied to a system to ensure that the load is receiving maximum power from the system?

and, conversely:

For a particular load, what conditions should be imposed on the source to ensure that it will deliver the maximum power available?

Even if a load cannot be set at the value that would result in maximum power transfer, it is often helpful to have some idea of the value that will draw maximum power so that you can compare it to the load at hand. For instance, if a design calls for a load of 100 Ω, to ensure that the load receives maximum power, using a resistor of 1 Ω or 1 kΩ results in a power transfer that is much less than the maximum possible. However, using a load of 82 Ω or 120 Ω probably results in a fairly good level of power transfer.

Fortunately, the process of finding the load that will receive maximum power from a particular system is quite straightforward due to the **maximum power transfer theorem,** which states the following:

A load will receive maximum power from a network when its resistance is exactly equal to the Thévenin resistance of the network applied to the load. That is,

$$\boxed{R_L = R_{Th}} \tag{9.2}$$

In other words, for the Thévenin equivalent circuit in Fig. 9.78, when the load is set equal to the Thévenin resistance, the load will receive maximum power from the network.

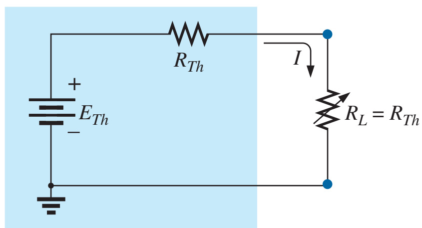


FIG. 9.78
Defining the conditions for maximum power to a load using the Thévenin equivalent circuit.

Using Fig. 9.78 with $R_L = R_{Th}$, the maximum power delivered to the load can be determined by first finding the current:

$$I_L = \frac{E_{Th}}{R_{Th} + R_L} = \frac{E_{Th}}{R_{Th} + R_{Th}} = \frac{E_{Th}}{2R_{Th}}$$

Then substitute into the power equation:

$$P_L = I_L^2 R_L = \left(\frac{E_{Th}}{2R_{Th}}\right)^2 (R_{Th}) = \frac{E_{Th}^2 R_{Th}}{4R_{Th}^2}$$

and

$$\boxed{P_{L_{max}} = \frac{E_{Th}^2}{4R_{Th}}} \tag{9.3}$$

To demonstrate that maximum power is indeed transferred to the load under the conditions defined above, consider the Thévenin equivalent circuit in Fig. 9.79.

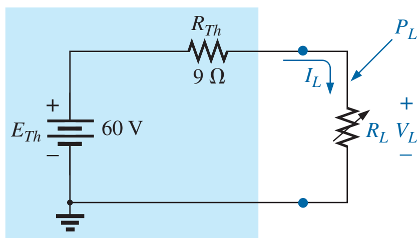


FIG. 9.79
Thévenin equivalent network to be used to validate the maximum power transfer theorem.

Before getting into detail, however, if you were to guess what value of R_L would result in maximum power transfer to R_L, you may think that the smaller the value of R_L, the better, because the current reaches a maximum when it is squared in the power equation. The problem is, however, that in the equation $P_L = I_L^2 R_L$, the load resistance is a multiplier. As it gets smaller, it forms a smaller product. Then again, you may suggest larger values of R_L, because the output voltage increases and power is determined by $P_L = V_L^2/R_L$. This time, however, the load resistance is in the denominator of the equation and causes the resulting power to decrease. A balance must obviously be made between the load resistance and the resulting current or voltage. The following discussion shows that

maximum power transfer occurs when the load voltage and current are one-half of their maximum possible values.

For the circuit in Fig. 9.79, the current through the load is determined by

$$I_L = \frac{E_{Th}}{R_{Th} + R_L} = \frac{60 \text{ V}}{9 \ \Omega + R_L}$$

The voltage is determined by

$$V_L = \frac{R_L E_{Th}}{R_L + R_{Th}} = \frac{R_L(60 \text{ V})}{R_L + R_{Th}}$$

and the power by

$$P_L = I_L^2 R_L = \left(\frac{60 \text{ V}}{9 \ \Omega + R_L}\right)^2 (R_L) = \frac{3600 R_L}{(9 \ \Omega + R_L)^2}$$

If we tabulate the three quantities versus a range of values for R_L from 0.1 Ω to 30 Ω, we obtain the results appearing in Table 9.1. Note in particular that when R_L is equal to the Thévenin resistance of 9 Ω, the power has a maximum value of 100 W, the current is 3.33 A or one-half its max-

TABLE 9.1

R_L (Ω)	P_L (W)	I_L (A)	V_L (V)
0.1	4.35	6.60	0.66
0.2	8.51	6.52	1.30
0.5	19.94	6.32	3.16
1	36.00	6.00	6.00
2	59.50	5.46	10.91
3	75.00	5.00	15.00
4	85.21	4.62	18.46
5	91.84	4.29	21.43
6	96.00	4.00	24.00
7	98.44 Increase	3.75 Decrease	26.25 Increase
8	99.65	3.53	28.23
9 (R_{Th})	**100.00 (Maximum)**	**3.33 ($I_{max}/2$)**	**30.00 ($E_{Th}/2$)**
10	99.72	3.16	31.58
11	99.00	3.00	33.00
12	97.96	2.86	34.29
13	96.69	2.73	35.46
14	95.27	2.61	36.52
15	93.75	2.50	37.50
16	92.16	2.40	38.40
17	90.53	2.31	39.23
18	88.89	2.22	40.00
19	87.24	2.14	40.71
20	85.61	2.07	41.38
25	77.86	1.77	44.12
30	71.00	1.54	46.15
40	59.98	1.22	48.98
100	30.30	0.55	55.05
500	6.95 Decrease	0.12 Decrease	58.94 Increase
1000	3.54	0.06	59.47

Consider an example where:

E_{th} = 60 V and

R_{th} = 9 Ω

$$P_L = \frac{E_{Th}^2 R_L}{(R_{Th} + R_L)^2} = \frac{3600 R_L}{(9\ \Omega + R_L)^2}$$

$$I_L = \frac{E_{Th}}{R_{Th} + R_L} = \frac{60\ \text{V}}{9\ \Omega + R_L}$$

$$V_L = \frac{R_L(60\ \text{V})}{R_{Th} + R_L} = \frac{R_L(60\ \text{V})}{9\ \Omega + R_L}$$

TABLE 9.1

R_L (Ω)	P_L (W)		I_L (A)		V_L (V)	
0.1	4.35		6.59		0.66	
0.2	8.51		6.52		1.30	
0.5	19.94		6.32		3.16	
1	36.00		6.00		6.00	
2	59.50		5.46		10.91	
3	75.00		5.00		15.00	
4	85.21		4.62		18.46	
5	91.84	Increase	4.29	Decrease	21.43	Increase
6	96.00		4.00		24.00	
7	98.44		3.75		26.25	
8	99.65 ↓		3.53 ↓		28.23 ↓	
9 (R_{Th})	**100.00 (Maximum)**		**3.33 ($I_{max}/2$)**		**30.00 ($E_{Th}/2$)**	
10	99.72		3.16		31.58	
11	99.00		3.00		33.00	
12	97.96		2.86		34.29	
13	96.69		2.73		35.46	
14	95.27		2.61		36.52	
15	93.75		2.50		37.50	
16	92.16		2.40		38.40	
17	90.53	Decrease	2.31	Decrease	39.23	Increase
18	88.89		2.22		40.00	
19	87.24		2.14		40.71	
20	85.61		2.07		41.38	
25	77.86		1.77		44.12	
30	71.00		1.54		46.15	
40	59.98		1.22		48.98	
100	30.30		0.55		55.05	
500	6.95		0.12		58.94	
1000	3.54 ↓		0.06 ↓		59.47 ↓	

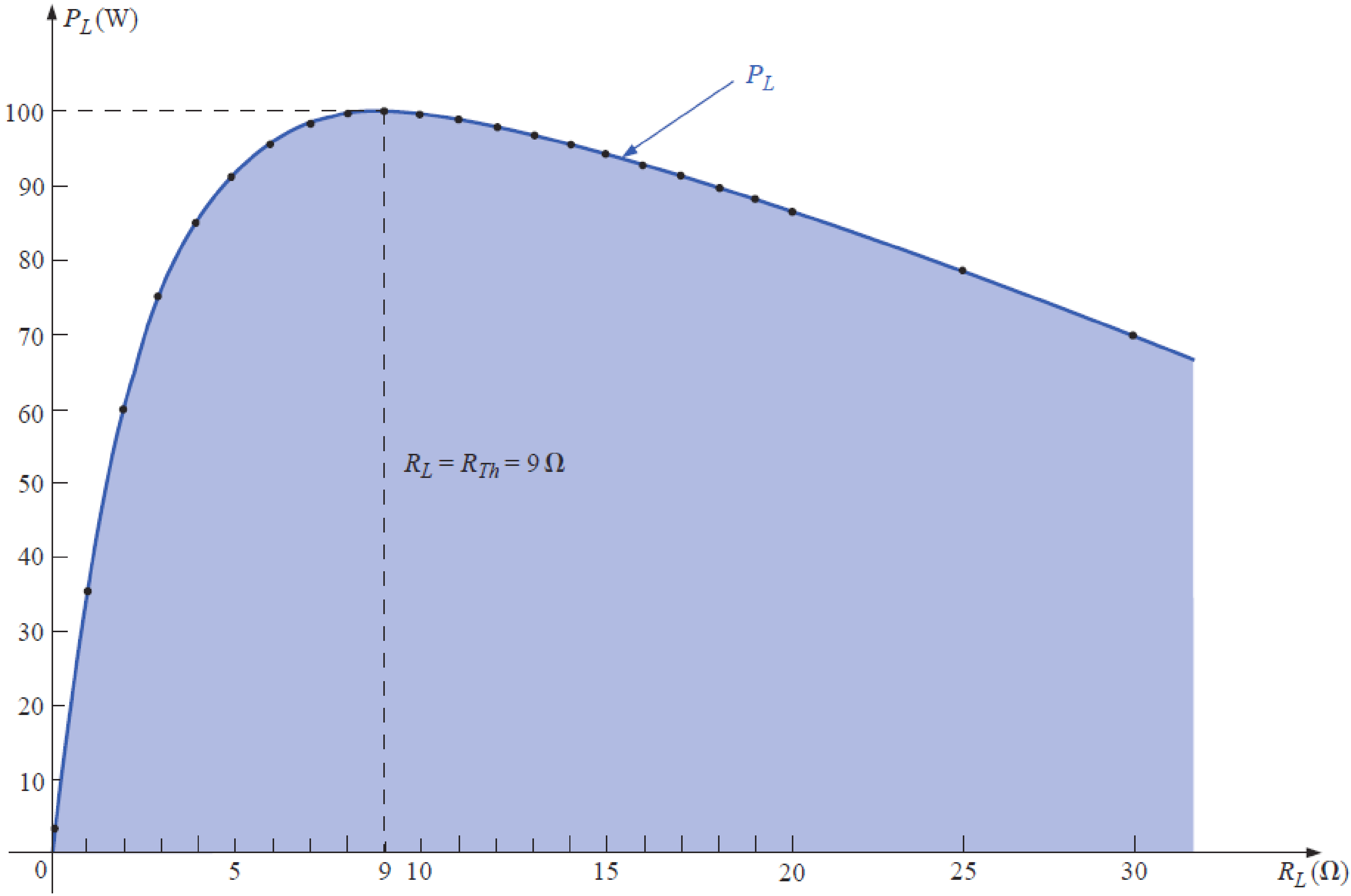


FIG. 9.80

P_L versus R_L for the network of Fig. 9.79.

Maximum Power Obtained When: $\boldsymbol{R_L} = \boldsymbol{R_{Th}} = 9\ \Omega$

- $\boldsymbol{R_L} < \boldsymbol{R_{Th}}$: The increase in power is very fast $\Longrightarrow$ small change in R_L give a large change in P_L.
- $\boldsymbol{R_L} > \boldsymbol{R_{Th}}$: The decrease in power is very slow $\Longrightarrow$ small change in R_L give a small change in P_L.

At $\boldsymbol{R_L} = \boldsymbol{R_{Th}}$: $I_L = \frac{I_{max}}{2}$ and $V_L = \frac{E_{Th}}{2}$

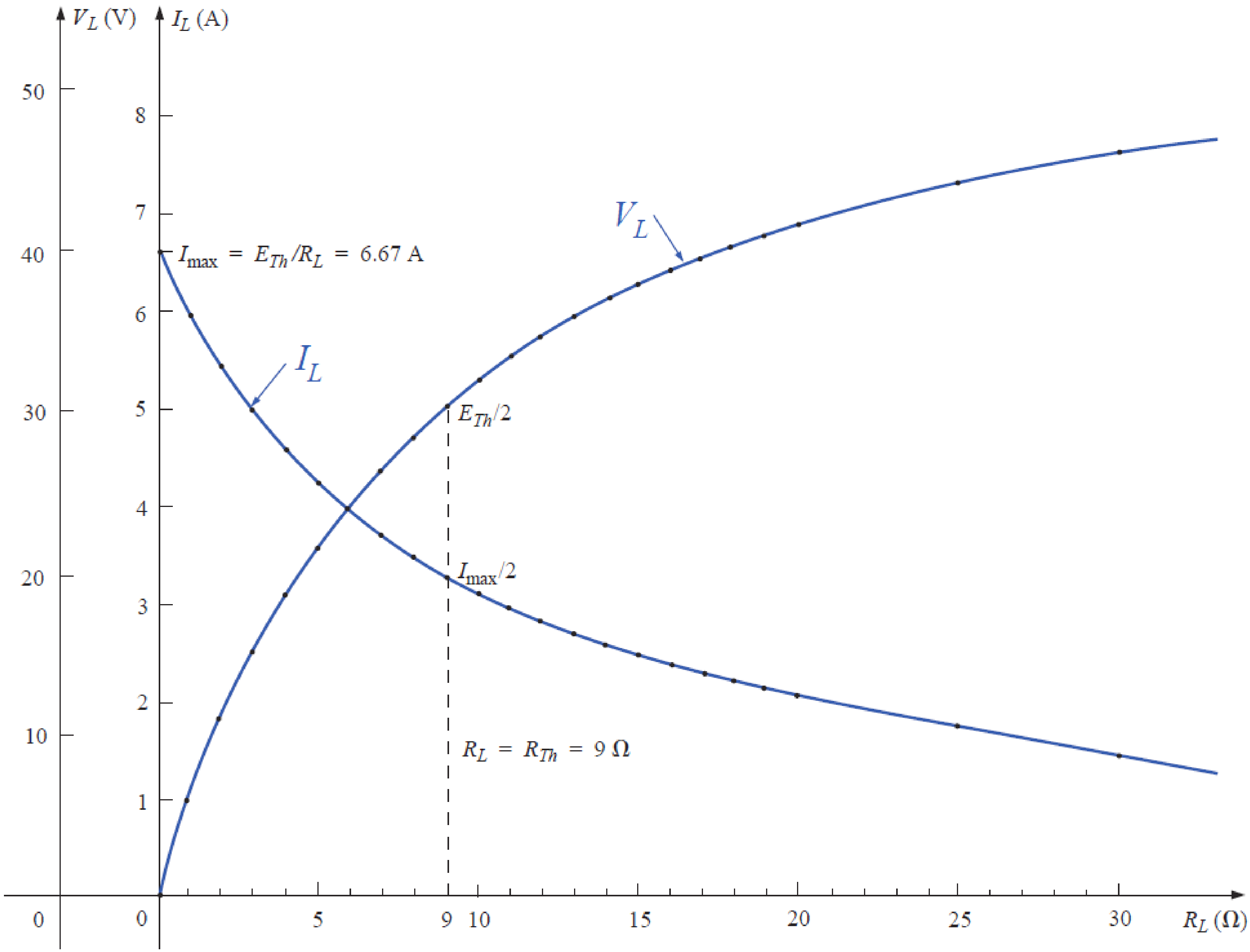


FIG. 9.81

V_L and I_L versus R_L for the network of Fig. 9.79.

The dc operating efficiency of a system is defined by the ratio of the power delivered to the load to the power supplied by the source; that is,

$$\boxed{\eta\% = \frac{P_L}{P_s} \times 100\%} \qquad \textbf{(9.5)}$$

For the situation defined by Fig. 9.77,

$$\eta\% = \frac{P_L}{P_s} \times 100\% = \frac{I_L^2 R_L}{I_L^2 R_T} \times 100\%$$

and

$$\eta\% = \frac{R_L}{R_{Th} + R_L} \times 100\%$$

For R_L that is small compared to R_{Th}, $R_{Th} \gg R_L$ and $R_{Th} + R_L \cong R_{Th}$, with

$$\eta\% \cong \frac{R_L}{R_{Th}} \times 100\% = \underbrace{\left(\frac{1}{R_{Th}}\right)}_{\text{Constant}} R_L \times 100\% = kR_L \times 100\%$$

$R_L \gg R_{Th}$ and $R_{Th} + R_L \cong R_L$.

$$\eta\% = \frac{R_L}{R_L} \times 100\% = 100\%$$

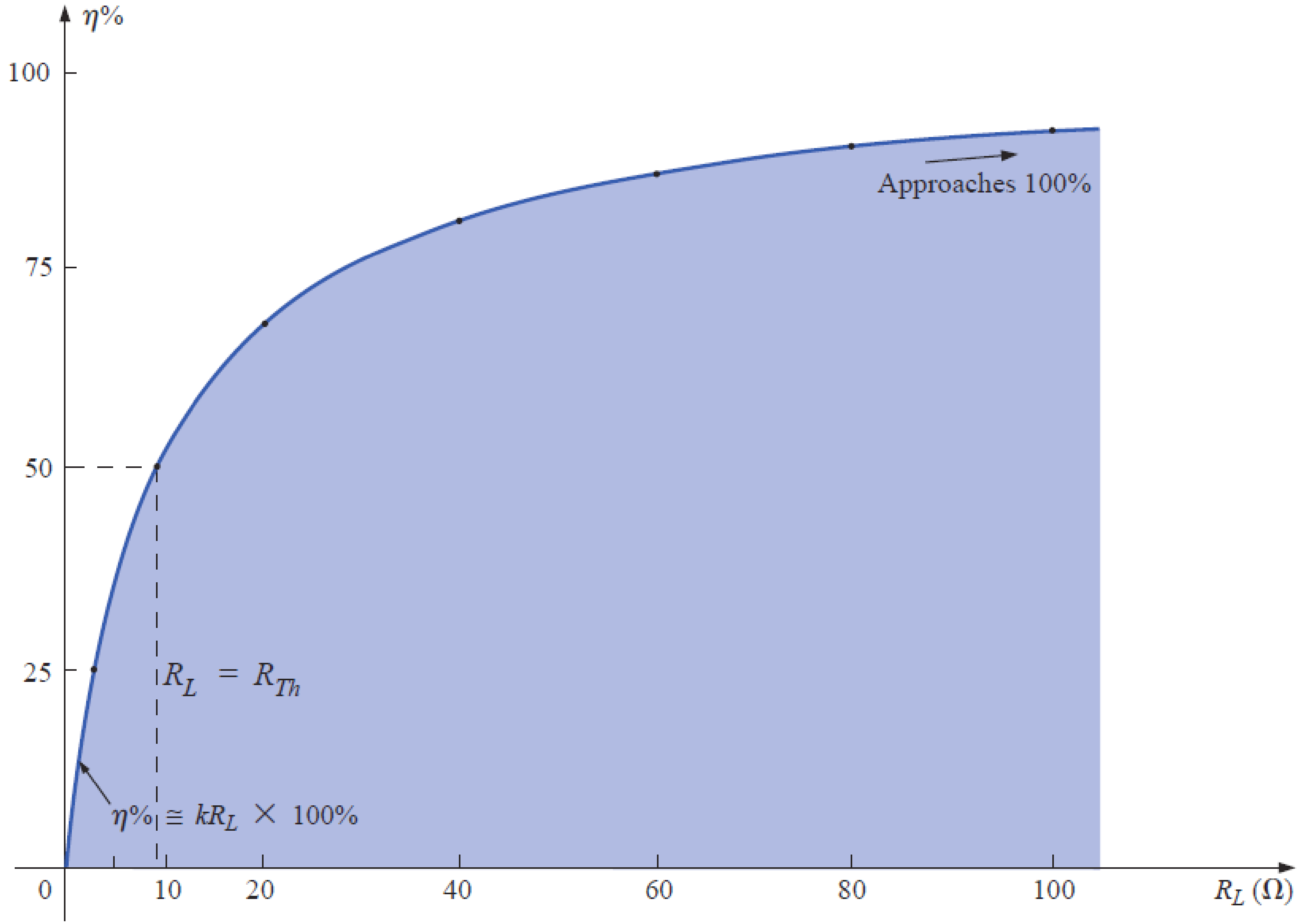


FIG. 9.82
Efficiency of operation versus increasing values of R_L.

The power delivered to R_L under maximum power conditions ($R_L = R_{Th}$) is

$$I = \frac{E_{Th}}{R_{Th} + R_L} = \frac{E_{Th}}{2R_{Th}}$$

$$P_L = I^2 R_L = \left(\frac{E_{Th}}{2R_{Th}}\right)^2 R_{Th} = \frac{E_{Th}^2 R_{Th}}{4R_{Th}^2}$$

and

$$\boxed{P_{L_{\max}} = \frac{E_{Th}^2}{4R_{Th}}} \quad \text{(watts, W)}$$

For the Norton circuit of Fig. 9.78,

$$\boxed{P_{L_{\max}} = \frac{I_N^2 R_N}{4}} \quad \text{(W)}$$

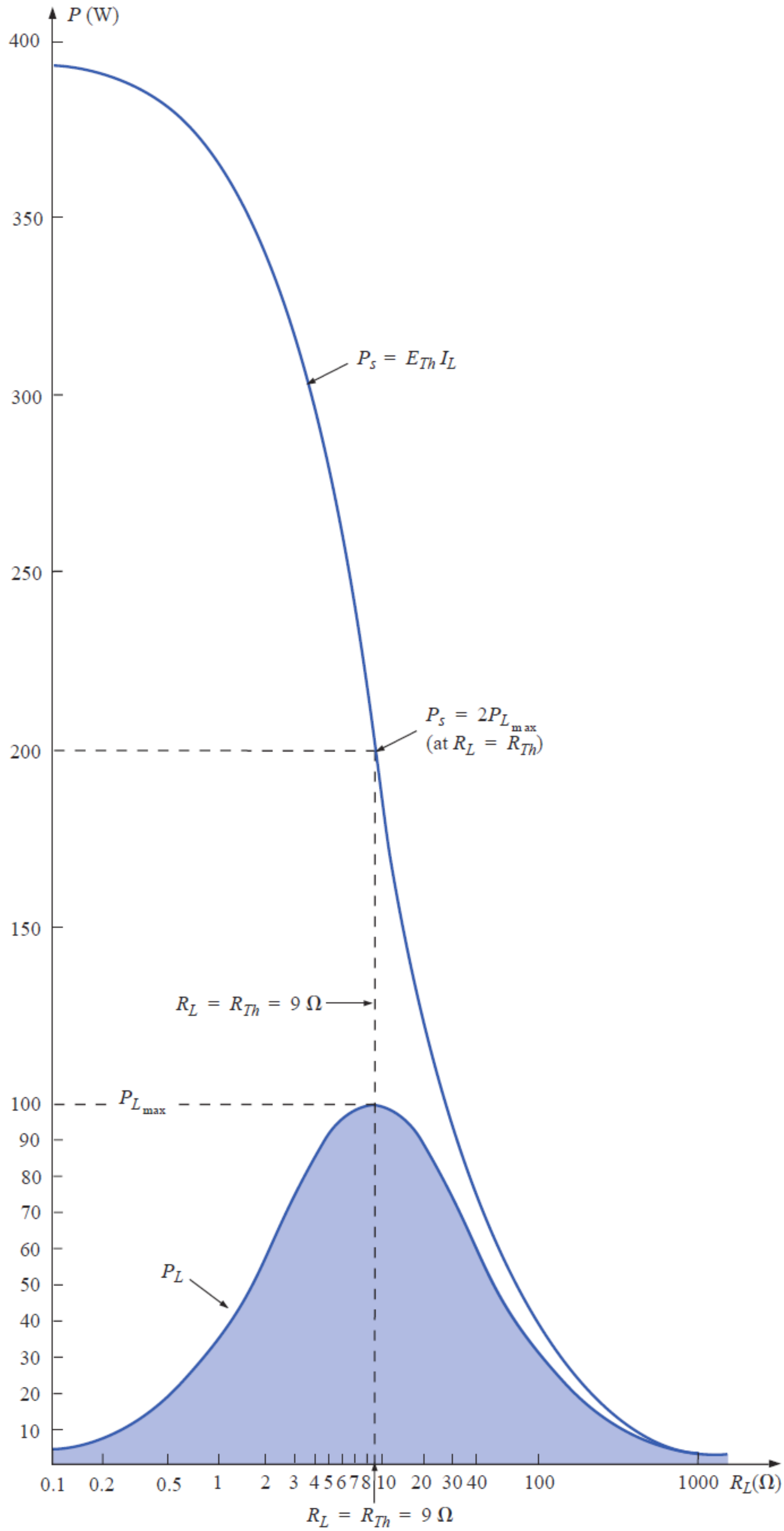


FIG. 9.83

P_s and P_L versus R_L for the network of Fig. 9.79.