Contents

Chapter 1

Discrete random variable

1.1 Discrete Probability Distributions

Definition 1 *The set of ordered pairs $(x, f(x))$ is a probability function, probability mass function, or probability distribution of the discrete random variable X if, for each possible outcome x,*

1. $f(x) \geq 0$,

2. $\sum_{x} f(x) = 1$,

3. $P(X = x) = f(x)$.

Definition 2 *The cumulative distribution function $F(x)$ of a discrete random variable X with probability distribution $f(x)$ is*

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), for - \infty < x < \infty$$

Definition 3 (Mean of a Random Variable) *Let X be a random variable with probability distribution $f(x)$. The mean, or expected value, of X is*

$$\mu = E(X) = \sum_{x} x f(x)$$

Example 4 *A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find* *the expected value of the number of good components in this sample.*

Example 5 *Let X represent the number of good components in the sample. The probability distribution of X is* $f(x) = \dfrac{\begin{pmatrix} 4 \\ x \end{pmatrix}\begin{pmatrix} 3 \\ 3-x \end{pmatrix}}{\begin{pmatrix} N \\ n \end{pmatrix}}$ *,* $x = 0, 1, 2, 3.$ *Simple calculations yield* $f(0) = 1/35$*,* $f(1) = 12/35$*,* $f(2) = 18/35$*, and* $f(3) = 4/35$*. Therefore,*

$$\mu = E(X) = (0)\frac{1}{35} + (1)\frac{12}{35} + (2)\frac{18}{35} + (3)\frac{4}{35} = 12/7 = 1.7$$

Thus, if a sample of size 3 is selected at random over and over again from a lot of 4 good components and 3 defective components, it will contain, on average, 1.7 good components.

Theorem 6 *Let X be a random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is*

$$\mu_{g(X)} = E[g(X)] = \sum_x g(x)f(x)$$

Example 7 *Suppose that the number of cars X that pass through a car wash between 4:00 P.M. and 5:00 P.M. on any sunny Friday has the following probability distribution:*

x	4	5	6	7	8	9
$f(x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Let $g(X) = 2X - 1$ represent the amount of money, in dollars, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.

Example 8 *Let X be a random variable with probability distribution as follows:*

x	0	1	2	3
$f(x)$	$\frac{1}{3}$	$\frac{1}{2}$	0	$\frac{1}{6}$

Find the expected value of $Y = (X - 1)^2$.

Theorem 9 (Variance of Random Variable) *Let X be a random variable with probability distribution $f(x)$ and mean μ. The variance of X is*

$$\sigma^2 = E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x)$$

The positive square root of the variance, σ, is called the standard deviation of X.

Example 10 *Calculate the variance of $g(X) = 2X + 3$, where X is a random variable with probability distribution*

x	0	1	2	3
$f(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

1.2 Some Discrete Probability Distributions

1.2.1 Discrete Uniform Random Variable

Definition 11 (Discrete Uniform Random Variable) *A random variable X is called discrete uniform if it has a finite number of possible values, say $x_1, x_2, ..., x_n$, and $\Pr(X = x_i) = 1/n$ for all i.*

1.2.2 Binomial Distribution

Definition 12 (Bernouilli Process) *Strictly speaking, the Bernoulli process must possess the following properties:*

1. The experiment consists of repeated trials.

2. Each trial results in an outcome that may be classified as a success or a failure.

3. The probability of success, denoted by p, remains constant from trial to trial.

4. The repeated trials are independent.

Definition 13 (Binomial Distribution) *A Bernoulli trial can result in a success with probability p and a failure withprobability $q = 1-p$. Then the probability distribution of the binomial random variable X, the number of successes in n independent trials, is*

$$\Pr(X = x) = \left(\begin{array}{c} n \\ x \end{array}\right) p^x q^{n-x}, \ x = 0, 1, 2, ..., n.$$

Example 14 *The probability that a certain kind of component will survive a shock test is 3/4. Find the probability that exactly 2 of the next 4 components tested survive.*

Solution 15 *Let X the number of components that will survive a shock test. Assuming that the tests are independent and $p = 3/4$ for each of the 4 tests, then X is a binomial distribution $Bin(4, 3/4)$. Hence,*

$$\Pr(X = 2) = \binom{4}{2}(3/4)^2(1/4)^2 \approx 0.21$$

Example 16 *The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that*

(a) at least 10 survive,

(b) from 3 to 8 survive,

and (c) exactly 5 survive?

Solution 17 *(a)* $\Pr(X \geq 10) = 1 - \Pr(X < 10)$

$= 1 - \sum_{x=0}^{9} b(x; 15, 0.4) = 1 - 0.9662 = 0.0338$

(b) $\Pr(3 \leq X \leq 8) = \sum_{x=3}^{8} b(x; 15, 0.4)$

$= \sum_{x=0}^{8} b(x; 15, 0.4) - \sum_{x=0}^{2} b(x; 15, 0.4)$

$= 0.9050 - 0.0271 = 0.8779$

(c) $\Pr(X = 5) = b(5; 15, 0.4) = \sum_{x=0}^{5} b(x; 15, 0.4)$

$- \sum_{x=0}^{4} b(x; 15, 0.4) = 0.4032 - 0.2173 = 0.1859$

Theorem 18 *The mean and variance of the binomial distribution $B(n, p)$ are*

$$\mu = np \text{ and } \sigma^2 = npq.$$

1.3 Hypergeometric Distribution

Definition 19 (Hypergeometric Distribution) *The probability distribution of the hypergeometric random variable X, the numberof successes in a random sample of size n selected from N items of which K are labeled success and $N-K$ labeled failure, is*

$$\Pr(X=x)=\frac{\begin{pmatrix} K \\ x \end{pmatrix}\begin{pmatrix} N-K \\ n-x \end{pmatrix}}{\begin{pmatrix} N \\ n \end{pmatrix}}$$

Theorem 20 *The mean and variance of the hypergeometric distribution $h(N,K,n)$ are*

$$\mu=n\frac{K}{N} \text{ and } \sigma^2=n\frac{K}{N}\left(1-n\frac{K}{N}\right)\frac{N-n}{N-1}.$$

Example 21 *Lots of 40 components each are deemed unacceptable if they contain 3 or more defectives. The procedure for sampling a lot is to select 5 components at random and to reject the lot if a defective is found. What is the probability that exactly 1 defective is found in the sample if there are 3 defectives in the entire lot?*

Solution 22 *Using the hypergeometric distribution with $n=5$, $N=40$, $k=3$, and $x=1$, we find the probability of obtaining 1 defective to be*

$$h(1;40,5,3)=\frac{\binom{3}{1}\binom{37}{4}}{\binom{40}{5}}=0.3011.$$

Theorem 23 (Approximation) *If n is small compared to N, then a binomial distribution $B(n,p=K/N)$ can be used to approximate the hypergeometric distribution $h(N,K,n)$.*

Example 24 *A manufacturer of automobile tires reports that among a shipment of 5000 sent to a local distributor, 1000 are slightly blemished. If one purchases 10 of these tires at random from the distributor, what is the probability that exactly 3 are blemished?*

Solution 25 *Since $N=5000$ is large relative to the sample size $n=10$, we shall approximate the desired probability by using the binomial distribution. The probability of obtaining a* *blemished tire is* 0.2*. Therefore, the probability of obtaining exactly* 3 *blemished tires is*

$$h(3;5000,10,1000) \approx b(3;10,0.2){=}0.8791-0.6778{=}0.2013.$$

1.3.1 Poisson Distribution

Definition 26 *Let* X *the number of outcomes occurring during a given time interval.* X *is called a Poisson random variable when its probability distribution is given by*

$$\Pr(X=x)=e^{-\lambda}\frac{\lambda^x}{x!},\ x=0,1,2,...,$$

where λ *is the average number of outcomes.*

Example 27 *During a laboratory experiment, the average number of radioactive particles passing through a counter in* 1 *millisecond is* 4*. What is the probability that* 6 *particles enter the counter in a given millisecond?*

Solution 28 *Using the Poisson distribution with* $x=6$ *and* $\lambda=4$ *and referring to Table A.2, we have*

$$p(6;4)=\frac{e^{-4}4^6}{6!}=0.1042.$$

Theorem 29 *Both the mean and the variance of the Poisson distribution* $P(\lambda)$ *are* λ*.*

Theorem 30 (Approximation) *Let* X *be a binomial random variable with probability distribution* $B(n,p)$*. When* n *is large (*$n\to\infty$*), and* p *small (*$p\to 0$*), then the poisson distribution can be used to approximate the binomial distribtion* $B(n,p)$ *by taking* $\lambda=np$*.*

Example 31 *In a certain industrial facility, accidents occur infrequently. It is known that the probability of an accident on any given day is* 0.005 *and accidents are independent of each other.*
(a) What is the probability that in any given period of 400 *days there will be an accident on one day?*
(b) What is the probability that there are at most three days with an accident?

Solution 32 *Let* X *be a binomial random variable with* $n=400$ *and* $p=0.005$*. Thus,* $np=2$*. Using the Poisson approximation, (a)* $\Pr(X=1)=e^{-2}2^1=0.271$ *and (b)* $\Pr(X\leq 3)=e^{-2}2^x/x!=0.857$*.*

Chapter 2

Continuous random variable

2.1 Probability density function

Definition 33 *The function $f(x)$ is a probability density function (pdf) for the continuous random variable X, defined over the set of real numbers, if*

1. $f(x) \geq 0$*, for all* $x \in R$*.*

2. $\int\limits_{-\infty}^{\infty} f(x)dx = 1.$

3. $\Pr(a \leq X \leq b) = \int\limits_{a}^{b} f(x)dx.$

Example 34 *Suppose that the error in the reaction temperature, in* $^\circ C$*, for a controlled laboratory experiment is a continuous random variable* X *having the probability density function*

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \textit{elsewhere} \end{cases}$$

(a) Verify that $f(x)$ *is a density function.*

(b) Find $\Pr(0 \leq X \leq 1)$*.*

(c) Find $\Pr(0 < X < 1)$*.*

Definition 35 *The cumulative distribution function $F(x)$ of a continuous random variable X with density function $f(x)$ is*

$$F(x) = \Pr(X \leq x) = \int_{-\infty}^{x} f(t)dt, \text{ for } -\infty < x < \infty.$$

Example 36 *For the density function of Example 2, find $F(x)$, and use it to evaluate* $\Pr(0 < X \leq 1)$.

Definition 37 (Mean of a Random Variable) *Let X be a random variable with probability distribution $f(x)$. The mean, or expected value, of X is*

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

Example 38 *For the density function of Example 2, find $E(X)$.*

Theorem 39 *Let X be a random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is*

$$\mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

Theorem 40 (Variance of Random Variable) *Let X be a random variable with probability distribution $f(x)$ and mean μ. The variance of X is*

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)$$

Theorem 41 *Let X a random variable. The variance of a random variable X is*

$$\sigma^2 = E(X^2) - E(X)^2.$$

Theorem 42 *Let X a random variable. If a and b are constants, then $E(aX+b) = aE(X)+b$.*

Theorem 43 *The expected value of the sum or difference of two or more functions of a random variable X is the sum or difference of the expected values of the functions. That is,*

$$E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)].$$

2.2 Some Continuous Probability Distributions

2.2.1 Continuous Uniform Distribution

Definition 44 (Uniform Distribution) *The density function of the continuous uniform random variable X on the interval $[a, b]$ is*

$$f(x) = \begin{cases} \frac{1}{b-a}, \; a \leq x \leq b \\ 0 \quad \textit{elsewhere.} \end{cases}$$

Example 45 *Suppose that a large conference room at a certain company can be reserved for no more than 4 hours. Both long and short conferences occur quite often. In fact, it can be assumed that the length X of a conference has a uniform distribution on the interval $[0, 4]$.*
a) What is the probability density function?
b) What is the probability that any given conference lasts at least 3 hours?

Theorem 46 *The mean and variance of the uniform distribution are*

$$\mu = E(X) = \frac{a+b}{2} \; \textit{and} \; \sigma^2 = \frac{(b-a)^2}{12}$$

The proofs of the theorems are left to the reader.

2.2.2 Normal Distribution

Definition 47 (Standard Normal Distribution) *The density of the standard normal distribution Z is*

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}x^2}, \; -\infty < x < \infty,$$

Theorem 48 *The mean and variance of standard normal distribution are* 0 *and* 1*, respectively. We denote the standard normal distribution by* $N(0,1)$*.*

Example 49 *Given a standard normal distribution* $N(0,1)$*, find the area under the curve that lies*
(a) to the right of $z = 1.84$
(b) between $z = -1.97$ *and* $z = 0.86$*.*

Solution 50 *(a)* 0.0329 *(b)* 0.7807*.*

Example 51 *Given a standard normal distribution* $N(0,1)$*, find the value of* k *such that*
(a) $\Pr(Z > k) = 0.3015$ *and*
(b) $P(k < Z < -0.18) = 0.4197$*.*

Solution 52 *(a)* $k = 0.52$ *(b)* $k = -2.37$*.*

Definition 53 (Normal Distribution) *The density of the normal random variable* X*, with mean* μ *and variance* σ^2*, and denoted by* $N(\mu, \sigma)$*, is*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-1}{2\sigma^2}(x-\mu)^2}, \quad -\infty < x < \infty,$$

where $\pi = 3.14159\ .\ .\ .$ *and* $e = 2.71828\ .\ .\ .\ .$

Theorem 54 *If* X *is normal random variable* $N(\mu, \sigma)$*, then the random variable* $(X-\mu)/\sigma$ *is a standard normal distribution* Z *with mean* 0 *and variance* 1*.*

Example 55 *Given a random variable* X *having a normal distribution with* $\mu = 50$ *and* $\sigma = 10$*, find the probability that* X *assumes a value between* 45 *and* 62*.*

Solution 56 *Using Table A.3, we have*
$\Pr(45 < X < 62) = \Pr(-0.5 < Z < 1.2) = \Pr(Z < 1.2) - \Pr(Z < -0.5)$
$= 0.8849 - 0.3085 = 0.5764.$

Example 57 *Given a normal distribution with* $\mu = 40$ *and* $\sigma = 6$*, find the value of* x *that has*
(a) 45% *of the area to the left*
(b) 14% *of the area to the right.*

Solution 58 *(a) We need to find a z value that leaves an area of* 0.45 *to the left. From Table A.3 we find* $\Pr(Z < -0.13) = 0.45$*, so the desired z value is* -0.13*. Hence,* $x = (6)(-0.13) + 40 = 39.22$*. (b) This time we require a z value that leaves* 0.14 *of the area to the right and hence an area of* 0.86 *to the left. Again, from Table A.3, we find* $P(Z < 1.08) = 0.86$*, so the desired z value is* 1.08 *and*

$$x = (6)(1.08) + 40 = 46.48.$$

Example 59 *An electrical firm manufactures light bulbs that have a life, before burn-out, that is normally distributed with mean equal to* 800 *hours and a standard deviation of* 40 *hours. Find the probability that a bulb burns between* 778 *and* 834 *hours.*

Solution 60 *The z values corresponding to* $x_1 = 778$ *and* $x_2 = 834$ *are*

$$z_1 = \frac{778 - 800}{40} = -0.55 \text{ and } z_2 = \frac{834 - 800}{40} = 0.85.$$

Hence,

$$\begin{aligned} \Pr(778 &< X < 834) = P(-0.55 < Z < 0.85) \\ &= P(Z < 0.85) - P(Z < -0.55) \\ &= 0.8023 - 0.2912 = 0.5111. \end{aligned}$$

Example 61 *A certain machine makes electrical resistors having a mean resistance of* 40 *ohms and a standard deviation of* 2 *ohms. Assuming that the resistance follows a normal distribution and can be measured to any degree of accuracy, what percentage of resistors will have a resistance exceeding* 43 *ohms?*

Solution 62 *A percentage is found by multiplying the relative frequency by* 100%*. Since the relative frequency for an interval is equal to the probability of a value falling in the interval, we must find the area to the right of* $x = 43$*. This can be done by transforming* $x = 43$ *to the* *corresponding z value, obtaining the area to the left of z from Table A.3, and then subtracting this area from* 1*. We find*

$$z = \frac{43 - 40}{2} = 1.5.$$

Therefore,$\Pr(X > 43) = \Pr(Z > 1.5) = 1 - \Pr(Z < 1.5) = 1 - 0.9332 = 0.0668$*. Hence,* 6.68% *of the resistors will have a resistance exceeding* 43 *ohms.*

2.2.3 Exponential Distribution

The exponential random variable is used when we are interested in the time of the first arrival or the time between arrival.

Definition 63 *The continuous random variable X has an exponential distribution, with parameter λ, if its density function is given by* $f(x) = \begin{cases} \lambda e^{-\lambda x}, \ x > 0 \\ 0 \qquad \textit{elsewhere} \end{cases}$ *where $\lambda > 0$.*

Theorem 64 *The mean and variance of the exponential distribution are $\mu = 1/\lambda$ and $\sigma^2 = 1/\lambda^2$.*

If X is the time of arrival of the first customer and if the average time is 30 minutes, then $\lambda = 1/30$.

Example 65 *Suppose that a system contains a certain type of component whose time, in years, to failure is given by T. The random variable T is modeled nicely by the exponential distribution with mean time to failure is* 5*.*
(a) If one component is installed, what is the probability that it is still functioning at the end of 8 *years?*
(b) If 5 *of these components are installed in different systems, what is the probability that at least* 2 *are still functioning at the end of* 8 *years? (Hint: use the binomial distribution).*

Solution 66 *(a)The probability that a given component is still functioning after 8 years is given by*

$$\Pr(T > 8) = \frac{1}{5}\int_{8}^{\infty} e^{-t/5} dt = e^{-8/5} \approx 0.2.$$

(b) Let X represent the number of components functioning after 8 years. X is binomial disctribution $Bin(8, 0.2)$. Then we have

$$\begin{aligned} \Pr(X &\geq 2) = \sum_{x=2}^{5} \Pr(X = x) = 1 - \left(\sum_{x=0}^{1} \Pr(X = x) \right) \\ &= 1 - 0.7373 = 0.2627. \end{aligned}$$

Chapter 3

Fundamental Sampling Distributions

3.1 Random sampling

Definition 67 *A population consists of the totality of the observations with which we are concerned*

Definition 68 *A sample is a subset of a population.*

In the field of statistical inference, statisticians are interested in arriving at conclusions concerning a population when it is impossible or impractical to observe the entire set of observations that make up the population. Therefore, we must depend on a subset of observations from the population to help us make **inferences** concerning that same population.

Definition 69 *A sample is a subset of a population.*

To eliminate any possibility of **bias** in the sampling procedure, it is desirable to choose a random sample in the sense that the observations are made independently and at random.

3.2 Some important statistics

Definition 70 *Any function of the random variables constituting a random sample is called a statistic.*

- Sample mean: $\overline{X} = \frac{1}{n}\sum\limits_{i=1}^{n} X_i$

- Sample median: $\widetilde{X} = \begin{cases} x(n+1)/2, \text{ if } n \text{ is odd,} \\ \frac{1}{2}(x_{n/2} + x_{n/2+1}), \text{ if } n \text{ is even.} \end{cases}$

- Sample variance: $S^2 = \frac{1}{n-1}\sum\limits_{i=1}^{n}(X_i - \overline{X})^2$

The computed value of S^2 for a given sample is denoted by s^2.

Theorem 71 *If S^2 is the variance of a random sample of size n, we may write*

$$S^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} X_i^2 - n\overline{X}^2\right]$$

- Sample standard deviation: $S = \sqrt{S^2}$

3.3 Sampling Distibutions

Let us consider a soft-drink machine designed to dispense, on average, 240 milliliters per drink. A company official who computes the mean of 40 drinks obtains $\overline{x} = 236$ milliliters. On the basis of this value, she decides that the machine is still dispensing drinks with an average content of $\mu = 240$ milliliters. The 40 drinks represent a sample from the infinite population of possible drinks that will be dispensed by this machine.The company official made the decision that the soft-drink machine dispenses drinks with an average content of 240 milliliters, even though the sample mean was 236 milliliters, because he knows from sampling theory that, if $\mu = 240$ milliliters, such a sample value could easily occur. In fact, if she ran similar tests, say every hour, she would expect the values of the statistic $\overline{x}$ to fluctuate above and below $\mu = 240$ milliliters. Only when the value of $\overline{x}$ is **substantially** different from 240 milliliters will the company official initiate action to adjust the machine.

Since a statistic is a random variable that depends only on the observed sample, it must have a probability distribution.

Definition 72 *The probability distribution of a statistic is called a sampling distribution.*

3.4 Sampling Distribution of Means and the Central Limit

Theorem 73 *If $X_1, X_2, ..., X_n$ are independent (?) random variables having normal distributions with means $\mu_1, \mu_2, ..., \mu_n$ and variances $\sigma_1^2, \sigma_2^2, ..., \sigma_n^2$, respectively, then the random variable $Y = a_1X_1 + a_2X_2 + \cdots + a_nX_n$ has a normal distribution with mean*

$$\mu_Y = a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n$$

and variance

$$\sigma_Y^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \cdots + a_n^2\sigma_n^2$$

Suppose that a random sample of n observations is taken from a normal population with mean μ and variance σ^2. Each observation X_i, $i = 1, 2, ..., n$, of the random sample will then have the same normal distribution. Hence, from Theorem 7, we conclude that

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

has a normal distribution with mean

$$\mu_{\overline{X}} = \frac{1}{n}\{\mu + \mu + ... + \mu\} = \sum_{i=1}^{n} \mu = \mu$$

and variance

$$\sigma_{\overline{X}}^2 = \frac{1}{n^2}\left\{\sigma^2 + \sigma^2 + ... + \sigma^2\right\} = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{\sigma^2}{n}.$$

Hence, we have

Corollary 74 *If $X_1, X_2, ..., X_n$ are independent random variables having normal distributions with means μ and variances σ, then the sample mean $\overline{X}$ is normally distributed with mean equal to μ and standard deviation equal to $\sigma/\sqrt{n}$. Consequently the random variable*

$$Z = \frac{(\overline{X} - \mu)}{\sigma/\sqrt{n}}$$

is a standard normal distribution.

Theorem 75 (Central Limit Theorem) *If $\overline{X}$ is the mean of a random sample of size n taken from a population with mean μ and finite variance σ^2, then the limiting form of the distribution of*

$$Z = \frac{(\overline{X} - \mu)}{\sigma/\sqrt{n}}$$

as $n \to \infty$, is the standard normal distribution $N(0, 1)$.

The normal approximation for $\overline{X}$ will generally be good if $n \geq 30$.

Example 76 *An electrical firm manufactures light bulbs that have a length of life that is approximately normally distributed, with mean equal to 800 hours and a standard deviation of 40 hours. Find the probability that a random sample of 16 bulbs will have an average life of less than 775 hours.*

Solution 77 *Here $\mu = 800$, $\sigma = 40$ and $n = 16$. The random variable $\overline{X}$ is normally distributed with mean $\mu_{\overline{X}} = \mu = 800$ and standard standard deviation $\sigma_{\overline{X}} = \sigma_X/\sqrt{n} = 10$.*
Then $(\overline{X} - 800)/10$ is a standard normal distribution $N(0, 1)$. Hence,

$$\begin{aligned} \Pr(\overline{X} &< 775) = P((\overline{X} - 800)/10 < (775 - 800)/10) \\ &= P(Z < -2.5) = 0.0062. \end{aligned}$$

Example 78 *Traveling between two campuses of a university in a city via shuttle bus takes, on average, 28 minutes with a standard deviation of 5 minutes. In a given week, a bus transported passengers 40 times. What is the probability that the average transport time was more than 30 minutes?*

Solution 79 *In this case, $\mu = 28$ and $\sigma = 3$. We need to calculate the probability $Pr(\overline{X} > 30)$ with $n = 40$. Hence,*

$$\begin{aligned} Pr(\overline{X} &> 30) = \Pr\left(\frac{\overline{X} - 28}{5/\sqrt{40}} \geq \frac{30 - 28}{5/\sqrt{40}}\right) = \Pr(Z \geq 2.53) \\ &= 1 - \Pr(Z \leq 2.53) = 1 - 0.9925 = 0.0075. \end{aligned}$$

There is only a slight chance that the average time of one bus trip will exceed 30 minutes.

3.5 Sampling Distribution of the Difference between Two Means

A scientist or engineer may be interested in a comparative experiment in which two manufacturing methods, 1 and 2, are to be compared. The basis for that comparison is $\mu_1 - \mu_2$, the difference in the population means. Suppose that we have two populations, the first with mean μ_1 and variance σ_1^2, and the second with mean μ_2 and variance σ_2^2. Let the statistic $\overline{X}_1$ represent the mean of a random sample of size n_1 selected from the first population, and the statistic $\overline{X}_2$ represent the mean of a random sample of size n_2 selected from the second population, independent of the sample from the first population. What can we say about the sampling distribution of the difference $\overline{X}_1 - \overline{X}_2$ for repeated samples of size n_1 and n_2? According to Theorem 8, the variables $\overline{X}_1$ and $\overline{X}_2$ are both approximately normally distributed with means μ_1 and μ_2 and variances σ_1^2/n_1 and σ_2^2/n_2, respectively. This approximation improves as n_1 and n_2 increase. We can conclude that $\overline{X}_1 - \overline{X}_2$ is approximately normally distributed with mean

$$\mu_{\overline{X}_1-\overline{X}_2} = \mu_{\overline{X}_1} - \mu_{\overline{X}_2} = \mu_1 - \mu_2$$

and variance

$$\sigma^2_{\overline{X}_1-\overline{X}_2} = \sigma^2_{\overline{X}_1} + \sigma^2_{\overline{X}_2} = \sigma_1^2/n_1 + \sigma_2^2/n_2$$

The Central Limit Theorem can be easily extended to the two-sample, two-population case.

Theorem 80 *If independent samples of size n_1 and n_2 are drawn at random from two populations, discrete or continuous, with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively, then the sampling distribution of the differences of means, $\overline{X}_1 - \overline{X}_2$, is approximately normally distributed with mean and variance given by*

$$\mu_{\overline{X}_1-\overline{X}_2} = \mu_1 - \mu_2 \text{ and } \sigma^2_{\overline{X}_1-\overline{X}_2} = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

Hence,

$$Z = \frac{(\overline{X}_1 - \overline{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$$

is approximately a standard normal variable.

If both n_1 and n_2 are greater than or equal to 30, the normal approximation for the distribution of $\overline{X}_1 - \overline{X}_2$ is good. Two independent experiments are run in which two different types of paint are compared.

Example 81 *Eighteen specimens are painted using type A, and the drying time, in hours, is recorded for each. The same is done with type B. The population standard deviations are both known to be* 1.0*. Assuming that the mean drying time is equal for the two types of paint, find* $P(\overline{X}_A - \overline{X}_B > 1.0)$*, where* $\overline{X}_A$ *and* $\overline{X}_B$ *are average drying times for samples of size* $n_A = n_B = 18$*.*

Solution 82 *From the sampling distribution of* $\overline{X}_A - \overline{X}_B$*, we know that the distribution is approximately normal with mean* $\mu_{\overline{X}_A - \overline{X}_B} = \mu_A - \mu_B = 0$ *and variance* $\sigma^2_{\overline{X}_A - \overline{X}_B} = \frac{\sigma_A^2}{n_A} + \frac{\sigma_B^2}{n_B} = 1/9$*. Corresponding to the value* $\overline{X}_A - \overline{X}_B$ *= 1.0, we have*

$$z = \frac{1 - (\mu_A - \mu_B)}{\sqrt{1/9}} = \frac{1 - 0}{\sqrt{1/9}} = 3$$

so

$$\Pr(Z > 3.0) = 1 - P(Z < 3.0) = 1 - 0.9987 = 0.0013.$$

Example 83 *The television picture tubes of manufacturer A have a mean lifetime of* 6.5 *years and a standard deviation of* 0.9 *year, while those of manufacturer B have a mean lifetime of* 6.0 *years and a standard deviation of* 0.8 *year. What is the probability that a random sample of* 36 *tubes from manufacturer A will have a mean lifetime that is at least* 1 *year more than the mean lifetime of a sample of* 49 *tubes from manufacturer B?*

Solution 84 *We are given the following information:*

Population 1	*Population 2*
$\mu_1 = 6.5$	$\mu_2 = 6.0$
$\sigma_1 = 0.9$	$\sigma_2 = 0.8$
$n_1 = 36$	$n_2 = 49$

If we use, the sampling distribution of $\overline{X}_1 - \overline{X}_2$ *will be approximately normal and will have a* *mean and standard deviation*

$$\mu_{\overline{X}_1-\overline{X}_2} = 6.5 - 6.0 \text{ and } \sigma_{\overline{X}_1-\overline{X}_2} = \sqrt{\tfrac{0.81}{36} + \tfrac{0.64}{49}} = 0.189$$

Hence,

$$\begin{aligned}\Pr(\overline{X}_1 - \overline{X}_2 &\geq 1.0) = P(Z > 2.65) = 1 - P(Z < 2.65) \\ &= 1 - 0.9960 = 0.0040.\end{aligned}$$

3.6 Sampling Distribution of S^2

Theorem 85 *If $X_1, X_2, ..., X_n$ an independent random sample that have the same standard normal distribution then $X = \sum_{i=1}^{n} X_i^2$ is chi-squared distribution, with $\nu = n$ degrees of freedom.*

Theorem 86 *The mean and variance of the chi-squared distribution χ^2 with ν degrees of freedom are $\mu = \nu$ and $\sigma^2 = 2\nu$.*

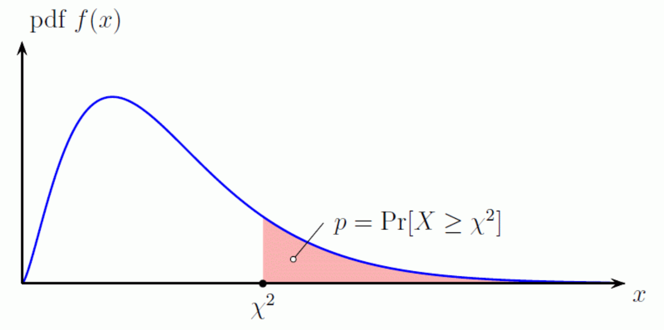
Table A.5 gives values of χ^2_α for various values of α and ν. Hence, the χ^2 value with 7 degrees of freedom, leaving an area of 0.05 to the right, is $\chi^2_{0.05} = 14.067$. Owing to lack of symmetry, we must also use the tables to find $\chi^2_{0.95} = 2.167$ for $\nu = 7$.

Example 87 *For a chi-squared distribution, find*
(a)$\chi^2_{0.025}$ when $\nu = 15$;
(b)$\chi^2_{0.01}$when $\nu = 7$;
(c)$\chi^2_{0.05}$ when $\nu = 24$.

Solution 88 *(a) 27.488.(b) 18.475.(c) 36.415*

For a chi-squared distribution X, find χ^2_α such that
(a) $P(X > \chi^2_\alpha) = 0.99$ when $\nu = 4$;
(b) $P(X > \chi^2_\alpha) = 0.025$ when $\nu = 19$;
(c) $P(37.652 < X < \chi^2_\alpha) = 0.045$ when $\nu = 25$.

Solution 89 *(a)* $\chi^2_\alpha = \chi^2_{0.99} = 0.297$.*(b)* $\chi^2_\alpha = \chi^2_{0.025} = 32.852$.*(c)* $\chi^2_{0.05} = 37.652$. *Therefore,* $\alpha = 0.05 - 0.045 = 0.005$. *Hence,* $\chi^2_\alpha = \chi^2_{0.005} = 46.928$.



Theorem 90 *If S^2 is the variance of a random sample of size n taken from a normal population having the variance σ^2, then the statistic*

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{\sigma^2}$$

has a chi-squared distribution with $\nu = n - 1$ degrees of freedom.

3.7 t-Distribution

Theorem 91 *Let Z be a standard normal random variable and V a chi-squared random variable with ν degrees of freedom. If Z and ν are independent, then the distribution of the random variable T, where*

$$T = \frac{Z}{\sqrt{V/\nu}}$$

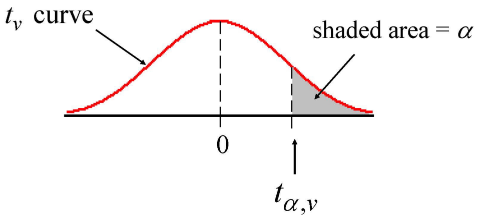
This is known as the t-distribution with ν degrees of freedom.

Corollary 92 *Let $X_1, X_2, ..., X_n$ be independent random variables that are all normal with mean μ and standard deviation σ. Let*

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \textit{and} \qquad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

Then the random variable $T = \frac{\overline{X}-\mu}{S/\sqrt{n}}$ has a t-distribution with $\nu = n - 1$ degrees of freedom.

Pictorial Definition of $t_{\alpha,v}$



Example 93 *The t-value with* $\nu = 14$ *degrees of freedom that leaves an area of* 0.025 *to the left, and therefore an area of* 0.975 *to the right, is*

$$t_{0.975} = -t_{0.025} = -2.145$$

Example 94 *Find* $\Pr(-t_{0.025} < T < t_{0.05})$.

Solution 95 *Since* $t_{0.05}$ *leaves an area of* 0.05 *to the right, and* $-t_{0.025}$ *leaves an area of* 0.025 *to the left, we find a total area of* $1 - 0.05 - 0.025 = 0.925$ *between* $-t_{0.025}$ *and* $t_{0.05}$. *Hence* $\Pr(-t_{0.025} < T < t_{0.05}) = 0.925$.

Example 96 *Find* k *such that* $\Pr(k < T < -1.761) = 0.045$ *for a random sample of size* 15 *selected from a normal distribution with* $T = \frac{\overline{X}-\mu}{S/\sqrt{n}}$.

Solution 97 *From Table A.4 we note that* 1.761 *corresponds to* $t_{0.05}$ *when* $\nu = 14$. *Therefore,*$-t_{0.05} = -1.761$. *Since* k *in the original probability statement is to the left of* $-t_{0.05} = -1.761$, *let* $k = -t_\alpha$. *Then, by using figure, we have*

$$0.045 = 0.05 - \alpha, \text{ or } \alpha = 0.005.$$

Hence, from Table A.4 with $\nu = 14$,

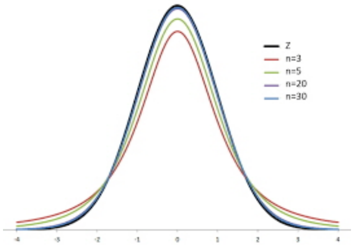


Figure 3-1: t-distribution

$$k = -t_{0.005} = -2.977 \ \textit{and} \ \Pr(-2.977 < T < -1.761) = 0.045.$$

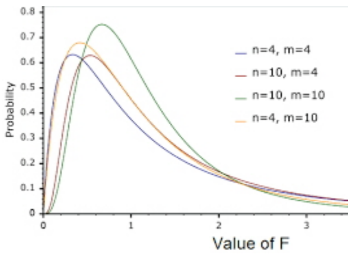


Figure 3-2: F-distribution

3.8 F-Distribution

The statistic F is defined to be the ratio of two independent chi-squared random variables, each divided by its number of degrees of freedom.

Theorem 98 *The random variable*

$$F = \frac{U/\nu_1}{V/\nu_2}$$

*where U and V are independent random variables having chi-squared distributions with ν_1 and ν_2 degrees of freedom, respectively, is the F-**distribution** with ν_1 and ν_2 degrees of freedom (d.f.).*

Theorem 99 *Writing $f_\alpha(\nu_1, \nu_2)$ for f_α with ν_1 and ν_2 degrees of freedom, we have*

$$f_{1-\alpha}(\nu_1, \nu_2) = \frac{1}{f_\alpha(\nu_2, \nu_1)}$$

Thus, the f-value with 6 and 10 degrees of freedom, leaving an area of 0.95 to the right, is $f_{0.95}(6, 10) = \frac{1}{f_{0.05}(10,6)} = \frac{1}{4.06} = 0.246$.

3.8.1 The F-Distribution with Two Sample Variances

Suppose that random samples of size n_1 and n_2 are selected from two normal populations with variances σ_1^2 and σ_2^2, respectively. From Theorem 16, we know that

$$\chi_1^2 = \frac{(n_1 - 1)S_1^2}{\sigma_1^2} \text{ and } \chi_2^2 = \frac{(n_2 - 1)S_2^2}{\sigma_2^2}$$

are random variables having chi-squared distributions with $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$ degrees of freedom. Furthermore, since the samples are selected at random, we are dealing with independent random variables. Then, using Theorem 24 with $\chi_1^2 = U$ and $\chi_2^2 = V$, we obtain the following result.

Theorem 100 *If S_1^2 and S_2^2 are the variances of independent random samples of size n_1 and n_2 taken from normal populations with variances σ_1^2 and σ_2^2, respectively, then*

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$$

has an F-distribution with $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$ degrees of freedom.

3.8.2 Example

For an F-distribution, find

(a) $f_{0.05}$ with $\nu_1 = 7$ and $\nu_2 = 15$;

(b) $f_{0.05}$ with $\nu_1 = 15$ and $\nu_2 = 7$:

(c) $f_{0.01}$ with $\nu_1 = 24$ and $\nu_2 = 19$;

(d) $f_{0.95}$ with $\nu_1 = 19$ and $\nu_2 = 24$;

(e) $f_{0.99}$ with $\nu_1 = 28$ and $\nu_2 = 12$.

Solution 101 *(a) 2.71.(b) 3.51.(c) 2.92.(d) 1/2.11 = 0.47.(e) 1/2.90 = 0.34.*

3.9 Sampling Distribution of Proportions and the Central Limit

In many situations the use of the sample proportion is easier and more reliable because, unlike the mean, the proportion does not depend on the population variance, which is usually an unknown quantity. We will represent the sample proportion by $\widehat{P}$ and the population proportion by p. Construction of the sampling distribution of the sample proportion is done in a manner similar to that of the mean. One has $\widehat{P} = X/n$ where X is a number of success for a sample of size n. It is clear that X is a binomial distribution $Bin(n, p)$. Its mean $\mu_X = np$ and its variance $\sigma_X^2 = np(1-p)$. Consequently:

Theorem 102 *The mean $\mu_{\widehat{p}}$ of the sample distribution $\widehat{P}$ is equal to the true population proportion p, and its variance $\sigma^2_{\widehat{p}}$ is equal to $p(1-p)/n$.*

Theorem 103 (Theorem Cemtral Limit) *If $np \geq 5$ and $n(1-p) \geq 5$, then the random variable $\widehat{P}$ is approximation a normal distribution with mean $\mu_{\widehat{p}} = p$ and standard deviation (or standard error) $\sigma_{\widehat{p}} = \sqrt{p(1-p)/n}$. Hence*

$$Z = \frac{\widehat{P} - p}{\sqrt{p(1-p)/n}}$$

is approxiamately a standard normal distribution.

Example 104 *In the mid seventies, according to a report by the National Center for Health Statistics,* 19.4 *percent of the adult U.S. male population was obese. What is the probability that in a simple random sample of size 150 from this population fewer than 15 percent will be obese?*

Solution 105 *Here $n = 150$, $p = 0.194$. Since $np \geq 5$ and $n(1-p) \geq 5$, hence*

$$Z = \frac{\widehat{P} - 0.194}{\sqrt{0.194(1-0.194)/150}} = \frac{\widehat{P} - 0.194}{0.032}$$

is approxiamately a standard normal distribution.
$\Pr(\widehat{P} \leq 0.15) = \Pr(\frac{\widehat{P}-0.194}{0.03} \leq \frac{0.15-0.194}{0.03}) \simeq \Pr(Z \leq -1.37) = 0.0853.$

3.10 Sampling Distribution of the Difference between Two Proportions

In some applications there are two actual physical dichotomous populations so that p_1 denotes the population success proportion for population one and p_2 denotes the population success proportion for population two. The sampling distribution of the difference between two sample proportions is constructed in a manner similar to the difference between two means. Independent random samples of size n_1 and n_2 are drawn from two populations of dichotomous variables where the proportions of observations with the character of interest in the two populations are p_1 and p_2, respectively.

Theorem 106 *The mean* $\mu_{\widehat{p}_1-\widehat{p}_2}$ *of the sample distribution of the difference between two sample proportions* $\widehat{P}_1 - \widehat{P}_2$ *is equal to the difference* $p_1 - p_2$ *between the true population proportions, and its variance* $\sigma^2_{\widehat{p}_1-\widehat{p}_2}$ *will be equal to* $p_1(1-p_1)/n_1 + p_1(1-p_2)/n_2$*.*

Theorem 107 *If* $n_1p_1 \geq 5$*,* $n_1(1-p_1) \geq 5$*,* $n_2p_2 \geq 5$*,* $n_2(1-p_2)$*, then the random variable* $\widehat{P}_1 - \widehat{P}_2$ *is approximation a normal distribution with mean* $\mu_{\widehat{p}_1-\widehat{p}_2} = p_1 - p_2$ *and standard deviation (or standard error)* $\sigma_{\widehat{p}} = \sqrt{p_1(1-p_1)/n_1 + p_1(1-p_2)/n_2}$*. Hence*

$$Z = \frac{\left(\widehat{P}_1 - \widehat{P}_2\right) - (p_1 + p_2)}{\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}}$$

is approxiamately a standard normal distribution.

Example 108 *Suppose that there are two large high schools, each with more than* 2000 *students, in a certain town. At School* 1*,* 70% *of students did their homework last night. Only* 50% *of the students at School* 2 *did their homework last night. The counselor at School 1 takes a sample random sample of* 100 *students and records the proportion that did homework. School 2's counselor takes a sample random sample of* 200 *students and records the proportion that did homework. Find the probability of getting a difference in sample proportion* $\widehat{P}_1 - \widehat{P}_2$ *of* 0.10 *or less from the two surveys.*

Solution 109 *Here* $p_1 = 0.7$*,* $p_2 = 0.5$*,* $n_1 = 100$ *and* $n_2 = 200$*. It is clear that* $n_1p_1 \geq 5$*,* $n_1(1-p_1) \geq 5$*,* $n_2p_2 \geq 5$*,* $n_2(1-p_2)$*. Also* $\mu_{\widehat{p}_1-\widehat{p}_2} = p_1 - p_2 = 0.2$ *and* $\sigma_{\widehat{p}} = \sqrt{p_1(1-p_1)/n_1 + p_1(1-p_2)/n_2} = 0.058$*. Hence,*

$$Z = \frac{\widehat{P}_1 - \widehat{P}_2 - p_1 - p_2}{\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}} = \frac{\widehat{P}_1 - \widehat{P}_2 - 0.2}{0.058}$$

is approximately a standard normal.

$\Pr(\widehat{P}_1 - \widehat{P}_2 \leq 0.10) = \Pr(\frac{\widehat{P}_1-\widehat{P}_2-0.2}{0.68} \leq \frac{0.10-0.2}{0.68}) \simeq \Pr(Z \leq -1.72) = 0.0427.$

Chapter 4

One and Two-Sample Estimation Problems

4.1 One- and Two-Sample Estimation Problems

4.1.1 Introduction

In previous chapters, we emphasized sampling properties of the sample mean and variance. The purpose of these presentations is to build a foundation that allows us to draw conclusions about the population parameters from experimental data.

4.1.2 Classical Methods of Estimation

A point estimate of some population parameter θ is a single value $\widehat{\theta}$ of a statistic $\widehat{\Theta}$. For example, the value $\overline{x}$ of the statistic $\overline{X}$, computed from a sample of size n, is a point estimate of the population parameter μ. Similarly, $\widehat{p} = x/n$ is a point estimate of the true proportion p for a binomial experiment.

An estimator is not expected to estimate the population parameter without error. We do not expect $\overline{X}$ to estimate μ exactly, but we certainly hope that it is not far off.

Unbiased Estimator

What are the desirable properties of a "good" decision function that would influence us to choose one estimator rather than another? Let $\widehat{\Theta}$ be an estimator whose value $\widehat{\theta}$ is a point estimate of some unknown population parameter θ. Certainly, we would like the sampling distribution of $\widehat{\Theta}$ to have a mean equal to the parameter estimated. An estimator possessing this property is said to be unbiased.

Definition 110 *A statistic $\widehat{\Theta}$ is said to be an unbiased estimator of the parameter θ if $\mu_{\widehat{\Theta}}$ $= E(\widehat{\Theta}) = \theta$.*

Example 111 *Show that S^2 is an unbiased estimator of the parameter σ^2. Hint: $(X_i - \overline{X}) = (X_i - \mu) - (\overline{X} - \mu)$.*

Variance of a Point Estimator

If $\widehat{\Theta}_1$ and $\widehat{\Theta}_2$ are two unbiased estimators of the same population parameter θ, we want to choose the estimator whose sampling distribution has the smaller variance.

Hence, if $\sigma^2_{\widehat{\theta}_1} < \sigma^2_{\widehat{\theta}_2}$, we say that $\widehat{\Theta}_1$ is a more efficient estimator of θ than $\widehat{\Theta}_1$.

Definition 112 *If we consider all possible unbiased estimators of some parameter θ, the one with the smallest variance is called the most efficient estimator of θ.*

Interval Estimation

Even the most efficient unbiased estimator is unlikely to estimate the population parameter exactly. There is no reason we should expect a **point estimate** from a given sample to be exactly equal to the population parameter it is supposed to estimate. There are many situations in which it is preferable to determine an interval within which we would expect to find the value of the parameter. Such an interval is called an interval estimate. An interval estimate of a population parameter θ is an interval of the form $\widehat{\theta}_L < \theta < \widehat{\theta}_U$, where $\widehat{\theta}_l$ and $\widehat{\theta}_U$ depend on the value of the statistic $\widehat{\Theta}$ for a particular sample and also on the sampling distribution of $\widehat{\Theta}$.

4.2 Single Sample: Estimating the Mean

The sampling distribution of $\overline{X}$ is centered at μ, and in most applications the variance is smaller than that of any other estimators of μ. Thus, the sample mean $\overline{x}$ will be used as a point estimate for the population mean μ.

Let us now consider the interval estimate of μ. If our sample is selected from a normal population or, failing this, if n is sufficiently large, we can establish a confidence interval for μ by considering the sampling distribution of $\overline{X}$.

Definition 113 (Confidence Interval on μ, σ^2 Known) *If $\overline{x}$ is the mean of a random sample of size n from a population with known variance σ^2, a $100(1-\alpha)\%$ confidence interval for μ is given by*

$$\overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}},$$

where $z_{\alpha/2}$ is the z-value leaving an area of $\alpha/2$ to the right.

Example 114 *The average zinc concentration recovered from a sample of measurements taken in 36 different locations in a river is found to be 2.6 grams per milliliter. Find the 95% and 99% confidence intervals for the mean zinc concentration in the river. Assume that the population standard deviation is 0.3 gram per milliliter.*

Solution 115 *The point estimate of μ is $\overline{x} = 2.6$. The z-value leaving an area of 0.025 to the right, and therefore an area of 0.975 to the left, is $z_{0.025} = 1.96$ (Table A.3). Hence, the 95% confidence interval is*

$$2.6 - (1.96)\left(\frac{0.3}{\sqrt{36}}\right) < \mu < 2.6 + (1.96)\left(\frac{0.3}{\sqrt{36}}\right)$$

which reduces to $2.50 < \mu < 2.70$. To find a 99% confidence interval, we find the z-value leaving an area of 0.005 to the right and 0.995 to the left. From Table A.3 again, $z_{0.005} = 2.575$, and the 99% confidence interval is

$$2.6 - (2.575)\left(\frac{0.3}{\sqrt{36}}\right) < \mu < 2.6 + (2.575)\left(\frac{0.3}{\sqrt{36}}\right)$$

or simply

$$2.47 < \mu < 2.73.$$

The error in estimating μ by $\overline{x}$ is the absolute value of the difference between μ and $\overline{x}$, and we can be $100(1-\alpha)\%$ confident that this difference will not exceed $z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$.

Theorem 116 *If $\overline{x}$ is used as an estimate of μ, we can be $100(1-\alpha)\%$ confident that the error will not exceed $z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$.*

Theorem 117 *If $\overline{x}$ is used as an estimate of μ, we can be $100(1-\alpha)\%$ confident that the error will not exceed a specified amount e when the sample size is*

$$n = \left(\frac{z_{\alpha/2}\sigma}{e}\right)^2$$

Example 118 *How large a sample is required if we want to be 95% confident that our estimate of μ in Example 5 is off by less than 0.05?*

Solution 119 *The population standard deviation is $\sigma = 0.3$. Then,*

$$n = \left[\frac{(1.96)(0.3)}{0.05}\right]^2 = 138.3.$$

Therefore, we can be 95% confident that a random sample of size 139 will provide an estimate $\overline{x}$ differing from μ by an amount less than 0.05.

The reader should recall learning in Chapter 3 that if we have a random sample from a normal distribution, then the random variable

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

has a Student t-distribution with $n-1$ degrees of freedom. Here S is the sample standard deviation. In this situation, with σ unknown, T can be used to construct a confidence interval on μ.

Definition 120 (Confidence Interval on μ, σ^2 unknown) *If $\overline{x}$ and s are the mean and standard deviation of a random sample of size n from a normal population with unknown variance σ^2, a $100(1-\alpha)\%$ confidence interval for μ is*

$$\overline{x} - t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \overline{x} + t_{\alpha/2}\frac{s}{\sqrt{n}},$$

where $t_{\alpha/2}$ is the t-value with $\nu = n-1$ degrees of freedom, leaving an area of $\alpha/2$ to the right.

Example 121 *The contents of seven similar containers of sulfuric acid are* $9.8, 10.2, 10.4, 9.8, 10.0, 10.2, 9.6$ *liters. Find a 95% confidence interval for the mean contents of all such containers, assuming an approximately normal distribution.*

Solution 122 *The sample mean and standard deviation for the given data are*

$$\overline{x} = 10.0 \text{ and } s = 0.283.$$

Using Table A.4, we find $t_{0.025} = 2.447$ for $v = 6$ degrees of freedom. Hence, the 95% confidence interval for μ is

$$10.0 - (2.447)\left(\frac{0.283}{\sqrt{7}}\right) < \mu < 10.0 + (2.447)\left(\frac{0.283}{\sqrt{7}}\right)$$

which reduces to $9.74 < \mu < 10.26$.

Concept of a Large-Sample Confidence Interval

Often statisticians recommend that even when normality cannot be assumed, σ is unknown, and $n \geq 30$, s can replace σ and the confidence interval

$$\overline{x} \pm z_{\alpha/2}\frac{s}{\sqrt{n}}$$

may be used. This is often referred to as a large-sample confidence interval.

Example 123 *Scholastic Aptitude Test (SAT) mathematics scores of a random sample of* 500 *high school seniors in the state of Texas are collected, and the sample mean and standard* *deviation are found to be* 501 *and* 112, *respectively. Find a* 99% *confidence interval on the mean SAT mathematics score for seniors in the state of Texas.*

4.3 Standard Error of a Point Estimate

We indicated earlier that a measure of the quality of an unbiased estimator is its variance. The variance of $\overline{X}$ is

$$\sigma^2_{\overline{X}} = \frac{\sigma^2}{n}$$

Thus, the standard deviation of $\overline{X}$, or standard error of $\overline{X}$, is $\sigma/\sqrt{n}$. Simply put, the standard error of an estimator is its standard deviation. For $\overline{X}$, the computed confidence limit

$$\overline{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \text{ is written } \overline{x} \pm z_{\alpha/2} \text{ s.e.}(\overline{x})$$

In the case where σ is unknown and sampling is from a normal distribution, s replaces σ and the estimated standard error $s/\sqrt{n}$ is involved. Thus, the confidence limits on μ are limit

$$\overline{x} \pm t_{\alpha/2}\frac{s}{\sqrt{n}} \text{ is written } \overline{x} \pm t_{\alpha/2} \text{ s.e.}(\overline{x})$$

4.4 Two Samples: Estimating the Difference between Two Means

Theorem 124 ***Confidence Interval for*** $\mu_1 - \mu_2$***,*** σ_1^2 ***and*** σ_2^2 ***known***
If $\overline{x}_1$ *and* $\overline{x}_2$ *are means of independent random samples of sizes* n_1 *and* n_2 *from populations with known variances* σ_1^2 *and* σ_2^2*, respectively, a* $100(1-\alpha)\%$ *confidence interval for* $\mu_1-\mu_2$ *is given by*

$$(\overline{x}_1 - \overline{x}_2) - z_{\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < \mu_1 - \mu_2 < (\overline{x}_1 - \overline{x}_2) + z_{\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}},$$

where $z_{\alpha/2}$ *is the* z*-value leaving an area of* $\alpha/2$ *to the right.*

Example 125 *A study was conducted in which two types of engines,* A *and* B*, were compared. Gas mileage, in miles per gallon, was measured. Fifty experiments were conducted using engine type* A *and* 75 *experiments were done with engine type* B*. The gasoline used and other conditions were held constant. The average gas mileage was* 36 *miles per gallon for engine* A *and* 42 *miles* *per gallon for engine B. Find a 96% confidence interval on $\mu_B-\ \mu_A$, where μ_A and μ_B are population mean gas mileages for engines A and B, respectively. Assume that the population standard deviations are 6 and 8 for engines and B, respectively.*

Solution 126 *The point estimate of $\mu_B-\ \mu_A$ is $\overline{x}_B - \overline{x}_A = 42 - 36 = 6$. Using $\alpha = 0.04$, we find $z_{0.02} = 2.05$ from Table A.3. Hence, with substitution in the formula above, the 96% confidence interval is*

$$6 - 2.05\sqrt{\frac{64}{75} + \frac{36}{50}} < \mu_B - \mu_A < 6 + 2.05\sqrt{\frac{64}{75} + \frac{36}{50}}$$

or simply $3.43 < \mu_B - \mu_A < 8.57$.

Variances Unknown but Equal

Consider the case where σ_1^2 and σ_2^2 are unknown and $\sigma_1^2 = \sigma_1^2$ $(= \sigma^2)$. A point estimate of the unknown common variance σ^2 can be obtained by pooling the sample variances. Denoting the pooled estimator by S_p^2, we have the following.

Definition 127 (of Variance) $S_p^2 = \frac{(n_1-1)S_1^2+(n_1-1)S_2^2}{(n_1+n_2-1)}$

Theorem 128 ***Confidence Interval for $\mu_1 - \mu_2$, $\sigma_1^2 = \sigma_2^2$ but Both Uknown***
If $\overline{x}_1$ and $\overline{x}_2$ are means of independent random samples of sizes n_1 and n_2, respectively, from approximately normal populations with unknown but equal variances, a $100(1-\alpha)\%$ confidence interval for $\mu_1 - \mu_2$ is given by

$$(\overline{x}_1 - \overline{x}_2) - t_{\frac{\alpha}{2}} s_p\sqrt{\tfrac{1}{n_1} + \tfrac{1}{n_2}} < \mu_1 - \mu_2 < (\overline{x}_1 - \overline{x}_2) + t_{\frac{\alpha}{2}} s_p\sqrt{\tfrac{1}{n_1} + \tfrac{1}{n_2}},$$

where where s_p is the pooled estimate of the population standard deviation and $t_{\alpha/2}$ is the t-value with $\nu = n_1 + n_2 - 2$ degrees of freedom, leaving an area of $\alpha/2$ to the right.

Example 129 *Two independent sampling stations, statoin 1 and station 2, were chosen for a study on pollution. For 12 monthly samples collected at station 1, the species diversity index had a mean value $\overline{x}_1 = 3.11$ and a standard deviation $s_1 = 0.771$, while 10 monthly samples collected at the station 2 had a mean index value $\overline{x}_2 = 2.04$ and a standard deviation $s_2 = 0.448$. Find a*

90% *confidence interval for the difference between the population means for the two locations, assuming that the populations are approximately normally distributed with equal variances.*

Solution 130 *Let μ_1 and μ_2 represent the population means, respectively, for the species diversity indices at the downstream and upstream stations. We wish to find a* 90% *confidence interval for $\mu_1 - \mu_2$. Our point estimate of $\mu_1 - \mu_2$ is*

$$\overline{x}_1 - \overline{x}_2 = 3.11 - 2.04 = 1.07.$$

The pooled estimate, s_p^2, of the common variance, σ^2, is

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_1 - 1)s_2^2}{(n_1 + n_2 - 1)} = \frac{(11)(0.7712)+(9)(0.4482)}{12+10-2} = 0.417.$$

Taking the square root, we obtain $s_p = 0.646$. Using $\alpha = 0.1$, we find in Table A.4 that $t_{0.05} = 1.725$ for $\nu = n_1 + n_2 - 2 = 20$ degrees of freedom. Therefore, the 90% *confidence interval for $\mu_1 - \mu_2$ is*

$$\begin{aligned} 1.07 + 1.725(0.646)\sqrt{\tfrac{1}{12} + \tfrac{1}{10}} &< \mu_1 - \mu_2 \\ &< 1.07 + 1.725(0.646)\sqrt{\tfrac{1}{12} + \tfrac{1}{10}} \end{aligned}$$

which simplifies to $0.593 < \mu_1 - \mu_2 < 1.547$.

4.5 Paired Observations

Now we shall consider estimation procedures for the difference of two means when the samples are not independent and the variances of the two populations are not necessarily equal. The situation considered here deals with a very special experimental condition, namely that of paired observations. For example, if we run a test on a new diet using 15 individuals, the weights before and after going on the diet form the information for our two samples. The two populations are "before" and "after," and the experimental unit is the individual. Obviously, the observations in a pair have something in common. To determine if the diet is effective, we consider the differences $d_1, d_2, ..., d_n$ in the paired observations. These differences are the values of a random sample $D_1, D_2, ..., D_n$ from a population of differences that we shall assume to be normally distributed with mean $\mu_D = \mu_1 - \mu_2$ and variance σ_D^2. We estimate σ_D^2 by σ_d^2, the variance of the differences that constitute our sample. The point estimator of μ_D is given by $\overline{D}$.

Theorem 131 ***Confidence Interval for*** $\mu_D = \mu_1 - \mu_2$, ***for Paired Observations***
If $\overline{d}$ *and* s_d *are the mean and standard deviation, respectively, of the normally distributed differences of* n *random pairs of measurements, a* $100(1-\alpha)\%$ *confidence interval for* $\mu_D = \mu_1 - \mu_2$ *is*

$$\overline{d} - t_{\alpha/2}\frac{s_d}{\sqrt{n}} < \mu < \overline{d} + t_{\alpha/2}\frac{s_d}{\sqrt{n}},$$

where $t_{\alpha/2}$ *is the* t*-value with* $\nu = n-1$ *degrees of freedom, leaving an area of* $\alpha/2$ *to the right.*

Example 132 *A study published in Chemosphere reported the levels of the dioxin TCDD of* 10 *Massachusetts Vietnam veterans who were possibly exposed to Agent Orange. The TCDD levels in plasma and in fat tissue are listed in Table 1. Find a* 95% *confidence interval for* $\mu_1 - \mu_2$, *where* μ_1 *and* μ_2 *represent the true mean TCDD levels in plasma and in fat tissue, respectively. Assume the distribution of the differences to be approximately normal.*

Veteran	TCDD levels in Plasma	TCDD levels in Fat Tissue	d_i
1	2.5	4.9	-2.4
2	3.1	5.9	-2.8
3	2.1	4.4	-2.3
4	3.5	6.9	-3.4
5	3.1	7.0	-3.9
6	1.8	4.2	-2.4
7	6.0	10.0	-4.0
8	3.0	5.5	-2.5
9	36.0	41.0	-5.0
10	4.7	4.4	0.3

Solution 133 *The point estimate of* μ_D *is* $\overline{d} = -2.84$. *The standard deviation,* s_d, *of the sample differences is* 1.42. *Using* $\alpha = 0.05$, *we find in Table A.4 that* $t_{0.025} = 2.262$ *for* $\nu =$

$n-1=9$ degrees of freedom. Therefore, the 95% confidence interval is

$$-2.84-(2.262)\left(\frac{1.42}{\sqrt{10}}\right)<\mu_D<-2.84+(2.262)\left(\frac{1.42}{\sqrt{10}}\right)$$

or simply $-3.85<\mu_D<-1.82$.

4.6 Single Sample: Estimating a Proportion

A point estimator of the proportion p in a binomial experiment is given by the statistic $\widehat{P}=X/n$, where X represents the number of successes in n trials. Therefore,

Definition 134 *the sample proportion $\widehat{p}=x/n$ will be used as the point estimate of the*

parameter p.

Theorem 135 (Large-Sample Confidence Intervals for p) *If $\widehat{p}$ is the proportion of successes in a random sample of size n and $\widehat{q}=1-\widehat{p}$, an approximate $100(1-\alpha)\%$ confidence interval, for the binomial parameter p is given by*

$$\widehat{p}-z_{\alpha/2}\sqrt{\frac{\widehat{pq}}{n}}<p<\widehat{p}+z_{\alpha/2}\sqrt{\frac{\widehat{pq}}{n}}$$

where $z_{\alpha/2}$ is the z-value leaving an area of $\alpha/2$ to the right.

Example 136 *In a random sample of $n=500$ families owning television sets in the city of Hamilton, Canada, it is found that $x=340$ subscribe to HBO. Find a 95% confidence interval for the actual proportion of families with television sets in this city that subscribe to HBO.*

Solution 137 *The point estimate of p is $\widehat{p}=340/500=0.68$. Using Table A.3, we find that $z_{0.025}=1.96$. Therefore, the 95% confidence interval for p is*

$$0.68-1.96\sqrt{\frac{(0.68)(0.32)}{500}}<p<0.68+1.96\sqrt{\frac{(0.68)(0.32)}{500}}$$

which simplifies to $0.6391<p<0.7209$.

Theorem 138 *If* $\widehat{p}$ *is used as an estimate of* p*, we can be* $100(1-\alpha)\%$ *confident that the error will not exceed* $z_{\alpha/2}\sqrt{\frac{\widehat{pq}}{n}}$.

Choice of Sample Size

Let us now determine how large a sample is necessary to ensure that the error in estimating p will be less than a specified amount e. By Theorem 23, we must choose n such that $z_{\alpha/2}\sqrt{\frac{\widehat{pq}}{n}} = e$.

Theorem 139 *If* $\widehat{p}$ *is used as an estimate of* p*, we can be* $100(1-\alpha)\%$ *confident that the error will be less than a specified amount e when the sample size is approximately*

$$n = \frac{z_{\alpha/2}^2 \widehat{p}\widehat{q}}{e^2}$$

Example 140 *How large a sample is required if we want to be 95% confident that our estimate of p in Example 21 is within* 0.02 *of the true value?*

Solution 141 *Let us treat the* 500 *families as a preliminary sample, providing an estimate* $\widehat{p}$ $= 0.68$*. Then,*

$$n = \frac{(1.96)^2(0.68)(0.32)}{0.02^2} = 2089.8 \approx 2090$$

Occasionally, it will be impractical to obtain an estimate of p to be used for determining the sample size for a specified degree of confidence. If this happens, we use the following theorem.

Theorem 142 *If* $\widehat{p}$ *is used as an estimate of* p*, we can be* $100(1-\alpha)\%$ *confident that the error will not exceed than a specified amount e when the sample size is approximately*

$$n = \frac{z_{\alpha/2}^2}{4e^2}$$

Example 143 *How large a sample is required if we want to be at least* 95% *confident that our estimate of p in Example 21 is within* 0.02 *of the true value?*

Solution 144 *Let assume that no preliminary sample has been taken to provide an estimate of p. Consequently, we can be at least* 95% *confident that our sample proportion will not differ from the true proportion by more than* 0.02 *if we choose a sample of size*

$$n = \frac{(1.96)^2}{4(0.02)^2} = 2401$$

Comparing the results of Examples 28 and 29, we see that information concerning p, provided by a preliminary sample or from experience, enables us to choose a smaller sample while maintaining our required degree of accuracy.

4.7 Two Samples: Estimating the Difference between Two Proportions

Consider the problem where we wish to estimate the difference between two binomial parameters p_1 and p_2. For example, p_1 might be the proportion of smokers with lung cancer and p_2 the proportion of nonsmokers with lung cancer, and the problem is to estimate the difference between these two proportions.

Theorem 145 ***Large-Sample Confidence Interval for*** $p_1 - p_2$
If $\widehat{p}_1$ *and* $\widehat{p}_2$ *are the proportions of successes in random samples of sizes* n_1 *and* n_2*, respectively,* $\widehat{q}_1 = 1 - \widehat{p}_1$*, and* $\widehat{q}_2 = 1 - \widehat{p}_2$*, an approximate* $100(1-\alpha)\%$ *confidence interval for the difference of two binomial parameters,* $p_1 - p_2$*, is given by*

$$(\widehat{p}_1 - \widehat{p}_2)\text{-}z_{\frac{\alpha}{2}}\sqrt{\frac{\widehat{p}_1\widehat{q}_1}{n_1}+\frac{\widehat{p}_2\widehat{q}_2}{n_1}}<p_1 - p_2 < (\widehat{p}_1 - \widehat{p}_2)+z_{\frac{\alpha}{2}}\sqrt{\frac{\widehat{p}_1\widehat{q}_1}{n_1}+\frac{\widehat{p}_2\widehat{q}_2}{n_1}}$$

Example 146 *A certain change in a process for manufacturing component parts is being considered. Samples are taken under both the existing and the new process so as to determine if the new process results in an improvement. If* 75 *of* 1500 *items from the existing process are found to be defective and* 80 *of* 2000 *items from the new process are found to be defective, find a* 90% *confidence interval for the true difference in the proportion of defectives between the existing and the new process.*

Solution 147 *Let* p_1 *and* p_2 *be the true proportions of defectives for the existing and new processes, respectively. Hence,* $\widehat{p}_1 = 75/1500 = 0.05$ *and* $\widehat{p}_2 = 80/2000 = 0.04$*, and the point estimate of* $p_1 -$ p_2 *is*

$$\widehat{p}_1 - \widehat{p}_2 = 0.05 - 0.04 = 0.01$$

Using Table A.3, we find $z_{0.05} = 1.645$. *Therefore, substituting into the formula, with*

$$1.645\sqrt{\frac{(0.05)(0.95)}{1500} + \frac{(0.04)(0.96)}{2000}} = 0.0117,$$

we find the 90% *confidence interval to be* $-0.0017 < p_1 - p_2 < 0.0217$.

4.8 Single Sample: Estimating the Variance

If a sample of size n is drawn from a normal population with variance σ^2 and the sample variance s^2 is computed, we obtain a value of the statistic S^2. This computed sample variance is used as a point estimate of σ^2. Hence, the statistic S^2 is called an estimator of σ^2. An interval estimate of σ^2 can be established by using the statistic

$$X = \frac{(n-1)S^2}{\sigma^2}$$

the statistic X has a chi-squared distribution with $n-1$ degrees of freedom when samples are chosen from a normal population.

Theorem 148 (Confidence Interval for σ^2) *If* s^2 *is the variance of a random sample of size n from a normal population, a* $100(1-\alpha)\%$ *confidence interval for* σ^2 *is*

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}$$

where $\chi^2_{\alpha/2}$ *and* $\chi^2_{1-\alpha/2}$ *are* χ^2*-values with* $\nu = n-1$ *degrees of freedom, leaving areas of* $\alpha/2$ *and* $1-$ $\alpha/2$, *respectively, to the right.*

An approximate $100(1-\alpha)\%$ confidence interval for σ is obtained by taking the square root of each endpoint of the interval for σ^2.

Example 149 *The following are the weights, in decagrams, of 10 packages of grass seed distributed by a certain company:* $46.4, 46.1, 45.8, 47.0, 46.1, 45.9, 45.8, 46.9, 45.2, 46.0$. *Find a* 95% *confidence interval for the variance of the weights of all such packages of grass seed distributed by this company, assuming a normal population.*

Solution 150 *First we find* s^2 = 0.286*. To obtain a* 95% *confidence interval, we choose* α = 0.05*. Then, using Table A.5 with* $\nu = 9$ *degrees of freedom, we find* $\chi^2_{.025} = 19.023$ *and* $\chi^2_{.975}$ = 2.700*. Therefore, the* 95% *confidence interval for* σ^2 *is*

$$\frac{(9)(0.286)}{19.023} < \sigma^2 < \frac{(9)(0.286)}{2.700}$$

or simply $0.135 < \sigma^2 < 0.953$.

4.9 Two Samples: Estimating the Ratio of Two Variances

A point estimate of the ratio of two population variances σ_1^2/σ_2^2 is given by the ratios s_1^2/s_2^2 of the sample variances. Hence, the statistic S_1^2/S_2^2 is called an estimator of σ_1^2/σ_2^2 . If σ_1^2 andσ_2^2 are the variances of normal populations, we can establish an interval estimate of σ_1^2/σ_2^2 by using the statistic

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$$

According to Theorem 25 of chapter 3, the random variable F has an F-distribution with ν_1 = $n_1 - 1$ and $\nu_2 = n_2 - 1$ degrees of freedom.

Theorem 151 (Confidence Interval for σ_1^2/σ_2^2) *If* s_1^2 *and* s_2^2 *are the variances of independent samples of sizes* n_1 *and* n_2*, respectively,from normal populations, then a* $100(1-\alpha)\%$ *confidence interval for* σ_1^2/σ_2^2 *is*

$$\frac{s_1^2}{s_2^2}\frac{1}{f_{\alpha/2}(\nu_1,\nu_2)} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{s_1^2}{s_2^2} f_{\alpha/2}(\nu_2,\nu_1)$$

where $f_{\alpha/2}(\nu_1,\nu_2)$ *is an* f*-value with* $\nu_1 = n_1 - 1$ *and* $\nu_2 = n_2 - 1$ *degrees of freedom, leaving an area of* $\alpha/2$ *to the right, and* $f_{\alpha/2}(\nu_2,\nu_1)$ *is a similar* f*-value with* $\nu_2 = n_2 - 1$ *and* $\nu_1 = n_1 - 1$ *degrees of freedom.*

an approximate $100(1-\alpha)\%$ confidence interval for σ_1/σ_2 is obtained by taking the square root of each endpoint of the interval for σ_1^2/σ_2^2.

Example 152 *A study was conducted to estimate the difference in the amounts of the chemical orthophosphorus measured at two different stations. Fifteen samples were collected from station*

1, and 12 *samples were obtained from station 2. The* 15 *samples from station 1 had an average orthophosphorus content of* 3.84 *milligrams per liter and a standard deviation of* 3.07 *milligrams per liter, while the* 12 *samples from station 2 had an average content of* 1.49 *milligrams per liter and a standard deviation of* 0.80 *milligram per liter. Determine a 98% confidence interval for* σ_1^2/σ_2^2 *and for* σ_1/σ_2*, where* σ_1^2 *and* σ_2^2 *are the variances of the populations of orthophosphorus contents at station* 1 *and station* 2*, respectively.*

Solution 153 *We have* $n = 15$*,* $n_2 = 12$*,* $s_1 = 3.07$*, and* $s_2 = 0.80$*. For a* 98% *confidence interval,* $\alpha = 0.02$*. Interpolating in Table A.6, we find* $f_{0.01}(14, 11) \approx 4.30$ *and* $f_{0.01}(11, 14) \approx 3.87$*. Therefore, the* 98% *confidence interval for* σ_1^2/σ_2^2 *is*

$$\left(\frac{3.07^2}{0.80^2}\right)\left(\frac{1}{4.30}\right) < \frac{\sigma_1^2}{\sigma_2^2} < \left(\frac{3.07^2}{0.80^2}\right)(3.87),$$

which simplifies to $3.425 < \frac{\sigma_1^2}{\sigma_2^2} < 56.991$*. Taking square roots of the confidence limits, we find that a* 98% *confidence interval for* σ_1/σ_2 *is*

$$1.851 < \frac{\sigma_1}{\sigma_2} < 7.549.$$

Since this interval does not allow for the possibility of σ_1/σ_2 *being equal to* 1*, we were correct in assuming that* $\sigma_1 \neq \sigma_2$ *(and* $\sigma_1^2 \neq \sigma_2^2$*).*

Chapter 5

One and Two-Sample Tests of Hypotheses

Let consider a medical researcher who should decide on the basis of experimental evidence whether coffee drinking increases the risk of cancer in humans or a sociologist who might wish to collect appropriate data to enable him or her to decide whether a person's blood type and eye color are independent variables. In each of these cases, the scientist postulates or conjectures something about a system. In addition, each must make use of experimental data and make a decision based on the data. In each case, the conjecture can be put in the form of a statistical hypothesis. Procedures that lead to the acceptance or rejection of statistical hypotheses such as these comprise a major area of statistical inference. First, let us define precisely what we mean by a statistical hypothesis.

Definition 154 *A **statistical hypothesis** is an assertion or conjecture concerning one or more populations.*

The truth or falsity of a statistical hypothesis is never known with absolute certainty unless we examine the entire population. This, of course, would be impractical in most situations. Instead, we take a random sample from the population of interest and use the data contained in this sample to provide evidence that either supports or does not support the hypothesis. Evidence from the sample that is inconsistent with the stated hypothesis leads to a rejection of the hypothesis.

The Role of Probability in Hypothesis Testing

It should be made clear to the reader that the decision procedure must include the probability of a wrong conclusion. For example, suppose that the hypothesis postulated by the engineer is that the fraction defective p in a certain process is 0.10. The experiment is to observe a random sample of the product in question. Suppose that 100 items are tested and 20 items are found defective. If, indeed, p = 0.10, the probability of obtaining 20 or more defectives is approximately 0.002. With the resulting small risk of a wrong conclusion, it would seem safe to reject the hypothesis that p = 0.10. As a result, the reader must be accustomed to understanding that rejection of a hypothesis implies that the sample evidence refutes it. Put another way, rejection means that there is a small probability of obtaining the sample information observed when, in fact, the hypothesis is true.

The formal statement of a hypothesis is often influenced by the structure of the probability of a wrong conclusion. If the scientist is interested in strongly supporting a contention, he or she hopes to arrive at the contention in the form of rejection of a hypothesis. If the medical researcher wishes to show strong evidence in favor of the contention that coffee drinking increases the risk of cancer, the hypothesis tested should be of the form “there is no increase in cancer risk produced by drinking coffee.” As a result, the contention is reached via a rejection. Similarly, to support the claim that one kind of gauge is more accurate than another, the engineer tests the hypothesis that there is no difference in the accuracy of the two kinds of gauges.

The Null and Alternative Hypotheses

The alternative hypothesis H_1 usually represents the question to be answered or the theory to be tested, and thus its specification is crucial. The null hypothesis H0 nullifies or opposes H_1 and is often the logical complement to H_1. Our conclusions will be:

fail to reject H_0 because of insufficient evidence in the data.

reject H_0 in favor of H_1 because of sufficiente vidence in the data or

fail to reject H_0 because of insufficient evidence in the data.

The Probability of a Type I Error

The decision procedure could lead to either of two wrong conclusions.

Definition 155 *Rejection of the null hypothesis when it is true is called a* ***type I error****.*

The probability of committing a type I error, also called the level of significance, is denoted by the Greek letter α.

Definition 156 *Nonrejection of the null hypothesis when it is false is called a* ***type II error****.*

The probability of committing a type II error, denoted by β, is impossible to compute unless we have a specific alternative hypothesis.

5.1 Single Sample: Tests Concerning a Single Mean

In this section, we formally consider tests of hypotheses on a single population mean.

Tests on a Single Mean (Variance Known)

Let $X_1, X_2, ..., X_n$ representing a random sample from a distribution with mean μ and variance $\sigma^2 > 0$. Consider first the hypothesis

$$\begin{aligned} H_0 &: \mu = \mu_0, \\ H_1 &: \mu \neq \mu_0. \end{aligned}$$

The appropriate test statistic should be based on the random variable $\overline{X}$. Recall that the random variable $\overline{X}$ has approximately a normal distribution with mean μ and variance σ^2/n for reasonably large sample sizes. So, $\mu_{\overline{X}} = \mu$ and $\sigma^2_{\overline{X}} = \sigma^2/n$.We can then determine a critical region based on the computed sample average, $\overline{x}$. We know that under H_0, that is, if $\mu = \mu_0$, $(\overline{X} - \mu_0)/(\sigma/\sqrt{n})$ follows an $N(0, 1)$ distribution, and hence the expressions

$$\Pr\left(-z_\alpha/2 \leq \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \leq z_\alpha/2\right) = 1 - \alpha$$

can be used to write an appropriate nonrejection region.

Theorem 157 (Test Procedure for a Single Mean (Variance Known)) *If* $-z_\alpha/2 < z < z_\alpha/2$, *do not reject* H_0. *Rejection of* H_0, *of course, implies acceptance of the alternative hypothesis* $\mu = \mu_0$. *With this definition of the critical region, it should be clear that there will be probability* α *of rejecting* H_0 *(falling into the critical region) when, indeed,* $\mu = \mu_0$.

Tests of one-sided hypotheses on the mean

For example, suppose that we seek to test

$$\begin{aligned} H_0 &: \mu = \mu_0, \\ H_1 &: \mu > \mu_0. \end{aligned}$$

The signal that favors H_1 comes from large values of z. Thus, rejection of H_0 results when the computed $z > z_\alpha$.

Example 158 *A random sample of* 100 *recorded deaths in the United States during the past year showed an average life span of* 71.8 *years. Assuming a population standard deviation of* 8.9 *years, does this seem to indicate that the mean life span today is greater than* 70 *years? Use a* 0.05 *level of significance.*

Solution 159 *1.* $H0 : \mu = 70$ *years.*
2. $H1 : \mu > 70$ *years.*
3. $\alpha = 0.05$.
4. Test statistic: $z = (\overline{x} - \mu_0)/(\sigma/\sqrt{n}) = (71.8 - 70)/(8.9/\sqrt{100}) = 2.02$.
5. Critical region: $z > z_\alpha$,*where* $z_\alpha = z_{0.05} = 1.645$
6. Decision: since $z = 2.02 > 1.645$, *reject* H_0 *and conclude that the mean life span today is greater than* 70 *years.*
The P-value corresponding to $z = 2.02$ *is given by the area on the right under the density of standard normal distribution. Using Table A.3, we have P-value* $= P(Z > 2.02) = 0.0217$. *As a result, the evidence in favor of* H_1 *is even stronger than that suggested by a* 0.05 *level of significance.*

Example 160 *A manufacturer of sports equipment has developed a new synthetic fishing line that the company claims has a mean breaking strength of* 8 *kilograms with a standard deviation* *of 0.5 kilogram. Test the hypothesis that* $\mu = 8$ *kilograms against the alternative that* $\mu \neq 8$ *kilograms if a random sample of* 50 *lines is tested and found to have a mean breaking strength of* 7.8 *kilograms. Use a* 0.01 *level of significance.*

Solution 161 *1.* $H_0 : \mu = 8$ *kilograms.*
2. $H_1 : \mu \neq 8$ *kilograms.*
3. $\alpha = 0.01$.
4. Critical region: $z > z_{\alpha/2}$ *and* $z < -z_{\alpha/2}$*, where* $z = (\overline{x} - \mu_0)/(\sigma/\sqrt{n})$ *and* $z_{\alpha/2} = 2.575$.
5. Computations: $\overline{x} = 7.8$*,* $\sigma = 0.5$*,* $n = 50$*, hence* $z = (\overline{x}-\mu_0)/(\sigma/\sqrt{n}) = (7.8-8)/(0.5/\sqrt{50}) = -2.83$
6. Decision: since $z = -2.83 < -2.575$*, hence reject* H_0 *and conclude that the average breaking strength is not equal to* 8 *but is, in fact, less than* 8 *kilograms.*
Since the test in this example is two tailed, the desired P-value is twice the area of the left of $z = -2.83$*. Therefore, using standard normal table, we have*

$$P\text{-}value = P(Z > |2.83|) = 2P(Z > 2.83) = 0.0046 < 0.01$$

which allows us to reject the null hypothesis that $\mu = 8$ *kilograms at a level of significance smaller than* 0.01.

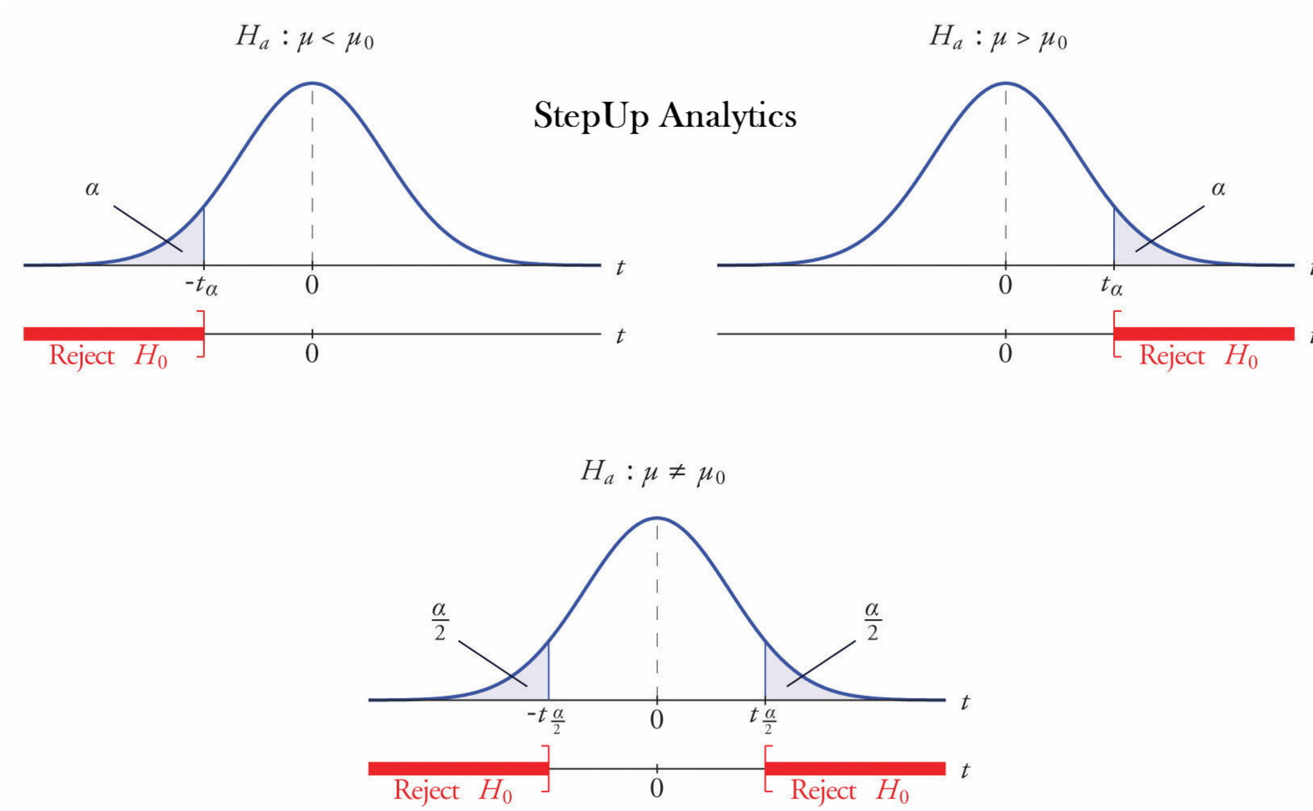
Tests on a Single Sample (Variance Unknown)

Result 162 (Single Mean (Variance Unknown)) *For the two-sided hypothesis*

$$\begin{aligned} H_0 &: \mu = \mu_0, \\ H_1 &: \mu \neq \mu_0 \end{aligned}$$

we reject H_0 *at significance level* α *when the computed t-statistic*

$$t = (\overline{x} - \mu_0)/(s/\sqrt{n})$$



5.jpg

Figure 5-1: Figure 5.1

exceeds $t_{\alpha/2,n-1}$ *or is less than* $-t_{\alpha/2,n-1}$.

For $H_1 : \mu > \mu_0$, rejection results when $t > t_{\alpha,n-1}$. For $H_1 : \mu < \mu_0$, the critical region is given by $t < -t_{\alpha,n-1}$.

Example 163 *The Edison Electric Institute has published figures on the number of kilowatt hours used annually by various home appliances. It is claimed that a vacuum cleaner uses an average of* 46 *kilowatt hours per year. If a random sample of* 12 *homes indicates that vacuum cleaners use an average of* 42 *kilowatt hours per year with a standard deviation of* 11.9 *kilowatt hours, does this suggest at the* 0.05 *level of significance that vacuum cleaners use, on average, less than* 46 *kilowatt hours annually? Assume the population of kilowatt hours to be normal.*

Solution 164 *1.* $H_0 : \mu = 46$ *kilowatt hours.*
2. $H_1 : \mu < 46$ *kilowatt hours.*
3. $\alpha = 0.05$.
4. Critical region: $t < -t_{\alpha,n-1}$, *where* $t = (\overline{x} - \mu_0)/(s/\sqrt{n})$ *with* 11 *degrees of freedom and* $t_{\alpha,n-1} = 1.796$.
5. Computations: $\overline{x} = 42$ *kilowatt hours,* $s = 11.9$ *kilowatt hours, and* $n = 12$.
Hence, $t = (42 - 46)/(11.9/\sqrt{12}) = -1.16$.
6. Since $t > -1.796$, *we do not reject* H_0 *and conclude that the average number of kilowatt hours used annually by home vacuum cleaners is not significantly less than* 46.
Also P-value $= \Pr(T < -1.16) = \Pr(T > 1.16) \approx 0.135$.

5.2 Two Samples: Tests on Two Means

The two-sided hypothesis on two means can be written generally as $H_0 : \mu_1 = \mu_2$. For σ_1 and σ_2 known, the test statistic is given by

$$z = \frac{\overline{x}_1 - \overline{x}_2}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$$

That is, reject H_0 in favor of $H_1 : \mu_1 \neq \mu_2$ if $z > z_{\alpha/2}$, and $z < -z_{\alpha/2}$.

One-tailed critical regions are used in the case of the one-sided alternatives. The reader should, as before, study the test statistic and be satisfied that for, say, $H_1 : \mu_1 > \mu_2$, the signal favoring H_1 comes from large values of z. Thus, the upper-tailed critical region applies.

Unknown But Equal Variances

If we assume that both distributions are normal and that $\sigma_1 = \sigma_2 = \sigma$, the two-sample t-test may be used. The test statistic is given by the following test procedure.

Result 165 (Two-Sample Pooled t-Test) *For the two-sided hypothesis*

$$\begin{aligned} H_0 &: \mu_1 = \mu_2 \\ H_1 &: \mu_1 \neq \mu_2, \end{aligned}$$

we reject H_0 at significance level α when the computed t-statistic

$$t = \frac{\overline{x}_1 - \overline{x}_2}{s_p\sqrt{1/n_1 + 1/n_2}}$$

where

$$s_p^2 = \frac{s_1^2(n_1 - 1) + s_2^2(n_2 - 1)}{n_1 + n_2 - 2}$$

exceeds $t_{\alpha/2,n_1+n_2-2}$ or is less than $-t_{\alpha/2,n_1+n_2-2}$.

For $H_1 : \mu_1 > \mu_2$, reject $H_0 : \mu_1 = \mu_2$ when $t > t_{\alpha,n_1+n_2-2}$. For $H_1 : \mu_1 < \mu_2$, reject $H_0 : \mu_1 = \mu_2$ when $t < -t_{\alpha,n_1+n_2-2}$.

Example 166 *An experiment was performed to compare the abrasive wear of two materials. Twelve pieces of material* 1 *were tested and ten pieces of material* 2 *were similarly tested. The samples of material* 1 *gave an average (coded) wear of* 85 *units with a sample standard deviation of* 4*, while the samples of material* 2 *gave an average of* 83 *with a sample standard deviation of* 5*. Can we conclude at the* 0.05 *level of significance that the abrasive wear of material* 1 *exceeds that of material* 2*? Assume the populations to be approximately normal with equal variances.*

Solution 167 *Let μ_1 and μ_2 represent the population means of the abrasive wear for material 1 and material 2, respectively.*

1. $H_0 : \mu_1 = \mu_2$.

2. $H_1 : \mu_1 > \mu_2$.

3. $\alpha = 0.05$.

4. Critical region: $t > t_{\alpha,n_1+n_2-2}$, *where* $t = \frac{\overline{x}_1 - \overline{x}_2}{s_p\sqrt{1/n_1+1/n_2}}$ *with* $\nu = 20$ *degrees of freedom and* $t_{\alpha,n_1+n_2-2} = t_{0.05,20} = 1.725$.

5. Computations:

$$\overline{x}_1 = 58 \quad s_1 = 4 \quad n_1 = 12$$

$$\overline{x}_2 = 81 \quad s_2 = 5 \quad n_2 = 10$$

Hence

$$\begin{aligned} s_p &= \sqrt{\frac{(11)(16) + (9)(25)}{12 + 10 - 2}} = 4.478 \\ t &= \frac{(85 - 83)}{4.478\sqrt{1/12 + 1/10}} = 1.04 < 1.725 \\ \textit{P-value} &= \Pr(T > 1.04) \approx 0.16.(\textit{See Table A.4.}) \end{aligned}$$

6. Decision: Do not reject H_0*. We are unable to conclude that the abrasive wear of material 1 exceeds that of material 2.*

Paired Observations

Testing of two means can be accomplished when data are in the form of paired observations, as discussed in Chapter 4. The statistical test for two means μ_1 and μ_2 in the situation with paired observations is based on the random variable

$$T = \frac{\overline{D} - \mu_D}{S_d/\sqrt{n}}$$

where $\overline{D}$ and S_d are random variables representing the sample mean and standard deviation of the differences of the observations in the experimental units. As in the case of the pooled t-test, the assumption is that the observations from each population are normal. This two-sample problem is essentially reduced to a one-sample problem by using the computed differences

$d_1, d_2, \ldots, d_n$. Critical regions are constructed using the t-distribution with $n-1$ degrees of freedom.

Example 168 ***Blood Sample Data****: A study was conducted to examine the influence of the drug succinylcholine on the circulation levels of androgens in the blood. Blood samples were taken from wild, free-ranging deer immediately after they had received an intramuscular injection of succinylcholine administered using darts and a capture gun. A second blood sample was obtained from each deer* 30 *minutes after the first sample, after which the deer was released. The levels of androgens at time of capture and* 30 *minutes later, measured in nanograms per milliliter (ng/mL), for* 15 *deer are given in Table 6.2. Assuming that the populations of androgen levels at time of injection and* 30 *minutes later are normally distributed, test at the* 0.05 *level of significance whether the androgen concentrations are altered after* 30 *minutes.*

	Androgen (ng/mL)		
Deer	At Time of Injection	30 Minutes after Injection	d_i
1	2.76	7.02	4.26
2	5.18	3.10	−2.08
3	2.68	5.44	2.76
4	3.05	3.99	0.94
5	4.10	5.21	1.11
6	7.05	10.26	3.21
7	6.60	13.91	7.31
8	4.79	18.53	13.74
9	7.39	7.91	0.52
10	7.30	4.85	−2.45
11	11.78	11.10	−0.68
12	3.90	3.74	−0.16
13	26.00	94.03	68.03
14	67.48	94.03	26.55
15	17.04	41.70	24.66

Solution 169 *Let μ_1 and μ_2 be the average androgen concentration at the time of injection and* 30 *minutes later, respectively. We proceed as follows:*

1. $H_0 : \mu_1 = \mu_2$ *or* $\mu_D = \mu_1 - \mu_2 = 0.$

2. $H_1 : \mu_1 \neq \mu_2$ *or* $\mu_D = \mu_1 - \mu_2 \neq 0.$

3. $\alpha = 0.05.$

4. Critical region: $t < -t_{\alpha/2,n-1}$ *and* $t > t_{\alpha/2,n-1}$*, where* $t = \frac{\overline{d}}{s_D/\sqrt{n}}$ *with* $\nu = 14$ *degrees of freedom and* $t_{\alpha/2,n-1} = 2.145.$

5. Computations: The sample mean and standard deviation for the d_i *are*

$$\overline{d} = 9.848 \quad and \quad s_d = 18.474$$

Therefore

$$t = \frac{9.848}{18.474/\sqrt{15}} = 2.06$$

6. Hence $-2.145 < t = 2.06 < 2.145$*. Though the t-statistic is not significant at the* 0.05 *level, from Table A.4,*

$$P = P(|T| > 2.06) \approx 0.06$$

As a result, there is no evidence that there is a difference in mean circulating levels of androgen.

5.3 One Sample: Test on a Single Proportion

Tests of hypotheses concerning proportions are required in many areas. We now consider the problem of testing the hypothesis that the proportion of successes in a binomial experiment equals some specified value. That is, we are testing the null hypothesis H_0 that $p = p_0$, where p is the parameter of the binomial distribution. The alternative hypothesis may be one of the usual one-sided or two-sided alternatives:

$$p < p_0 \quad p > p_0 \quad \text{or} \quad p \neq p_0$$

We know that if $np_0 \geq 5$ and $n(1 - p_0) \geq 5$, then the random variable $\widehat{P}$ is approximately a normal distribution with mean p_0 and standard deviation $\sigma_{\widehat{P}} = \sqrt{p_0(1 - p_0)/n}$. The **z-value**

for testing $\mathbf{p} = \mathbf{p}_0$ is given by

$$z = \frac{\widehat{p} - p_0}{\sqrt{p_0(1 - p_0)/n}}$$

Hence, for a two-tailed test at the α-level of significance, the critical region is $z < -z_{\alpha/2}$ or $z > z_{\alpha/2}$. For the one-sided alternative $p < p_0$, the critical region is $z < -z_\alpha$, and for the alternative $p > p_0$, the critical region is $z > z_\alpha$.

Example 170 *A commonly prescribed drug for relieving nervous tension is believed to be only* 60% *effective. Experimental results with a new drug administered to a random sample of* 100 *adults who were suffering from nervous tension show that* 70 *received relief. Is this sufficient evidence to conclude that the new drug is superior to the one commonly prescribed? Use a* 0.05 *level of significance.*

Solution 171 *1.* $H_0 : p = 0.6$.
2. $H_1 : p > 0.6$.
3. $\alpha = 0.05$.
4. Critical region: $Z > z_\alpha$*, where* $z_\alpha = 1.645$*. Then, the critical region:* $z > 1.645$.
5. Computations: $x = 70$,$n = 100$, $\widehat{p} = 70/100 = 0.7$, *and*

$$\begin{aligned} z &= \frac{0.7 - 0.6}{\sqrt{\frac{(0.6)(0.4)}{100}}} = 2.04 \\ z &= 2.04 > 1.645 \\ P\text{-}value &= \Pr(Z > 2.04) < 0.0207. \end{aligned}$$

6. Decision: Reject H_0 *and conclude that the new drug is superior.*

5.4 Two Samples: Tests on Two Proportions

Situations often arise where we wish to test the hypothesis that two proportions are equal. That is, we are testing $p_1 = p_2$ against one of the alternatives $p_1 < p_2$, $p_1 > p_2$, or $p_1 = p_2$. The statistic on which we base our decision is the random variable $\widehat{P}_1 - \widehat{P}_2$. When $H_0 : p_1 = p_2$ $(= p)$ is true, we know that

$$Z = \frac{\widehat{P}_1 - \widehat{P}_2}{\sqrt{pq(1/n_1 + 1/n_2)}}$$

To compute a value of Z, however, we must estimate the parameters p and q that appear in the radical. Under H_0, both $\widehat{P}_1$ and $\widehat{P}_2$ are estimators of p. we use the pooled estimate of the proportion p, which is

$$\widehat{p} = \frac{x_1 + x_2}{n_1 + n_2}$$

where x_1 and x_2 are the numbers of successes in each of the two samples. Substituting $\widehat{p}$ for p and $\widehat{q} = 1 - \widehat{p}$ for q, the z-value for testing $p_1 = p_2$ is determined from the formula

$$z = \frac{\widehat{p}_1 - \widehat{p}_2}{\sqrt{\widehat{p}\widehat{q}(1/n_1 + 1/n_2)}}$$

The critical regions for the appropriate alternative hypotheses are set up as before, using critical points of the standard normal curve.

Example 172 *A vote is to be taken among the residents of a town and the surrounding county to determine whether a proposed chemical plant should be constructed. To determine if there is a significant difference in the proportions of town voters and county voters favoring the proposal, a poll is taken. If* 120 *of* 200 *town voters favor the proposal and* 240 *of* 500 *county residents favor it, would you agree that the proportion of town voters favoring the proposal is higher than the proportion of county voters? Use an* $\alpha = 0.05$ *level of significance.*

Solution 173 *Let* p_1 *and* p_2 *be the true proportions of voters in the town and county, respectively, favoring the proposal.* $\widehat{p}_1 = x_1/n_1 = 120/200 = 0.6$, $\widehat{p}_2 = x_2/n_2 = 240/500 = 0.48$, *and the pooled estimate* $\widehat{p} = (x_1 + x_2)/(n_1 + n_2) = (120 + 240)/(200 + 500) = 0.51$.
1. $H_0 : p_1 = p_2$.
2. $H_1 : p_1 > p_2$.
3. $\alpha = 0.05$.
4. The test statistic

$$z = \frac{\widehat{p}_1 - \widehat{p}_2}{\sqrt{\widehat{p}\widehat{q}(1/n_1 + 1/n_2)}} = \frac{0.60 - 0.48}{(0.51)(0.49)(1/200 + 1/500)} = 2.9$$

5. Critical region: $z > 1.645$. *P-value* $= P(Z > 2.9) = 0.0019$.
6. Decision: Reject H_0 *and agree that the proportion of town voters favouring the proposal is*

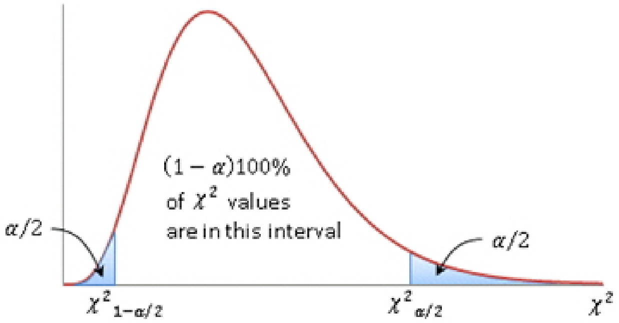


Figure 5-2: Figure 5.2

higher than the proportion of county voters.

5.5 One- and Two-Sample Tests Concerning Variances

In this section, we are concerned with testing hypotheses concerning population variances or standard deviations. Let us first consider the problem of testing the null hypothesis H_0 that the population variance σ^2 equals a specified value σ_0^2 against one of the usual alternatives σ^2 $< \sigma_0^2$, $\sigma^2 > \sigma_0^2$, or $\sigma^2 \neq \sigma_0^2$. If we assume that the distribution of the population being sampled is normal, the chi-squared value for testing $\sigma^2 = \sigma_0^2$ is given by

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

where n is the sample size, s^2 is the sample variance, and σ_0^2 is the value of σ^2 given by the null hypothesis. If H_0 is true, χ^2 is a value of the chi-squared distribution with $\nu = n - 1$ degrees of freedom. Hence, for a two-tailed test at the α-level of significance, the critical region is $\chi^2 < \chi^2_{1-\alpha/2}$ or $\chi^2 > \chi^2_{\alpha/2}$ (see figure 5.2). For the one-sided alternative $\sigma^2 < \sigma_0^2$, the critical region is $\chi^2 < \chi^2_{1-\alpha}$, and for the one-sided alternative $\sigma^2 > \sigma_0^2$, the critical region is $\chi^2 > \chi^2_\alpha$.

Example 174 *A manufacturer of car batteries claims that the life of the company's batteries is approximately normally distributed with a standard deviation equal to* 0.9 *year. If a random sample of* 10 *of these batteries has a standard deviation of* 1.2 *years, do you think that* $\sigma > 0.9$ *year? Use a* 0.05 *level of significance.*

Solution 175 *1.* $H_0 : \sigma^2 = 0.81$.

2. $H_1 : \sigma^2 > 0.81$.

3. $\alpha = 0.05$.

4. Critical region: The null hypothesis is rejected when $\chi^2 > 16.919$, *where* $\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$, *with* $\nu = 9$ *degrees of freedom.*

5. Computations: $s^2 = 1.44$, $n = 10$, *and*

$$\chi^2 = \frac{(9)(1.44)}{0.81} = 16.0, \; P \approx 0.07.$$

6. Decision: The χ^2*-statistic is not significant at the* 0.05 *level. However, based on the P-value* 0.07, *there is evidence that* $\sigma > 0.9$.

Now let us consider the problem of testing the equality of the variances σ_1^2 and σ_2^2 of two populations. That is, we shall test the null hypothesis H_0 that $\sigma_1^2 = \sigma_2^2$ against one of the usual alternatives $\sigma_1^2 < \sigma_2^2$, $\sigma_1^2 > \sigma_2^2$, or $\sigma_1^2 \neq \sigma_2^2$. For independent random samples of sizes n_1 and n_2, respectively, from the two populations, the f-value for testing $\sigma_1^2 = \sigma_2^2$ is the ratio

$$f = \frac{s_1^2}{s_2^2}$$

where s_1^2 and s_2^2 are the variances computed from the two samples. If the two populations are approximately normally distributed and the null hypothesis is true, then the ratio $f = s_1^2/s_2^2$ is a value of the F-distribution with $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$ degrees of freedom. Therefore, the critical regions of size α corresponding to the one-sided alternatives $\sigma_1^2 < \sigma_2^2$ and $\sigma_1^2 > \sigma_2^2$ are, respectively, $f < f_{1-\alpha}(\nu 1, \nu 2)$ and $f > f_\alpha(\nu 1, \nu 2)$. For the two-sided alternative $\sigma_1^2 \neq \sigma_2^2$, the critical region is $f < f_{1-\alpha/2}(\nu 1, \nu 2)$ or $f > f_{\alpha/2}(\nu 1, \nu 2)$.

Example 176 *In testing for the difference in the abrasive wear of the two materials in Example 166, we assumed that the two unknown population variances were equal. Were we justified in making this assumption? Use a* 0.10 *level of significance.*

Solution 177 *Let* σ_1^2 *and* σ_2^2 *be the population variances for the abrasive wear of material* 1 *and material* 2, *respectively.*

1. $H_0 : \sigma_1^2 = \sigma_2^2$.

2. $H_1 : \sigma_1^2 \neq \sigma_2^2$.

3. $\alpha = 0.10$.

4. Critical region: We have $f_{0.05}(11, 9) = 3.11$, and, by using Theorem 99, we find $f_{0.95}(11, 9) = \frac{1}{f0.05(9,11)} = 0.34$. Therefore, the null hypothesis is rejected when $f < 0.34$ or $f > 3.11$, where $f = s_1^2/s_2^2$ with $\nu_1 = 11$ and $\nu_2 = 9$ degrees of freedom.

5. Computations: $s_1^2 = 16$, $s_2^2 = 25$, hence $f = 16/25 = 0.64$.

6. Decision: Do not reject H_0. Conclude that there is insufficient evidence that the variances differ.

Chapter 6

Chi square tests

6.1 Goodness-of-Fit Test

we consider a test to determine if a population has a specified theoretical distribution. The test is based on how good a fit we have between the frequency of occurrence of observations in an observed sample and the expected frequencies obtained from the hypothesized distribution. To illustrate, we consider the tossing of a die. We hypothesize that the die is honest, which is equivalent to testing the hypothesis that the distribution of outcomes is the discrete uniform distribution

$$f(x) = \frac{1}{6},\ x, 2, 3, 4, 5, 6$$

Suppose that the die is tossed 120 times and each outcome is recorded. Theoretically, if the die is balanced, we would expect each face to occur 20 times. The results are given in the table 6.1.

Face	1	2	3	4	5	6
Observed	20	22	17	18	19	24
Expected	20	20	20	20	20	20

By comparing the observed frequencies with the corresponding expected frequencies, the hypothesis, H_0 : the die is fair, should be rejected or not. A **goodness-of-fit test** between observed and expected frequencies is based on the quantity

$$\chi^2 = \sum_{i=1}^{k} \frac{(o_i - e_i)^2}{e_i}$$

where χ^2 is a value of a random variable whose sampling distribution is approximated very closely by the chi-squared distribution with $\upsilon = k - 1$ degrees of freedom. The symbols o_i and e_i represent the observed and expected frequencies, respectively, for the ith cell.

If the observed frequencies are close to the corresponding expected frequencies, the χ^2-value will be small, indicating a good fit. If the observed frequencies differ considerably from the expected frequencies, the χ^2-value will be large and the fit is poor. A good fit leads to the acceptance of H_0, whereas a poor fit leads to its rejection. The critical region will, therefore, fall in the right tail of the chi-squared distribution. For a level of significance equal to α, we find the critical value χ^2_α from Table A.5, and then $\chi^2 > \chi^2_\alpha$ constitutes the critical region. The decision criterion described here should not be used unless each of the expected frequencies is at least equal to 5. This restriction may require the combining of adjacent cells, resulting in a reduction in the number of degrees of freedom.

From Table 6.1, we find the χ^2-value to be

$$\begin{aligned}\chi^2_\alpha &= \tfrac{(20-20)^2}{20} + \tfrac{(22-20)^2}{20} + \tfrac{(17-20)^2}{20} + \tfrac{(18-20)^2}{20} \\ &+ \tfrac{(19-20)^2}{20} + \tfrac{(24-20)^2}{20} = 1.7\end{aligned}$$

Using chi-squared table, we find $\chi^2_{0.05} = 11.070$ for $\upsilon = 5$ degrees of freedom. Since 1.7 is less than the critical value, we fail to reject H_0. We conclude that there is insufficient evidence that the die is not balanced.

A second example is to test the hypothesis that the frequency distribution of battery lives given in Table 6.2 may be approximated by a normal distribution with mean $\mu = 3.5$ and standard deviation $\sigma = 0.7$. The expected frequencies for the 7 classes (cells), listed in Table 6.2, are obtained by computing the areas under the hypothesized normal curve that fall between the various class boundaries.

Table 6.2: Frequency Distribution of Battery Life.

Class Boundaries	Frequency

1.45-1.95	2
1.95-2.45	1
2.45-2.95	4
2.95-3.45	15
3.45-3.95	10
3.95-4.45	5
4.45-4.95	3

The z-values corresponding to the boundaries of the first class are $z_1 = (1.45 - 3.5)/0.7 = -2.93$ and $z_2 = (1.95 - 3.5)/0.7 = -2.21$. From standard normal table, we find the area between $z_1 = -2.93$ and $z_2 = -2.21$ to be area equal to $\Pr(-2.93 < Z < -2.21) = \Pr(Z < -2.21) - \Pr(Z < -2.93) = 0.0136 - 0.0017 = 0.0119$. Hence, the expected frequency for the first class is $e_1 = 40(0.0119) = 0.5$. Similarly, weget

$e_2 = 40[P((1.95 - 3.5)/0.7 < Z < (2.45 - 3.5)/0.7)] = 40[P(-2.21 < Z < -1.5)] = 2.1$

$e_3 = 40[P((2.45 - 3.5)/0.7 < Z < (2.95 - 3.5)/0.7)] = 40[P(-1.5 < Z < -0.79)] = 5.9$

$e_4 = 40[P((2.95 - 3.5)/0.7 < Z < (3.45 - 3.5)/0.7)] = 40[P(-0.79 < Z < -0.07)] = 10.3$

$e_5 = 40[P((3.45 - 3.5)/0.7 < Z < (3.95 - 3.5)/0.7)] = 40[P(-0.07 < Z < 0.64)] = 10.7$

$e_6 = 40[P((3.95 - 3.5)/0.7 < Z < (4.45 - 3.5)/0.7)] = 40[P(0.64 < Z < 1.36)] = 7.0$

$e_7 = 40[P((4.45 - 3.5)/0.7 < Z < (4.95 - 3.5)/0.7)] = 40[P(1.36 < Z < 2.07)] = 3.5$

It is customary to round these frequencies to one decimal. Note that we have to combined adjacent classes in Table 16.2 where the expected frequencies are less than 5 (a rule of thumb in the goodness-of-fit test). Consequently, the total number of intervals is reduced from 7 to 4, resulting in $\upsilon = 3$ degrees of freedom. The χ^2-value is then given by

$\chi^2 = \frac{(7-8.5)\hat{}2}{8.5} + \frac{(15-10.3)\hat{}2}{10.3} + \frac{(10-10.7)\hat{}2}{10.7} + \frac{(8-10.5)\hat{}2}{10.5} = 3.05$

Since the computed χ^2-value is less than $\chi^2_{0.05} = 7.815$ for 3 degrees of freedom, we have no reason to reject the null hypothesis and conclude that the normal distribution with $\mu = 3.5$ and $\sigma = 0.7$ provides a good fit for the distribution of battery lives.

6.2 Test for Independence (Categorical Data)

The chi-squared test procedure discussed in Section 6.1 can also be used to test the hypothesis H_0 of independence of two variables of classification. A test of independence tests the null hypothesis that there is no association between the two variables in a contingency table where the data is all drawn from one population.

Suppose that we wish to determine whether the opinions of the voting residents of the state of Illinois concerning a new tax reform are independent of their levels of income. Members of a random sample of 1000 registered voters from the state of Illinois are classified as to whether they are in a low, medium, or high income bracket and whether or not they favor the tax reform. The observed frequencies are presented in Table 6.2.

Tax Reform	Low	Medium	High	Total
For	182	213	203	598
Against	154	138	110	402
Total	336	351	313	1000

To find these expected frequencies, let us define the following events:

L: A person selected is in the low-income level.

M: A person selected is in the medium-income level.

H: A person selected is in the high-income level.

F: A person selected is for the tax reform.

A: A person selected is against the tax reform.

We have

$P(L) = 336/1000, P(M) = 351/1000, P(H) = 313/1000, P(F) = 598/1000, P(A) = 402/1000.$

If H_0 is true and the two variables are independent, we should have

$$P(L \cap F) = P(L)P(F) = \left(\frac{336}{1000}\right)\left(\frac{598}{1000}\right)$$

$$P(L \cap A) = P(L)P(A) = \left(\frac{336}{1000}\right)\left(\frac{402}{1000}\right)$$

$$P(M \cap F) = P(M)P(F) = \left(\frac{351}{1000}\right)\left(\frac{598}{1000}\right)$$

$$P(M \cap A) = P(M)P(A) = \left(\frac{351}{1000}\right)\left(\frac{402}{1000}\right)$$

$$P(H \cap F) = P(H)P(F) = \left(\frac{313}{1000}\right)\left(\frac{598}{1000}\right)$$

$$P(H \cap A) = P(H)P(A) = \left(\frac{313}{1000}\right)\left(\frac{402}{1000}\right)$$

The expected frequencies are obtained by multiplying each cell probability by

the total number of observations 1000.

The general rule for obtaining the expected frequency of any cell is given by the following formula:

$$\text{expected frequency} = \frac{(\text{column total}) \times (\text{row total})}{\text{grand total}}$$

A simple formula providing the correct number of degrees of freedom is

$$\upsilon = (c - 1) \times (r - 1)$$

Table 6.3: Observed and Expected Frequencies

	Income Level			
Tax Reform	Low	Medium	High	Total
For	182 (200.9)	213 (209.9)	203 (187.2)	598
Against	154 (135.1)	138 (141.1)	110 (125.8)	402
Total	336	351	313	1000

Hence, the expected frequency for the first class is

$$e_{11} = (336 \times 598)/1000 = 200.9.$$

Here $\upsilon = (2 - 1)(3 - 1) = 2$ degrees of freedom. If $\chi^2 > \chi^2_\alpha$, reject the null hypothesis of independence at the α-level of significance; otherwise, fail to reject the null hypothesis.

1. H_0: the two random variables (voter's opinion concerning the tax reform and his or her level of income) are independent.

2. H_1: voter's opinion concerning the tax reform and his or her level of income are not independent.

3. $\alpha = 0.05$.

4. The test statistic:

$$\chi^2{=}\sum_{i,j}\frac{(o_{ij}-e_{ij})^2}{e_ij} = \frac{(182-200.9)^2 2}{200.9} + \frac{(213-209.9)^2}{209.9} + \frac{(203-187.2)^2}{187.2}$$
$$+\frac{(154-135.1)^2}{135.1} + \frac{(138-141.1)^2}{141.1} + \frac{(110-125.8)^2}{125.8} = 7.85, \text{ P} \simeq 0.02.$$

5. From chi-square table we find that $\chi^2_{0.05} = 5.991$ for $\upsilon = (2-1)(3-1) = 2$ degrees of freedom. The null hypothesis is rejected and we conclude that a voter's opinion concerning the tax reform and his or her level of income are not independent.

6.3 Test for Homogeneity

Now, rather than test for independence, this test determines if two or more populations have the same distribution of a single categorical variable. The key difference from the test of independence is that there are multiple populations that the data is drawn from. Suppose, for example, that we decide in advance to select 200 Democrats, 150 Republicans, and 150 Independents from the voters of the state of North Carolina and record whether they are for a proposed abortion law, against it, or undecided. We assume that the row totals in table 6.4 were random, while the column totals were presumably fixed in advance, since they represented numbers of voters sampled from different political affiliations.

Table 6.4: Observed Frequencies

	Political Affiliation			
Abortion Law	Democrat	Republican	Independent	Total
For	82	70	62	214
Against	93	62	67	222
Undecided	25	18	21	64
Total	200	150	150	500

we want to test the hypothesis that the proportions of Democrats, Republicans, and Independents favoring the abortion law are the same; the proportions of each political affiliation against the law are the same; and the proportions of each political affiliation that are undecided are the same. Then the null hypothesis is

$$H_0 : \text{For each row } i,\, p_{i1} = \cdots = p_{ic}$$

The alternative hypothesis H_1 is that at least one of the null hypothesis statements is false. Such a test is called a test for homogeneity.

If homogeneity holds we should have

$$\Pr(\text{For }|\text{Democrat}) = \Pr(\text{For }|\text{Republic}) = \Pr(\text{For }|\text{Independent})$$

But $\Pr(\text{For}) = \Pr(\text{For}|\text{Democrat})\Pr(\text{Democrat}) + \Pr(\text{For}|\text{Republic})\Pr(\text{Republic})$

$+\Pr(\text{For}|\text{Independent})\Pr(\text{Independent}) = \Pr(\text{For}|\text{Democrat})$

Consequently

$$\Pr(\text{For }|\text{Democrat}) = \Pr(\text{For }|\text{Republic}) = \Pr(\text{For }|\text{Independent}) = \Pr(\text{For}) \text{ estimated by } \frac{214}{500}$$

But $\Pr(\text{For}\cap\text{Democrat}) = \Pr(\text{For}|\text{Democrat})\Pr(\text{Democrat}) \simeq \frac{214}{500}\frac{200}{500}$. Hence $e_{11} = \left(\frac{214}{500}\right)\left(\frac{200}{500}\right)(500) = \frac{214\times200}{500} = 85.6$. The expected cell frequencies is then obtained by multiplying the corresponding row and column totals and then dividing by the grand total. The analysis then proceeds using the same chi-squared statistic as before. Use of the chi-square distribution is appropriate whenever the expected values are all greater than or equal to 5.

1. H_0: For each opinion, the proportions of Democrats, Republicans, and Independents are the same.

2. H_1: For at least one opinion, the proportions of Democrats, Republicans, and Independents are not the same.

3. $\alpha = 0.05$.

4. Critical region: $\chi^2 > \chi^2_\alpha$ with $\nu = (3-1)(3-1) = 4$ degrees of freedom.

Then the critical region: $\chi^2 > 9.488$.

5. Computations: The observed and expected cell frequencies are displayed in Table 6.6.

Table 6.5: Observed and Expected Frequencies

Political Affiliation

Abortion Law	Democrat	Republican	Independent	Total
For	82 (85.6)	70 (64.2)	62 (64.2)	214
Against	93 (88.8)	62 (66.6)	67 (66.6)	222
Undecided	25 (25.6)	18 (19.2)	21 (19.2)	64
Total	200	150	150	500

$\chi^2 = 1.53$.

6. Decision: Do not reject H_0. There is insufficient evidence to conclude that the proportions of Democrats, Republicans, and Independents differ for each stated opinion.

Chapter 7

Simple linear regression and correlation

7.1 Joint Probability Distributions

If X and Y are two discrete random variables, the probability distribution for their simultaneous occurrence can be represented by a function with values $f(x,y)$ for any pair of values (x,y) within the range of the random variables X and Y. It is customary to refer to this function as the joint probability distribution of X and Y.

Hence, in the discrete case,

$$f(x,y) = \Pr(X = x, Y = y)$$

Definition 178 *The function $f(x,y)$ is a joint probability distribution or probability mass function of the discrete random variables X and Y if*
1. $f(x,y) \geq 0$ for all (x,y),
2. $\underset{x}{\Sigma}\underset{y}{\Sigma}\ f(x,y) = 1$,
3. $\Pr(X = x, Y = y) = f(x,y)$

When X and Y are continuous random variables, we have

Definition 179 *The function $f(x,y)$ is a joint density function of the continuous random* *variables X and Y if*

1. $f(x,y) \geq 0$ *for all* (x,y),

2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$,

3. $\Pr[(X,Y) \in A] = \displaystyle\int\int_A f(x,y)dxdy$, *for any region A in the xy plane.*

Example 180 *Let a joint density function of X and Y given by*

$$f(x) = \begin{cases} \frac{2}{5}(2x+3y), & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \textit{elsewhere.} \end{cases}$$

1) Verify condition 2 of Definition 179.

2) Find $\Pr[(X,Y) \in A]$, *where* $A = \{(x,y) \left| 0 < x < \frac{1}{2}, \frac{1}{4} < y < \frac{1}{2} \right.\}$

Definition 181 *The marginal distributions of X alone and of Y alone are*

$$g(x) = \sum_y f(x,y) \text{ and } h(y) = \sum_x f(x,y)$$

for the discrete case, and

$$g(x) = \int_{-\infty}^{\infty} f(x,y)dy \text{ and } h(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

for the continuous case.

Example 182 *Find $g(x)$ and $h(y)$ for the joint density function of Example 180.*

Definition 183 *Let X and Y be two random variables, discrete or continuous, with joint probability distribution $f(x,y)$ and marginal distributions g(x) and h(y), respectively. The random variables X and Y are said to be statistically independent if and only if*

$$f(x,y) = g(x)h(y)$$

for all (x,y).

Definition 184 *Let X be a random variable with probability distribution $f(x)$ and mean μ. The variance of X is*

$$\begin{aligned}\sigma^2 &= E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x), \text{ if } X \text{ is discrete, and}\\ \sigma^2 &= E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx, \text{ if } X \text{ is continuous.}\end{aligned}$$

The positive square root of the variance, σ, is called the standard deviation of X.

Example 185 *The weekly demand for a drinking-water product, in thousands of liters, from a local chain of efficiency stores is a continuous random variable X having the probability density*

$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the mean and variance of X.

Definition 186 *Let X be a random variable with probability distribution $f(x)$. The variance of the random variable $g(X)$ is*

$$\sigma^2_{g(X)} = E[(g(X) - \mu_{g(X)})^2] = \sum_x (g(x) - \mu)^2 f(x)$$

if X is discrete, and

$$\sigma^2_{g(X)} = E[(g(X) - \mu_{g(X)})^2] = \int_{-\infty}^{\infty} (g(x) - \mu)^2 f(x)dx$$

if X is continuous.

Definition 187 *Let X and Y be random variables with joint probability distribution $f(x,y)$. The mean, or expected value, of the random variable $g(X,Y)$ is*

$$\mu_{g(X,Y)} = E[g(X,Y)] = \sum_x \sum_y g(x,y) f(x,y)$$

if X and Y are discrete, and

$$\mu_{g(X,Y)} = E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)dxdy$$

f X and Y are continuous.

Exercise 188 *Let X and Y be the random variables with joint probability distribution indicated in the following table. Find the expected value of $g(X,Y) = XY$.*

$y \backslash x$	0	1	2	
0	$\frac{3}{28}$	$\frac{9}{29}$	$\frac{3}{28}$	$\frac{15}{28}$
1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
	$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

Solution 189 *By definition,*

$$\begin{aligned} E(XY) &= \sum_{x=0}^{2}\sum_{y=0}^{2} xyf(x,y) \\ &= (0)(0)f(0,0) + (0)(1)f(0,1) \\ &\quad +(1)(0)f(1,0) + (1)(1)f(1,1) + (2)(0)f(2,0) \\ &= f(1,1) = \frac{3}{14}. \end{aligned}$$

Definition 190 *Let X and Y be random variables with joint probability distribution $f(x,y)$. The covariance of X and Y is*

$$\sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \sum_x\sum_y (x-\mu_X)(y-\mu_Y)f(x,y)$$

if X and Y are discrete, and

$$\sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y)f(x,y)dxdy$$

if X and Y are continuous.

Theorem 191 *The covariance of two random variables X and Y with means μ_X and μ_Y ,* *respectively, is given by*

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y$$

Example 192 *Let the joint density function given by*

$$f(x,y) = \begin{cases} 8xy, & 0 \leq y \leq x \leq 1, \\ 0, & \textit{elsewhere.} \end{cases}$$

Find the covariance of X and Y .

Although the covariance between two random variables does provide information regarding the nature of the relationship, the magnitude of σ_{XY} does not indicate anything regarding the strength of the relationship, since σ_{XY} is not scale-free. Its magnitude will depend on the units used to measure both X and Y . There is a scale-free version of the covariance called the correlation coefficient that is used widely in statistics.

Definition 193 *Let X and Y be random variables with covariance σ_{XY} and standard deviations σ_X and σ_Y , respectively. The correlation coefficient of X and Y is*

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

It should be clear to the reader that ρ_{XY} is free of the units of X and Y. The correlation coefficient satisfies the inequality $-1 \leq \rho_{XY} \leq 1$. It assumes a value of zero when $\sigma_{XY} = 0$. Where there is an exact linear dependency, say $Y \equiv a + bX$,$\rho_{XY} = 1$ if $b > 0$ and $\rho_{XY} = -1$ if $b < 0$.

7.2 Means and Variances of Linear Combinations of Random Variables

Theorem 194 *If a and b are constants, then*

$$E(aX + b) = aE(X) + b.$$

Proof. As exercise. ■

Corollary 195 *Setting $a = 0$, we see that $E(b) = b$.*

Corollary 196 *Setting $b = 0$, we see that $E(aX) = aE(X)$.*

Theorem 197 *The expected value of the sum or difference of two or more functions of a random variable X is the sum or difference of the expected values of the functions. That is,*

$$E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)].$$

Example 198 *Let X be a random variable with probability distribution as follows:*

x	0	1	2	3
$f(x)$	$\frac{1}{3}$	$\frac{1}{2}$	0	$\frac{1}{6}$

Find the expected value of $Y = (X - 1)^2$.

Example 199 *The weekly demand for a certain drink, in thousands of liters, at a chain of convenience stores is a continuous random variable $g(X) = X2 + X - 2$, where X has the density function*

$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2, \\ 0, & \textit{elsewhere.} \end{cases}$$

Find the expected value of the weekly demand for the drink.

Theorem 200 *The expected value of the sum or difference of two or more functions of the random variables X and Y is the sum or difference of the expected values of the functions. That is,*

$$E[g(X,Y) \pm h(X,Y)] = E[g(X,Y)] \pm E[h(X,Y)].$$

Corollary 201 *Setting $g(X,Y) = g(X)$ and $h(X,Y) = h(Y)$, we see that*

$$E[g(X) \pm h(Y)] = E[g(X)] \pm E[h(Y)].$$

Corollary 202 *Setting $g(X,Y) = X$ and $h(X,Y) = Y$, we see that*

$$E[X \pm Y] = E[X] \pm E[Y].$$

Theorem 203 *Let X and Y be two independent random variables. Then*

$$E(XY) = E(X)E(Y).$$

Proof. By Definition

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy$$

Since X and Y are independent, we may write

$$f(x,y) = g(x)h(y),$$

where $g(x)$ and $h(y)$ are the marginal distributions of X and Y , respectively. Hence,

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyg(x)h(y)dxdy = \int_{-\infty}^{\infty} g(x)dx\int_{-\infty}^{\infty} h(y)dy = E(X)E(Y)$$

■

Corollary 204 *Let X and Y be two independent random variables. Then $cov(X,Y) = 0$.*

Theorem 205 *If X and Y are random variables with joint probability distribution $f(x,y)$ and a, b, and c are constants, then $var(aX + bY + c) = a^2var(X)$ $+b^2var(Y)+$ $2abcov(X,Y)$.*

Corollary 206 *Setting $b = 0$, we see that*

$$var(aX + c) = a^2var(X).$$

Corollary 207 *Setting $a = 1$ and $b = 0$, we see that*

$$var(X + c) = var(X).$$

Corollary 208 *Setting $b = 0$ and $c = 0$, we see that*

$$var(aX) = var(X)$$

Corollary 209 *If X and Y are independent random variables, then*

$$var(aX + bY) = a^2 var(X) + b^2 var(Y).$$

Corollary 210 *If $X_1, X_2, ..., X_n$ are independent random variables, then*

$$var(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1^2 var(X_1) + a_2^2 var(X_2) + \cdots + a_n^2 var(X_n).$$

Example 211 *If X and Y are random variables with variances $var(X) = 2$ and $var(Y) = 4$ and covariance $cov(X, Y) = -2$, find the variance of the random variable $Z = 3X - 4Y + 8$.*

7.3 Correlation

We consider the problem of measuring the relationship between the two variables X and Y. We want to to determine whether large values of X are associated with large values of Y, and vice versa. Correlation analysis attempts to measure the strength of such relationships between two variables X and Y by means of a single number called a correlation coefficient.

Definition 212 (Coefficient) *The measure of linear association ρ between two variables X and Y is estimated by the sample correlation coefficient r, where*

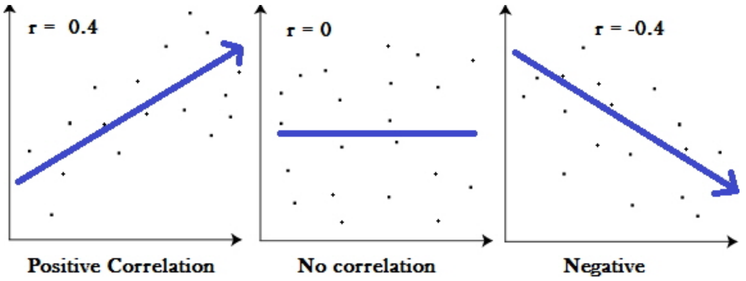
$$r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$$

with $S_{xy} = \sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})$, $S_{xx} = \sum_{i=1}^{n}(x_i - \overline{x})^2$ and $S_{yy} = \sum_{i=1}^{n}(y_i - \overline{y})^2$.

Example 213 *Let consider the following grades of 6 students selected at random*

Mathematics grade	70	92	80	74	65	83
English grade	74	84	63	87	78	90

We have $n = 6$, $S_{xy} = 115.33, S_{xx} = 471.33, S_{yy} = 491.33$. Hence $r = \frac{115.33}{\sqrt{(471.33)(491.33)}} = 0.24$.



7.3.1 Properties of r

$r = 1$ iff all (x_i, y_i) pairs lie on straight line with positive slope, and $r = -1$ iff all (x_i, y_i) pairs lie on a straight line with negative slope.

7.4 Simple linear regression

The form of a relationship between the response Y (the dependent or the response variable) and the regressor X (the independent variable) is in mathematically the linear relationship

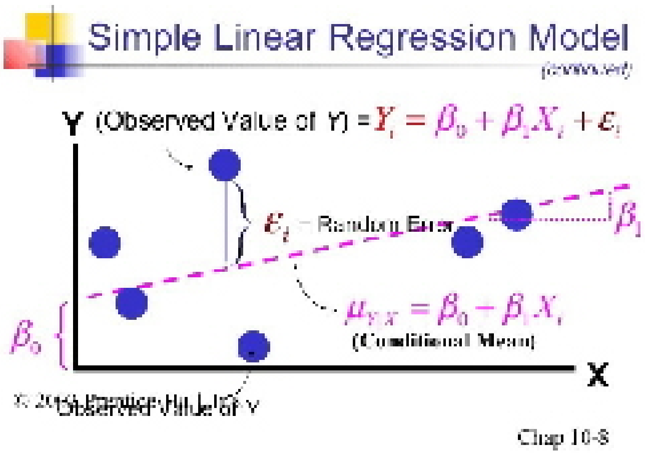
$$Y = \beta_0 + \beta_1 X$$

where, β_0 is the intercept and β_1 is the slope. The relationship is illustrated in Figure 7.1.

An important aspect of regression analysis is to estimate the parameters β_0 and β_1 (i.e., estimate the so-called regression coefficients). The method of estimation will be discussed in the next section. Suppose we denote the estimates b_0 for β_0 and b_1 for β_1. Then the estimated or fitted regression line is given by

$$\hat{y} = b_0 + b_1 x$$

where $\hat{y}$ is the predicted or fitted value.


Simple Linear Regression Model
(continued)
Y (Observed Value of Y) = $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$
ε_i = Random Error
β_1
$\mu_{Y|X} = \beta_0 + \beta_1 X_i$
(Conditional Mean)
β_0
X
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Observed Value of Y
Chap 10-8

Figure 7-1: Figure 7.1

7.4.1 Least Squares and the Fitted Model

We shall find b_0 and b_1, the estimates of β_0 and β_1, so that the sum of the squares of the residuals is a minimum. The residual sum of squares is often called the sum of squares of the errors about the regression line and is denoted by SSE. This minimization procedure for estimating the parameters is called the method of least squares. Hence, we shall find a and b so as to minimize

$$\text{the error sum of squares} = SSE = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y - \hat{y}_i)^2 = \sum_{i=1}^{n} (y - b_0 - b_1 x_i)^2$$

Differentiating SSE with respect to b_0 and b_1, we have

Theorem 214 *Given the sample $\{(x_i, y_i); i = 1, 2, ..., n\}$, the least squares estimates b_0 and b_1 of the regression coefficients β_0 and β_1 are computed from the formulas*

$$\begin{aligned} b_1 &= \frac{\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{n}(x_i - \overline{x})^2} = \frac{\sum_{i=1}^{n} x_i y_i - n\overline{x}\overline{y}}{\sum_{i=1}^{n} x_i^2 - n\overline{x}^2} \\ b_0 &= \overline{y} - b_1 \overline{x} \end{aligned}$$

Example 215 *Consider the experimental data in Table 7.1, which were obtained from 33 sam-*

ples of chemically treated waste in a study conducted at Virginia Tech. Readings on x, the percent reduction in total solids, and y, the percent reduction in chemical oxygen demand, were recorded.

Solids Reduction, x (%)	*Oxygen Demand Reduction, y (%)*	*Solids Reduction, x (%)*	*Oxygen Demand Reduction, y (%)*
3	*5*	*36*	*34*
7	*11*	*37*	*36*
11	*21*	*38*	*38*
15	*16*	*39*	*37*
18	*16*	*39*	*36*
27	*28*	*39*	*45*
29	*27*	*40*	*39*
30	*25*	*41*	*41*
30	*35*	*42*	*40*
31	*30*	*42*	*44*
31	*40*	*43*	*37*
32	*32*	*44*	*44*
33	*34*	*45*	*46*
33	*32*	*46*	*46*
34	*34*	*47*	*49*
36	*37*	*50*	*51*
36	*38*		

The estimated regression line is given by

$$\hat{y} = 3.8296 + 0.9036x.$$

Using the regression line, we would predict a 31% reduction in the chemical oxygen demand when the reduction in the total solids is 30%. The 31% reduction in the chemical oxygen demand may be interpreted as an estimate of the population mean $\mu_{Y|30}$ or as an estimate of a *new observation when the reduction in total solids is 30%.*

7.4.2 Properties of the Least Squares Estimators

We are interested in the expectation and variance the estimator B_1 of β_1 and the expectation of B_0 the estimator of β_0.

Theorem 216 $E(B_0) = \beta_0$, $E(B_1) = \beta_1$, $var(B_1) = \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \overline{x})^2}$

Theorem 217 *An unbiased estimate of σ^2 is*

$$\widehat{\sigma}^2 = \frac{SSE}{n-2} = \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{n-2}$$

7.4.3 Inferences Concerning the Regression Coefficients

Theorem 218 *A* $100(1-\alpha)\%$ *confidence interval for the parameter β_1 in the regression line*

$$b_1 - t_{\alpha/2}\frac{\widehat{\sigma}^2}{\sum_{i=1}^{n}(x_i - \overline{x})^2} < \beta_1 < b_1 + t_{\alpha/2}\frac{\widehat{\sigma}^2}{\sum_{i=1}^{n}(x_i - \overline{x})^2}$$

where $t_{\alpha/2}$ is a value of the t-distribution with $n-2$ degrees of freedom.

Find a 95% confidence interval for β_1 in the regression line , based on the pollution data of Table 7.1. We show that

$$\widehat{\sigma}^2 = \frac{SSE}{n-2} = \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{n-2} = 0.4299.$$

Therefore, taking the square root, we obtain $\widehat{\sigma} = 3.2295$. Also, $S_{xx} = \sum_{i=1}^{n}(x_i - \overline{x})^2 = 4152.18$. Using Table A.4, we find $t_{0.025} \approx 2.045$ for 31 degrees of freedom. Therefore, a 95% confidence interval for β_1 is

$$0.903643 - (2.045)\frac{3.2295}{\sqrt{4152.18}} < \beta_1 < 0.903643 + (2.045)\frac{3.2295}{\sqrt{4152.18}}$$

which simplifies to

$$0.8012 < \beta_1 < 1.0061.$$

Hypothesis Testing on the Slope

To test the null hypothesis H_0 that $\beta_1 = \beta_{10}$ against a suitable alternative, we again use the t-distribution with $n-2$ degrees of freedom to establish a critical region and then base our decision on the value of

$$t = \frac{b_1 - \beta_{10}}{\widehat{\sigma}/\sqrt{S_{xx}}}$$

Example 219 *Using the estimated value $b_1 = 0.903643$ of Example 7.1, test the hypothesis that $\beta_1 = 1.0$ against the alternative that $\beta_1 < 1.0$.*

Solution 220 *The hypotheses are $H_0 : \beta_1 = 1.0$ and $H_1 : \beta_1 < 1.0$. So*

$$t = \frac{0.903643 - 1.0}{3.2295/\sqrt{4152.18}} = -1.92,$$

with $n - 2 = 31$ degrees of freedom ($P \approx 0.03$).
Decision: P-value < 0.05, suggesting strong evidence that $\beta_1 < 1.0$

One important t-test on the slope is the test of the hypothesis $H_0 : \beta_1 = 0$ versus $H_1 : \beta_1 \neq 0$. When the null hypothesis is not rejected, the conclusion is that there is no significant linear relationship between $E(y)$ and the independent variable x. Rejection of H_0 above implies that a significant linear regression exists.

Measuring Goodness-of-Fit: the Coefficient of Determination

A goodness-of-fit statistic is a quantity that measures how well a model explains a given set of data. A linear model fits well if there is a strong linear relationship between x and y.

Definition 221 *The coefficient of determination, R^2, is given by*

$$R^2 = 1 - \frac{SSE}{SST}$$

where $SSE = \sum_{i=1}^{n}(y - \hat{y}_i)^2$ and $SST = \sum_{i=1}^{n}(y - \overline{y})^2$.

Note that if the fit is perfect, all residuals $y - \hat{y}_i$ are zero, and thus $R^2 = 1.0$. But if SSE is only slightly smaller than SST, $R^2 \approx 0$. In the example of table 7.1, the coefficient of determination $R^2 = 0.913$, suggests that the model fit to the data explains 91.3% of the variability observed in the response, the reduction in chemical oxygen demand.

Theorem 222 *The square r^2 of the sample correlation coefficient gives the value of the coefficient of determination R^2 that would result from fitting the simple linear regression model.*

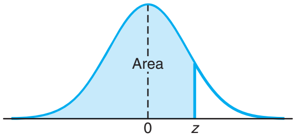

Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Table A.3 (continued) Areas under the Normal Curve

z	**.00**	**.01**	**.02**	**.03**	**.04**	**.05**	**.06**	**.07**	**.08**	**.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

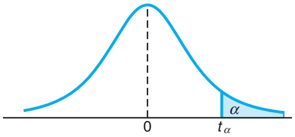

Table A.4 Critical Values of the t-Distribution

	α						
v	**0.40**	**0.30**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**
1	0.325	0.727	1.376	1.963	3.078	6.314	12.706
2	0.289	0.617	1.061	1.386	1.886	2.920	4.303
3	0.277	0.584	0.978	1.250	1.638	2.353	3.182
4	0.271	0.569	0.941	1.190	1.533	2.132	2.776
5	0.267	0.559	0.920	1.156	1.476	2.015	2.571
6	0.265	0.553	0.906	1.134	1.440	1.943	2.447
7	0.263	0.549	0.896	1.119	1.415	1.895	2.365
8	0.262	0.546	0.889	1.108	1.397	1.860	2.306
9	0.261	0.543	0.883	1.100	1.383	1.833	2.262
10	0.260	0.542	0.879	1.093	1.372	1.812	2.228
11	0.260	0.540	0.876	1.088	1.363	1.796	2.201
12	0.259	0.539	0.873	1.083	1.356	1.782	2.179
13	0.259	0.538	0.870	1.079	1.350	1.771	2.160
14	0.258	0.537	0.868	1.076	1.345	1.761	2.145
15	0.258	0.536	0.866	1.074	1.341	1.753	2.131
16	0.258	0.535	0.865	1.071	1.337	1.746	2.120
17	0.257	0.534	0.863	1.069	1.333	1.740	2.110
18	0.257	0.534	0.862	1.067	1.330	1.734	2.101
19	0.257	0.533	0.861	1.066	1.328	1.729	2.093
20	0.257	0.533	0.860	1.064	1.325	1.725	2.086
21	0.257	0.532	0.859	1.063	1.323	1.721	2.080
22	0.256	0.532	0.858	1.061	1.321	1.717	2.074
23	0.256	0.532	0.858	1.060	1.319	1.714	2.069
24	0.256	0.531	0.857	1.059	1.318	1.711	2.064
25	0.256	0.531	0.856	1.058	1.316	1.708	2.060
26	0.256	0.531	0.856	1.058	1.315	1.706	2.056
27	0.256	0.531	0.855	1.057	1.314	1.703	2.052
28	0.256	0.530	0.855	1.056	1.313	1.701	2.048
29	0.256	0.530	0.854	1.055	1.311	1.699	2.045
30	0.256	0.530	0.854	1.055	1.310	1.697	2.042
40	0.255	0.529	0.851	1.050	1.303	1.684	2.021
60	0.254	0.527	0.848	1.045	1.296	1.671	2.000
120	0.254	0.526	0.845	1.041	1.289	1.658	1.980
∞	0.253	0.524	0.842	1.036	1.282	1.645	1.960

Table A.4 (continued) Critical Values of the t-Distribution

	α						
v	**0.02**	**0.015**	**0.01**	**0.0075**	**0.005**	**0.0025**	**0.0005**
1	15.894	21.205	31.821	42.433	63.656	127.321	636.578
2	4.849	5.643	6.965	8.073	9.925	14.089	31.600
3	3.482	3.896	4.541	5.047	5.841	7.453	12.924
4	2.999	3.298	3.747	4.088	4.604	5.598	8.610
5	2.757	3.003	3.365	3.634	4.032	4.773	6.869
6	2.612	2.829	3.143	3.372	3.707	4.317	5.959
7	2.517	2.715	2.998	3.203	3.499	4.029	5.408
8	2.449	2.634	2.896	3.085	3.355	3.833	5.041
9	2.398	2.574	2.821	2.998	3.250	3.690	4.781
10	2.359	2.527	2.764	2.932	3.169	3.581	4.587
11	2.328	2.491	2.718	2.879	3.106	3.497	4.437
12	2.303	2.461	2.681	2.836	3.055	3.428	4.318
13	2.282	2.436	2.650	2.801	3.012	3.372	4.221
14	2.264	2.415	2.624	2.771	2.977	3.326	4.140
15	2.249	2.397	2.602	2.746	2.947	3.286	4.073
16	2.235	2.382	2.583	2.724	2.921	3.252	4.015
17	2.224	2.368	2.567	2.706	2.898	3.222	3.965
18	2.214	2.356	2.552	2.689	2.878	3.197	3.922
19	2.205	2.346	2.539	2.674	2.861	3.174	3.883
20	2.197	2.336	2.528	2.661	2.845	3.153	3.850
21	2.189	2.328	2.518	2.649	2.831	3.135	3.819
22	2.183	2.320	2.508	2.639	2.819	3.119	3.792
23	2.177	2.313	2.500	2.629	2.807	3.104	3.768
24	2.172	2.307	2.492	2.620	2.797	3.091	3.745
25	2.167	2.301	2.485	2.612	2.787	3.078	3.725
26	2.162	2.296	2.479	2.605	2.779	3.067	3.707
27	2.158	2.291	2.473	2.598	2.771	3.057	3.689
28	2.154	2.286	2.467	2.592	2.763	3.047	3.674
29	2.150	2.282	2.462	2.586	2.756	3.038	3.660
30	2.147	2.278	2.457	2.581	2.750	3.030	3.646
40	2.123	2.250	2.423	2.542	2.704	2.971	3.551
60	2.099	2.223	2.390	2.504	2.660	2.915	3.460
120	2.076	2.196	2.358	2.468	2.617	2.860	3.373
∞	2.054	2.170	2.326	2.432	2.576	2.807	3.290

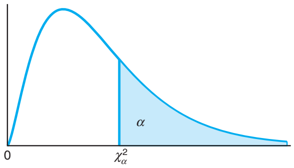

Table A.5 Critical Values of the Chi-Squared Distribution

	α									
v	**0.995**	**0.99**	**0.98**	**0.975**	**0.95**	**0.90**	**0.80**	**0.75**	**0.70**	**0.50**
1	0.0^4393	0.0^3157	0.0^3628	0.0^3982	0.00393	0.0158	0.0642	0.102	0.148	0.455
2	0.0100	0.0201	0.0404	0.0506	0.103	0.211	0.446	0.575	0.713	1.386
3	0.0717	0.115	0.185	0.216	0.352	0.584	1.005	1.213	1.424	2.366
4	0.207	0.297	0.429	0.484	0.711	1.064	1.649	1.923	2.195	3.357
5	0.412	0.554	0.752	0.831	1.145	1.610	2.343	2.675	3.000	4.351
6	0.676	0.872	1.134	1.237	1.635	2.204	3.070	3.455	3.828	5.348
7	0.989	1.239	1.564	1.690	2.167	2.833	3.822	4.255	4.671	6.346
8	1.344	1.647	2.032	2.180	2.733	3.490	4.594	5.071	5.527	7.344
9	1.735	2.088	2.532	2.700	3.325	4.168	5.380	5.899	6.393	8.343
10	2.156	2.558	3.059	3.247	3.940	4.865	6.179	6.737	7.267	9.342
11	2.603	3.053	3.609	3.816	4.575	5.578	6.989	7.584	8.148	10.341
12	3.074	3.571	4.178	4.404	5.226	6.304	7.807	8.438	9.034	11.340
13	3.565	4.107	4.765	5.009	5.892	7.041	8.634	9.299	9.926	12.340
14	4.075	4.660	5.368	5.629	6.571	7.790	9.467	10.165	10.821	13.339
15	4.601	5.229	5.985	6.262	7.261	8.547	10.307	11.037	11.721	14.339
16	5.142	5.812	6.614	6.908	7.962	9.312	11.152	11.912	12.624	15.338
17	5.697	6.408	7.255	7.564	8.672	10.085	12.002	12.792	13.531	16.338
18	6.265	7.015	7.906	8.231	9.390	10.865	12.857	13.675	14.440	17.338
19	6.844	7.633	8.567	8.907	10.117	11.651	13.716	14.562	15.352	18.338
20	7.434	8.260	9.237	9.591	10.851	12.443	14.578	15.452	16.266	19.337
21	8.034	8.897	9.915	10.283	11.591	13.240	15.445	16.344	17.182	20.337
22	8.643	9.542	10.600	10.982	12.338	14.041	16.314	17.240	18.101	21.337
23	9.260	10.196	11.293	11.689	13.091	14.848	17.187	18.137	19.021	22.337
24	9.886	10.856	11.992	12.401	13.848	15.659	18.062	19.037	19.943	23.337
25	10.520	11.524	12.697	13.120	14.611	16.473	18.940	19.939	20.867	24.337
26	11.160	12.198	13.409	13.844	15.379	17.292	19.820	20.843	21.792	25.336
27	11.808	12.878	14.125	14.573	16.151	18.114	20.703	21.749	22.719	26.336
28	12.461	13.565	14.847	15.308	16.928	18.939	21.588	22.657	23.647	27.336
29	13.121	14.256	15.574	16.047	17.708	19.768	22.475	23.567	24.577	28.336
30	13.787	14.953	16.306	16.791	18.493	20.599	23.364	24.478	25.508	29.336
40	20.707	22.164	23.838	24.433	26.509	29.051	32.345	33.66	34.872	39.335
50	27.991	29.707	31.664	32.357	34.764	37.689	41.449	42.942	44.313	49.335
60	35.534	37.485	39.699	40.482	43.188	46.459	50.641	52.294	53.809	59.335

Table A.5 (continued) Critical Values of the Chi-Squared Distribution

	α									
v	**0.30**	**0.25**	**0.20**	**0.10**	**0.05**	**0.025**	**0.02**	**0.01**	**0.005**	**0.001**
1	1.074	1.323	1.642	2.706	3.841	5.024	5.412	6.635	7.879	10.827
2	2.408	2.773	3.219	4.605	5.991	7.378	7.824	9.210	10.597	13.815
3	3.665	4.108	4.642	6.251	7.815	9.348	9.837	11.345	12.838	16.266
4	4.878	5.385	5.989	7.779	9.488	11.143	11.668	13.277	14.860	18.466
5	6.064	6.626	7.289	9.236	11.070	12.832	13.388	15.086	16.750	20.515
6	7.231	7.841	8.558	10.645	12.592	14.449	15.033	16.812	18.548	22.457
7	8.383	9.037	9.803	12.017	14.067	16.013	16.622	18.475	20.278	24.321
8	9.524	10.219	11.030	13.362	15.507	17.535	18.168	20.090	21.955	26.124
9	10.656	11.389	12.242	14.684	16.919	19.023	19.679	21.666	23.589	27.877
10	11.781	12.549	13.442	15.987	18.307	20.483	21.161	23.209	25.188	29.588
11	12.899	13.701	14.631	17.275	19.675	21.920	22.618	24.725	26.757	31.264
12	14.011	14.845	15.812	18.549	21.026	23.337	24.054	26.217	28.300	32.909
13	15.119	15.984	16.985	19.812	22.362	24.736	25.471	27.688	29.819	34.527
14	16.222	17.117	18.151	21.064	23.685	26.119	26.873	29.141	31.319	36.124
15	17.322	18.245	19.311	22.307	24.996	27.488	28.259	30.578	32.801	37.698
16	18.418	19.369	20.465	23.542	26.296	28.845	29.633	32.000	34.267	39.252
17	19.511	20.489	21.615	24.769	27.587	30.191	30.995	33.409	35.718	40.791
18	20.601	21.605	22.760	25.989	28.869	31.526	32.346	34.805	37.156	42.312
19	21.689	22.718	23.900	27.204	30.144	32.852	33.687	36.191	38.582	43.819
20	22.775	23.828	25.038	28.412	31.410	34.170	35.020	37.566	39.997	45.314
21	23.858	24.935	26.171	29.615	32.671	35.479	36.343	38.932	41.401	46.796
22	24.939	26.039	27.301	30.813	33.924	36.781	37.659	40.289	42.796	48.268
23	26.018	27.141	28.429	32.007	35.172	38.076	38.968	41.638	44.181	49.728
24	27.096	28.241	29.553	33.196	36.415	39.364	40.270	42.980	45.558	51.179
25	28.172	29.339	30.675	34.382	37.652	40.646	41.566	44.314	46.928	52.619
26	29.246	30.435	31.795	35.563	38.885	41.923	42.856	45.642	48.290	54.051
27	30.319	31.528	32.912	36.741	40.113	43.195	44.140	46.963	49.645	55.475
28	31.391	32.620	34.027	37.916	41.337	44.461	45.419	48.278	50.994	56.892
29	32.461	33.711	35.139	39.087	42.557	45.722	46.693	49.588	52.335	58.301
30	33.530	34.800	36.250	40.256	43.773	46.979	47.962	50.892	53.672	59.702
40	44.165	45.616	47.269	51.805	55.758	59.342	60.436	63.691	66.766	73.403
50	54.723	56.334	58.164	63.167	67.505	71.420	72.613	76.154	79.490	86.660
60	65.226	66.981	68.972	74.397	79.082	83.298	84.58	88.379	91.952	99.608

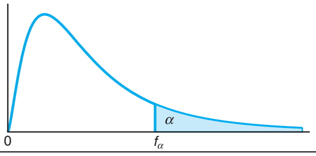

Table A.6 Critical Values of the *F*-Distribution

	$f_{0.05}(v_1, v_2)$								
	v_1								
v_2	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88

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Table A.6 (continued) Critical Values of the F-Distribution

	$f_{0.05}(v_1, v_2)$									
	v_1									
v_2	**10**	**12**	**15**	**20**	**24**	**30**	**40**	**60**	**120**	∞
1	241.88	243.91	245.95	248.01	249.05	250.10	251.14	252.20	253.25	254.31
2	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
6	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

Table A.6 (continued) Critical Values of the *F*-Distribution

	$f_{0.01}(v_1, v_2)$								
	v_1								
v_2	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	4052.18	4999.50	5403.35	5624.58	5763.65	5858.99	5928.36	5981.07	6022.47
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41

Table A.6 (continued) Critical Values of the F-Distribution

	$f_{0.01}(v_1, v_2)$									
	v_1									
v_2	**10**	**12**	**15**	**20**	**24**	**30**	**40**	**60**	**120**	∞
1	6055.85	6106.32	6157.28	6208.73	6234.63	6260.65	6286.78	6313.03	6339.39	6365.86
2	99.40	99.42	99.43	99.45	99.46	99.47	99.47	99.48	99.49	99.50
3	27.23	27.05	26.87	26.69	26.60	26.50	26.41	26.32	26.22	26.13
4	14.55	14.37	14.20	14.02	13.93	13.84	13.75	13.65	13.56	13.46
5	10.05	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
6	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
7	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74	5.65
8	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86
9	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
10	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
11	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69	3.60
12	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
13	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25	3.17
14	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09	3.00
15	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87
16	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93	2.84	2.75
17	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83	2.75	2.65
18	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66	2.57
19	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58	2.49
20	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
21	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55	2.46	2.36
22	3.26	3.12	2.98	2.83	2.75	2.67	2.58	2.50	2.40	2.31
23	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.45	2.35	2.26
24	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
25	3.13	2.99	2.85	2.70	2.62	2.54	2.45	2.36	2.27	2.17
26	3.09	2.96	2.81	2.66	2.58	2.50	2.42	2.33	2.23	2.13
27	3.06	2.93	2.78	2.63	2.55	2.47	2.38	2.29	2.20	2.10
28	3.03	2.90	2.75	2.60	2.52	2.44	2.35	2.26	2.17	2.06
29	3.00	2.87	2.73	2.57	2.49	2.41	2.33	2.23	2.14	2.03
30	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
40	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80
60	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
120	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
∞	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.00