الفصل الأول	316 ريض	قسم الرياضيات
2011-2010	حلول الاختبار النهائي	جامعة الملك سعود

1.

$$
\begin{aligned}
\left\langle ae^{x}+be^{-x},x\right\rangle &= a\int_{-1}^{1}xe^{x}dx+b\int_{-1}^{1}xe^{-x}dx\\
&= a\left[xe^{x}\Big|_{-1}^{1}-\int_{-1}^{1}e^{x}dx\right]+b\left[-xe^{-x}\Big|_{-1}^{1}+\int_{-1}^{1}e^{-x}dx\right]\\
&= a\left[e^{1}+e^{-1}-e^{1}+e^{-1}\right]+\left[-e^{-1}-e^{1}-e^{-1}+e^{1}\right]\\
&= 2e^{-1}a-2e^{-1}b=0 \qquad \Rightarrow a=b
\end{aligned}
$$

$$
\begin{aligned}
\left\|ae^{x}+be^{-x}\right\|^{2} &= a^{2}\int_{-1}^{1}(e^{2x}+2+e^{-2x})\,dx=a^{2}\left[\tfrac{1}{2}(e^{2}-e^{-2})+2(2)-\tfrac{1}{2}(e^{-2}-e^{2})\right]\\
&= a^{2}\left[e^{2}-e^{-2}+4\right]=1
\end{aligned}
$$

$$
\Rightarrow a=\pm\frac{1}{\sqrt{2\sinh 2+4}}
$$

2.

$$
\begin{aligned}
a_{0} &= \tfrac{1}{2}\int_{-1}^{1}(x+1)\,dx=\frac{1}{2}\cdot 2=1\\
a_{n} &= \int_{-1}^{1}(x+1)\cos n\pi x\,dx=(x+1)\frac{1}{n\pi}\sin n\pi x\Bigg|_{-1}^{1}-\frac{1}{n\pi}\int_{-1}^{1}\sin n\pi x\,dx\\
&= \frac{1}{n^{2}\pi^{2}}\cos n\pi x\Bigg|_{-1}^{1}=0 \quad \forall\, n=1,2,\ldots\\
b_{n} &= \int_{-1}^{1}(x+1)\sin n\pi\,dx=-(x+1)\frac{1}{n\pi}\cos n\pi x\Bigg|_{-1}^{1}+\frac{1}{n\pi}\int_{-1}^{1}\cos n\pi x\,dx\\
&= \frac{-2}{n\pi}\cos n+\frac{1}{n^{2}\pi^{2}}\sin n\pi x\Bigg|_{-1}^{1}=\frac{2}{n\pi}(-1)^{n+1}
\end{aligned}
$$

$$
\Rightarrow x+1=1+\frac{2}{\pi}\sum_{1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi x
$$

التقارب غير منتظم لأن $f(-1)\neq f(1)$.

3.

$$\left\langle e^{-x/2}, L_n \right\rangle = \int_0^\infty e^{-\frac{x}{2}} L_n(x) e^{-x}\, dx$$
$$= \frac{1}{n!}\int_0^\infty e^{-\frac{x}{2}} \frac{d^n}{dx^n}(x^n e^{-x})\, dx$$
$$= \frac{1}{n!}\left[e^{-\frac{x}{2}} \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x}) \Big|_0^\infty + \frac{1}{2}\int_0^\infty e^{-\frac{x}{2}} \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})\, dx \right]$$
$$= \cdots$$
$$= \frac{1}{n!2^n}\int_0^\infty e^{-\frac{x}{2}} x^n e^{-x} dx$$
$$= \frac{1}{n!2^n}\int_0^\infty x^n e^{\frac{-3x}{2}}\, dx$$

$$\int_0^\infty x^n e^{\frac{-3x}{2}}\, dx = -\frac{2}{3} x^n e^{\frac{-3x}{2}} \Big|_0^\infty + \frac{2}{3} n \int_0^\infty x^{n-1} e^{\frac{-3x}{2}}\, dx$$
$$= \cdots$$
$$= \left(\frac{2}{3}\right)^n n! \int_0^\infty e^{\frac{-3x}{2}}\, dx$$
$$= \left(\frac{2}{3}\right)^{n+1} n!$$

$$\therefore\ \left\langle e^{-\frac{x}{2}}, L_n \right\rangle = \frac{2}{3^{n+1}}$$

$$e^{-x/2} = \sum_{n=0}^{\infty} \frac{\left\langle e^{-x/2}, L_n \right\rangle}{\|L_n\|^2} L_n(x) = 2\sum_{n=0}^{\infty} 3^{-n-1} L_n(x).$$

4.

$$y(x) = x^{-\frac{1}{2}} u$$
$$y' = -\frac{1}{2} x^{-\frac{3}{2}} u + x^{-\frac{1}{2}} u'$$
$$y'' = \frac{3}{4} x^{-\frac{5}{2}} u - x^{-\frac{3}{2}} u' + x^{-\frac{1}{2}} u''$$

$$\therefore\ x^2\left(\frac{3}{4} x^{-\frac{5}{2}} u - x^{-\frac{3}{2}} u' + x^{-\frac{1}{2}} u''\right) + x\left(-\frac{1}{2} x^{-\frac{3}{2}} u + x^{-\frac{1}{2}} u'\right) + \left(x^2 - \frac{1}{4}\right) x^{-\frac{1}{2}} u = 0$$

$$x^{\frac{3}{2}}\left(u'' + u\right) = 0 \Rightarrow u'' + u = 0$$

$$u(x) = c_1 \cos x + c_2 \sin x \qquad \therefore\ y(x) = c_1 \frac{\cos x}{\sqrt{x}} + c_2 \frac{\sin x}{\sqrt{x}}$$

5.

$$\begin{aligned}
J_{-\frac{1}{2}}(x) &= \left(\frac{x}{2}\right)^{-\frac{1}{2}} \sum_{0}^{\infty} \frac{(-1)^m}{m!\,\Gamma\left(m+\frac{1}{2}\right)} \left(\frac{x}{2}\right)^{2m} \\
&= \sqrt{\frac{2}{x}} \sum_{0}^{\infty} \frac{(-1)^m}{m!\left(m-\frac{1}{2}\right)\left(m-\frac{3}{2}\right)\cdots\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \left(\frac{x}{2}\right)^{2m} \\
&= \sqrt{\frac{2}{x}} \sum_{0}^{\infty} \frac{(-1)^m x^{2m}}{m!(2m-1)(2m-3)\cdots(1)\sqrt{\pi}} \frac{1}{2^m} \\
&= \sqrt{\frac{2}{\pi x}} \sum_{0}^{\infty} \frac{(-1)^m x^{2m}}{2m(2m-2)(2m-3)\cdots(2)\cdot(2m-1)\cdots(1)} \\
&= \sqrt{\frac{2}{\pi x}} \sum_{0}^{\infty} \frac{(-1)}{(2m)!} x^{2m} \\
&= \sqrt{\frac{2}{\pi x}} \cos x.
\end{aligned}$$

6.

odd function, $\therefore\ A(\xi) = 0$

$$\begin{aligned}
B(\xi) &= 2\int_0^{\pi} \sin x \sin x\xi \, d\xi = \int_0^{\pi} \left[\cos(1-\xi)x + \cos(1+\xi)x\right] dx \\
&= \left[\frac{\sin(1-\xi)x}{1-\xi} - \frac{\sin(1+\xi)x}{1+\xi}\right]_0^{\pi} = -\frac{\cos\pi \sin\pi\xi}{1-\xi} - \frac{\cos\pi \sin\pi\xi}{1+\xi} = 2\frac{\sin\pi\xi}{1-\xi^2}
\end{aligned}$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{\sin\pi\xi}{1-\xi^2} \sin x\xi \, d\xi$$

at $x = \pi/2$, $\quad 1 = \frac{2}{\pi}\int_0^{\infty} \frac{\sin\pi\xi \sin\frac{1}{2}\pi\xi}{1-\xi^2} d\xi \quad \Rightarrow \quad \pi = 2\int_0^{\infty} \frac{\sin\pi\xi \sin\frac{1}{2}\pi\xi}{1-\xi^2} d\xi\,.$