KING SAUD UNIVERSITY
College of Sciences, Mathematics Department
Bachelor AFM, Riyadh

Academic Year 2016/2017
Stochastic Processes: Math: 380
Mhamed Eddahbi

Exercises with solutions on Markov chain and applications

1. Recall the definition of a Markov chain (do it at home)

2. Give the form of the transition matrix of a general Markov chain and give an example (do it at home)

3. Give the form of the transition matrix of an homogeneous Markov chain and give an example (do it at home)

4. Consider a Markov chain with $E = \{0, 1, 2\}$ and

$$P(6,7) = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}$$

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(a) Find: (i) $P(X_7 = j \mid X_6 = 0)$, $j = 0, 1, 2$. (ii) $P(X_7 = 1 \mid X_6 = j)$, $j = 0, 1, 2$.

(b) Which of the following are legitimate one–step transition probability matrices

$$\text{i) } P_0 = \begin{pmatrix} -1 & 2 \\ 0.5 & 0.5 \end{pmatrix}, \quad \text{ii) } Q_0 = \begin{pmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{pmatrix} \quad \text{and} \quad R_0 = \begin{pmatrix} 0.3 & 0.7 \\ 0.5 & 0.6 \end{pmatrix}$$

Solution: a. (i) We have that $P(X_7 = 0 \mid X_6 = 0) = 0.2$, $P(X_7 = 1 \mid X_6 = 0) = 0.3$ and $P(X_7 = 2 \mid X_6 = 0) = 0.5$.
(ii) We have that $P(X_7 = 1 \mid X_6 = 0) = 0.3$, $P(X_7 = 1 \mid X_6 = 1) = 0.5$ and $P(X_7 = 1 \mid X_6 = 2) = 0.1$.
b. (i) The matrix is not a legitimate transition probability because the entry $(1, 1)$ is negative.
ii) Yes Q_0 is a transition probability matrix.
iii) The matrix is not a legitimate transition probability because the elements of the second row do not add to one.

5. A computer system can operate in two different modes. Every hour, it remains in the same mode or switches to a different mode according to the transition probability matrix

$$P = \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix}$$

(a) Compute the 2–step transition probability matrix.
Solution: Since the Markov chain is homogeneous 2–step transition probability matrix is given by

$$P^{(2)} = P^2 = \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.52 & 0.48 \\ 0.48 & 0.52 \end{pmatrix}$$

(b) If the system is in Mode I at 5:30 pm, what is the probability that it will be in Mode I at 8:30 pm on the same day?
Solution: We have

$$P(X_{8:30} = j \mid X_{5:30} = i) = P(X_3 = j \mid X_0 = i) = P_{i,j}^{(3)}$$

therefore

$$P^{(3)} = P^3 = \begin{pmatrix} 0.52 & 0.48 \\ 0.48 & 0.52 \end{pmatrix} \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.496 & 0.504 \\ 0.504 & 0.496 \end{pmatrix}.$$

So $P_{i,i}^{(3)} = 0.496$ for $i = 1, 2$.

6. Consider a Markov chain with $E = \{0, 1, 2\}$, $\alpha_0 = (0.3, 0.4, 0.3)$,

$$P(0,1) = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix} \quad \text{and} \quad P(1,2) = \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.5 & 0.3 \end{pmatrix}$$

(a) Calculate $P(X_0 = 1)$, $P(X_1 = 1)$, $P(X_2 = 1)$,
Solution: We have $\alpha_0 = (P(X_0 = 0), P(X_0 = 1), P(X_0 = 2))$ then $P(X_0 = 1) = 0.4$. We know that $P(X_1 = 1) = \alpha_1(1)$ where

$$\begin{aligned} \alpha_1 &= \alpha_0 P(0,1) = (0.3, 0.4, 0.3) \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix} \\ \alpha_1 &= (0.21, 0.32, 0.47) \quad \text{and } P(X_1 = 1) = 0.32 \end{aligned}$$

Also $P(X_2 = 1) = \alpha_2(1)$ where

$$\begin{aligned} \alpha_2 &= \alpha_1 P(1,2) = (0.21, 0.32, 0.47) \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.5 & 0.3 \end{pmatrix} \\ \alpha_2 &= (0.211, 0.416, 0.373) \quad \text{and } P(X_2 = 1) = 0.416 \end{aligned}$$

(b) Calculate $P(X_0 = 1, X_1 = 2)$, $P(X_0 = 1, X_1 = 0)$, $P(X_0 = 0, X_2 = 2)$.
Solution: We have

$$\begin{aligned} P(X_0 = 1, X_1 = 2) &= P(X_1 = 2 \mid X_0 = 1) P(X_0 = 1) \\ &= P_{1,2}(0,1)\alpha_0(1) = 0.2 \times 0.4 = 0.08 \end{aligned}$$

and

$$\begin{aligned} P(X_0 = 1, X_1 = 0) &= P(X_1 = 0 \mid X_0 = 1) P(X_0 = 1) \\ &= P_{1,0}(0,1)\alpha_0(1) = 0.3 \times 0.4 = 0.12 \end{aligned}$$

$$\begin{aligned} P(X_0 = 0, X_2 = 2) &= P(X_2 = 2 \mid X_0 = 0) P(X_0 = 0) \\ &= P_{0,2}(0,2)\alpha_0(0) \end{aligned}$$

and

$$\begin{aligned} P(0,2) &= P(0,1)P(1,2) \\ &= \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.5 & 0.3 \end{pmatrix} \\ &= \begin{pmatrix} 0.21 & 0.42 & 0.37 \\ 0.22 & 0.38 & 0.4 \\ 0.2 & 0.46 & 0.34 \end{pmatrix} \end{aligned}$$

then $P_{0,2}(0,2)\alpha_0(0) = 0.37 \times 0.3 = 0.111$

(c) Calculate $P(X_0 = 1, X_1 = 2, X_2 = 2)$, $P(X_0 = 2, X_1 = 1, X_2 = 0)$.
Solution: We can write

$$\begin{aligned} P(X_0 = 1, X_1 = 2, X_2 = 2) &= P(X_2 = 2 \mid X_0 = 1, X_1 = 2) P(X_0 = 1, X_1 = 2) \\ &= P(X_2 = 2 \mid X_1 = 2) \times 0.08 \\ &= P_{2,2}(1,2) \times 0.08 = 0.3 \times 0.08. \end{aligned}$$

And

$$\begin{aligned} P(X_0 = 2, X_1 = 1, X_2 = 0) &= P(X_2 = 0 \mid X_0 = 2, X_1 = 1) P(X_1 = 1 \mid X_0 = 2) P(X_0 = 2) \\ &= P(X_2 = 0 \mid X_1 = 1) P(X_1 = 1 \mid X_0 = 2) \alpha_0(2) \\ &= P_{1,0}(1,2) \times P_{2,1}(0,1) \times \alpha_0(2) = 0.1 \times 0.1 \times 0.3 = 0.003. \end{aligned}$$

7. Consider a Markov chain with $E = \{1, 2\}$,

$$P(0,1) = \begin{pmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{pmatrix}, \quad P(1,2) = \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix}$$

Suppose that $X_0 = 1$. Find the probability that at stage 2 the chain is in state 2.
Solution: We should find $P(X_2 = 2) = \alpha_2(2)$ then we need α_2 which is given by $\alpha_2 = \alpha_1 P(1,2) = \alpha_0 P(0,1) P(1,2)$. But $\alpha_0 = (P(X_0 = 1), P(X_0 = 2)) = (1, 0)$

$$(1,0) \begin{pmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix} = (0.48, 0.52)$$

Then $P(X_2 = 2) = 0.52$.

8. Consider a Markov chain with $E = \{1, 2\}$, $\alpha_3 = (0.2, 0.8)$,

$$P(3,4) = \begin{pmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{pmatrix}, \quad P(4,5) = \begin{pmatrix} 0.5 & 0.5 \\ 0.8 & 0.2 \end{pmatrix}$$

(a) Find $P(X_3 = 2, X_4 = 1)$, $P(X_3 = 1, X_4 = 1, X_5 = 2)$, $P(X_5 = 2)$.
Solution: We have

$$P(X_3 = 2, X_4 = 1) = P(X_4 = 1 \mid X_3 = 2) P(X_3 = 2) = P_{2,1}(3,4) \alpha_3(2) = 0.6 \times 0.8.$$

We can write by definition.

$$\begin{aligned} P(X_3 = 1, X_4 = 1, X_5 = 2) &= P(X_5 = 2 \mid X_4 = 1, X_3 = 1) P(X_4 = 1, X_3 = 1) \\ &= P(X_5 = 2 \mid X_4 = 1) P(X_4 = 1 \mid X_3 = 1) P(X_3 = 1) \\ &= P_{1.2}(4,5) P_{1,1}(3,4) \alpha_3(1) = 0.5 \times 0.3 \times 0.2. \end{aligned}$$

To compute $P(X_5 = 2) = \alpha_5(2)$ we need α_5 which is given by

$$\alpha_5 = \alpha_4 P(4,5) = \alpha_3 P(3,4) P(4,5)$$

Then

$$\alpha_5 = (0.2, 0.8) \begin{pmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{pmatrix} \begin{pmatrix} 0.5 & 0.5 \\ 0.8 & 0.2 \end{pmatrix} = (0.638, 0.362)$$

and $\alpha_5(2) = 0.362$.

9. Consider a Markov chain with $E = \{1, 2\}$,

$$P(3,4) = \begin{pmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{pmatrix}, \quad P(4,5) = \begin{pmatrix} 0.5 & 0.5 \\ 0.8 & 0.2 \end{pmatrix}$$

(a) Find $P(X_4 = 2 \mid X_3 = 1)$, $P(X_4 = 2, X_5 = 1 \mid X_3 = 1)$, $P(X_5 = 1 \mid X_3 = 1)$.
Solution: We have

$$P(X_4 = 2 \mid X_3 = 1) = P_{1,2}(3,4) = 0.7$$

and

$$\begin{aligned} P(X_4 = 2, X_5 = 1 \mid X_3 = 1) &= P(X_5 = 1 \mid X_4 = 2, X_3 = 1)\, P(X_4 = 2 \mid X_3 = 1) \\ &= P(X_5 = 1 \mid X_4 = 2)\, P(X_4 = 2 \mid X_3 = 1) \\ &= P_{2,1}(4,5) P_{1,2}(3,4) = 0.8 \times 0.7 = 0.56 \end{aligned}$$

and

$$P(X_5 = 1 \mid X_3 = 1) = P_{1,1}(3,5)$$

so we need the matrix $P(3,5)$ which is given by

$$\begin{aligned} P(3,5) &= P(3,4)P(4,5) \\ &= \begin{pmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{pmatrix} \begin{pmatrix} 0.5 & 0.5 \\ 0.8 & 0.2 \end{pmatrix} \\ &= \begin{pmatrix} 0.71 & 0.29 \\ 0.62 & 0.38 \end{pmatrix} \end{aligned}$$

then $P(X_5 = 1 \mid X_3 = 1) = 0.71$.

(b) Set

$$P(7,8) = \begin{pmatrix} 0.3 & 0.7 \\ 0.2 & 0.8 \end{pmatrix}, \quad P(8,9) = \begin{pmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{pmatrix}$$

Find $P(X_9 = 1 \mid X_5 = 2,\ X_7 = 1)$.
Solution: From the definition of the Markov chain we can write

$$P(X_9 = 1 \mid X_5 = 2,\ X_7 = 1) = P(X_9 = 1 \mid X_7 = 1) = P_{1,1}(7,9)$$

so we need the matrix $P(7,9)$ which is given by

$$\begin{aligned} P(7,9) &= P(7,8)P(8,9) \\ &= \begin{pmatrix} 0.3 & 0.7 \\ 0.2 & 0.8 \end{pmatrix} \begin{pmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{pmatrix} \\ &= \begin{pmatrix} 0.53 & 0.47 \\ 0.52 & 0.48 \end{pmatrix} \end{aligned}$$

then $P(X_9 = 1 \mid X_5 = 2,\ X_7 = 1) = 0.53$.

10. Suppose that an homogeneous Markov chain has state space $E = \{1, 2, 3\}$, transition matrix

$$P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

and initial distribution $\alpha_0 = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$.

(a) Find: $P\left(X_2=2\right)$, $P\left(X_0=1, X_3=3\right)$,
Solution: $P\left(X_2=2\right)=\alpha_2(2)$ which can be calculated easily from the formula

$$\alpha_2=\alpha_1 P(1,2)=\alpha_0 P(0,1) P(1,2)=\alpha_0 P^2$$

then

$$\left(\begin{array}{ccc}\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2}\end{array}\right)^2=\left(\begin{array}{ccc}\frac{4}{9} & \frac{2}{9} & \frac{1}{3} \\ \frac{7}{24} & \frac{11}{24} & \frac{1}{4} \\ \frac{1}{3} & \frac{7}{24} & \frac{3}{8}\end{array}\right)$$

and

$$\alpha_2=\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)\left(\begin{array}{ccc}\frac{4}{9} & \frac{2}{9} & \frac{1}{3} \\ \frac{7}{24} & \frac{11}{24} & \frac{1}{4} \\ \frac{1}{3} & \frac{7}{24} & \frac{3}{8}\end{array}\right)=\left(\frac{3}{8}, \frac{5}{16}, \frac{5}{16}\right)$$

then : $\alpha_2(2)=\frac{5}{16}$.

$$P\left(X_0=1, X_3=3\right)=P\left(X_3=3 \mid X_0=1\right) P\left(X_0=1\right)=\alpha_0(1) P_{1,3}(0,3)$$

hence we need to compute $P(0,3)=P(0,1) P(1,2) P(2,3)=P^3$. Now we have

$$\left(\begin{array}{ccc}\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2}\end{array}\right)^3=\left(\begin{array}{ccc}\frac{37}{108} & \frac{41}{108} & \frac{5}{18} \\ \frac{7}{18} & \frac{37}{144} & \frac{17}{48} \\ \frac{101}{288} & \frac{91}{288} & \frac{1}{3}\end{array}\right)$$

hence $P\left(X_0=1, X_3=3\right)=\frac{1}{2} \times \frac{5}{18}=\frac{5}{36}$.

(b) Find $P\left(X_1=2, X_2=3, X_3=1 \mid X_0=1\right)$, $P\left(X_2=3 \mid X_3=1\right)$
Solution:

$$\begin{aligned}
& P\left(X_1=2, X_2=3, X_3=1 \mid X_0=1\right) \\
= & \frac{P\left(X_1=2, X_2=3, X_3=1, X_0=1\right)}{P\left(X_0=1\right)} \\
= & \frac{P\left(X_3=1 \mid X_0=1, X_1=2, X_2=3\right) P\left(X_2=3 \mid X_0=1, X_1=2\right) P\left(X_0=1, X_1=2\right)}{P\left(X_0=1\right)} \\
= & P\left(X_3=1 \mid X_2=3\right) P\left(X_2=3 \mid X_1=2\right) P\left(X_1=2 \mid X_0=1\right) \\
= & P_{3,1} P_{2,3} P_{1,2}=\frac{1}{4} \times \frac{1}{2} \times \frac{2}{3}=\frac{1}{12}
\end{aligned}$$

From the definition of conditional probability we have

$$\begin{aligned}
P\left(X_2=3 \mid X_3=1\right) & =\frac{P\left(X_2=3,\ X_3=1\right)}{P\left(X_3=1\right)}=\frac{P\left(X_3=1 \mid X_2=3\right) P\left(X_2=3\right)}{P\left(X_3=1\right)} \\
& =\frac{1}{\alpha_3(1)} P_{3,1}(2,3) \alpha_2(3) .
\end{aligned}$$

Moreover

$$\alpha_3=\alpha_2 P=\left(\frac{3}{8}, \frac{5}{16}, \frac{5}{16}\right)\left(\begin{array}{ccc}\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2}\end{array}\right)=\left(\frac{23}{64}, \frac{21}{64}, \frac{5}{16}\right)$$

thence

$$P\left(X_2=3 \mid X_3=1\right)=\frac{1}{\frac{23}{64}} \frac{1}{4} \frac{5}{16}=\frac{5}{23}$$

(c) Find $P(X_{12}=1 \mid X_5=3, X_{10}=1)$, $P(X_3=3, X_5=1 \mid X_0=1)$, $P(X_3=3 \mid X_0=1)$
Solution: The the Markov property of the chain leads to

$$\begin{aligned} P(X_{12}=1 \mid X_5=3, X_{10}=1) &= P(X_{12}=1 \mid X_{10}=1) \\ &= P_{1,1}(10,12). \end{aligned}$$

This the (1,1) entry of the 2–step transition probability matrix $P(10,12) = P(10,11)P(11,12) = P^2$ which was computed in a. then $P_{1,1}(10,12) = P^2_{1,1} = \frac{37}{108}$.
From the properties of the conditional probability we can write

$$\begin{aligned} P(X_3=3, X_5=1 \mid X_0=1) &= P(X_5=1 \mid X_3=3,\ X_0=1)\, P(X_3=3 \mid X_0=1) \\ &= P(X_5=1 \mid X_3=3)\, P(X_3=3 \mid X_0=1) \\ &= P_{3,1}(3,5)P_{1,3}(0,3) \end{aligned}$$

So we need the two matrix $P(3,5)$ and $P(0,3)$, but we know that $P(3,5) = P(3,4)P(4,5) = P^2$ and $P(0,3) = P(0,1)P(1,2)P(2,3) = P^3$ since the chain is homogeneous, therefore $P_{3,1}(3,5) = P^2_{3,1} = \frac{1}{3}$ and $P_{1,3}(0,3) = P^3_{1,3} = \frac{5}{18}$, finally $P(X_3=3, X_5=1 \mid X_0=1) = \frac{1}{3}\frac{5}{18} = \frac{5}{54}$.
$P(X_3=3 \mid X_0=1) = P_{1,3}(0,3) = P^3_{1,3} = \frac{5}{18}$.

11. A Markov chain $\{X_n, n \geq 0\}$ with states $0, 1, 2$, has the transition probability matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

(a) Draw a state transition diagram of the Markov chain

(b) If $P(X_0=0) = P(X_0=1) = \frac{1}{4}$, find $E[X_3]$.
Solution: We have

$$\begin{aligned} E[X_3] &= \sum_{i=0}^{2} iP(X_3=i) \\ &= P(X_3=1) + 2P(X_3=2) \\ &= \alpha_3(1) + 2\alpha_3(2) \end{aligned}$$

$$P^3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}^3 = \begin{pmatrix} \frac{13}{36} & \frac{11}{54} & \frac{47}{108} \\ \frac{4}{9} & \frac{4}{27} & \frac{11}{27} \\ \frac{5}{12} & \frac{2}{9} & \frac{13}{36} \end{pmatrix}$$

then $\alpha_3 = \alpha_0 P^3$

$$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right) \begin{pmatrix} \frac{13}{36} & \frac{11}{54} & \frac{47}{108} \\ \frac{4}{9} & \frac{4}{27} & \frac{11}{27} \\ \frac{5}{12} & \frac{2}{9} & \frac{13}{36} \end{pmatrix} = \left(\frac{59}{144}, \frac{43}{216}, \frac{169}{432}\right)$$

then $\alpha_3(1) = \frac{43}{216}$ and $\alpha_3(2) = \frac{169}{432}$, then $E[X_3] = \frac{43}{216} + 2\frac{169}{432} = \frac{373}{144}$.

12. A system has three possible states, 0, 1 and 2. Every hour it makes a transition to a different state, which is determined by a coin flip. For example, from state 0, it makes a transition to state 1 or state 2 with probabilities 0.5 and 0.5.

(a) Find the transition probability matrix.
Solution: If the coin is fair the transition matrix is given by

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

If the coin is not fair the transition matrix is given by

$$P = \begin{pmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{pmatrix}$$

(b) Find the three–step transition probability matrix.
Solution: If the coin is fair three–step transition probability matrix is given by

$$P^3 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}^3 = \begin{pmatrix} \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \end{pmatrix}$$

If the coin is not fair three–step transition probability matrix

$$P^3 = \begin{pmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{pmatrix}^3 = \begin{pmatrix} p^3-(p-1)^3 & -3p^2(p-1) & 3p(p-1)^2 \\ 3p(p-1)^2 & p^3-(p-1)^3 & -3p^2(p-1) \\ -3p^2(p-1) & 3p(p-1)^2 & p^3-(p-1)^3 \end{pmatrix}$$

13. Let $\{X_n, n \geq 0\}$ be a sequence of random variables taking values is $\{1, 2, 3\}$. Let $X_0 = 2$ and let Q be a matrix given by

$$Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

(a) Can you deduce $P(X_n = 1 \mid X_{n-1} = 2)$ and $P(X_n = 3 \mid X_{n-1} = 3)$ from the matrix Q ?

(b) Find if possible $P(X_5 = 1 \mid X_4 = 3)$ and $P(X_2 = 2 \mid X_1 = 2)$?
Solution: There no thing saying that $\{X_n, n \geq 0\}$ is a Markov chain and Q is its transition probability matrix.

14. Consider the Markov chain on states 1, 2, 3 with transition probability matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

(a) Find the communicating classes of this M.C.
Solution: We have $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 2 \longrightarrow 1$ hence $\{1, 2, 3\}$ is the communicating.

(b) Is it irreducible ?
Solution: There only one communicating class of this Markov chain. Therefore the chain recurrent and irreducible.

15. Consider the Markov chain on states 1, 2, 3, 4 with transition probability matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Find the communicating classes of this M.C.

Solution: We have $1 \longrightarrow 2 \longrightarrow 1$ and $3 \longrightarrow 4$ hence $\{1,2\}$, $\{3\}$ and $\{4\}$ are the three communicating classes of the Markov chain. The class $\{1,2\}$ is aperiodic and the classes $\{1,2\}$, $\{4\}$ recurrent, the state 4 is absorbing, and $\{3\}$ is transient.

(b) Is it irreducible ?
Solution: The Markov chain is not irreducible since it has three communicating classes.

16. Let $\{X_n, n \geq 0\}$ be a Markov chain taking values is $\{1,2,3\}$. Consider the following matrix

$$P = \begin{pmatrix} a & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & b & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & c \end{pmatrix}$$

(a) Find a, b and c such that P is a transition probability matrix $\{X_n, n \geq 0\}$.
Solution: We should have $a = b = c = 0$.

(b) Specify communicating classes of the Markov chain $\{X_n, n \geq 0\}$.
Solution: We have $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 1$ hence the chain has one communicating class $\{1,2,3\}$, then is irreducible.

(c) Determine whether theses classes are transient or recurrent
Solution: The class $\{1,2,3\}$ is recurrent.

(d) Do the same for the following matrices corresponding to some Markov chains

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

For P: Assume states are $1,2,3,4$. We have $1 \longrightarrow 4 \longrightarrow 3 \longrightarrow 2 \longrightarrow 4 \longrightarrow 3 \longrightarrow 1$ there is only one communicating class $\{1,2,3,4\}$. The Markov chain is irreducible
For Q: Assume states are $1,2,3,4,5$. We have $5 \longrightarrow 1 \longrightarrow 2 \longrightarrow 1$ and $4 \longrightarrow 3$ the classes are $\{1,2\}$, $\{3\}$ $\{4\}$, $\{5\}$. The class $\{1,2\}$ is recurrent and aperiodic $\{4\}$ and $\{5\}$ are transient $\{3\}$ is recurrent (absorbing), so the Markov chain is not irreducible
For R: Assume states are $1,2,3,4,5$. We have $1 \longrightarrow 3 \longrightarrow 1$ and $4 \longrightarrow 5 \longrightarrow 4$ the communicatings classes are $\{1,3\}$, $\{2\}$ $\{4,5\}$. So the chain is not irreducible. The two classes recurrent $\{1,3\}$ and $\{4,5\}$ are aperiodic and recurrent and the class $\{2\}$ is transient.