KING SAUD UNIVERSITY
College of Sciences, Mathematics Department
Bachelor AFM, Riyadh

Academic year 2016/2017
Course: QMF Actu. 468
Resp.: M. Eddahbi

Homework 3 Multi–asset one period model

We specify below the basic elements of a financial market with T periods:

- A finite probability space $\Omega = \{\omega_1, \ldots, \omega_k\}$ with k elements.
- A probability measure $\mathbb{P}$ on Ω, such that $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$.
- A riskless asset (a saving account) $S_t^0, t \in \{0, 1, 2, \ldots N\}$ such that $S_0^0 = 1$ with a constant interest rate r.
- A d–dimensional price process S_t, $t \in \{0, 1, 2, \ldots N\}$ where $S_t = (S_t^0, S_t^1, \ldots, S_t^d)$ and S_t^i stands for the price of the asset i at time t.

1. Consider the following model $k = 3$, $d = 1$, $r = \frac{1}{9}$

n	S_n^0	S_n^1		
		ω_1	ω_2	ω_3
0	1	5	5	5
1	$\frac{10}{9}$	$\frac{20}{3}$	$\frac{40}{9}$	$\frac{30}{9}$

Question: Is this model arbitrage free ?
Solution : If a RNPM $Q = (q_1, q_2, 1 - q_1 - q_2)$ exists then it should satisfy

$$\begin{aligned} E_Q\left[S_1^1\right] &= \frac{10}{9}S_0^1 \iff \frac{20}{3}q_1 + \frac{40}{9}q_2 + \frac{30}{9}(1 - q_1 - q_2) = 5\frac{10}{9} \\ &\iff 6q_1 + 4q_2 + 3(1 - q_1 - q_2) = 5 \\ &\iff 3q_1 + q_2 + 3 = 5 \iff q_2 = 2 - 3q_1. \end{aligned}$$

Then $Q = (q, 2 - 3q, 1 - q - 2 + 3q) = (q, 2 - 3q, 2q - 1)$ provided that

$$\begin{aligned} 0 &< q < 1;\ 0 < 2 - 3q < 1 \ \text{ and } \ 0 < 2q - 1 < 1 \\ &\Updownarrow \\ 0 &< q < 1;\ \frac{1}{3} < q < \frac{2}{3} \ \text{ and } \ \frac{1}{2} < q < 1. \end{aligned}$$

Therefore a RNPM $Q = (q, 2 - 3q, 2q - 1)$ exists if and only if $\frac{1}{2} < q < \frac{2}{3}$. So the model is arbitrage free. Remark that there is infinitely many RNPM.

2. Consider now, the following model: given by $k = 3$, $d = 2$, $r = \frac{1}{9}$ and the discounted price

n	S_n^0	$\widetilde{S}_n^1$			$\widetilde{S}_n^2$		
		ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
0	1	5	5	5	10	10	10
1	$\frac{10}{9}$	6	6	3	12	8	8

Question: Is this model arbitrage free ?
Solution : If a RNPM $Q = (q_1, q_2, 1 - q_1 - q_2)$ exists then it should satisfy

$$\text{i)} \quad E_Q\left[\widetilde{S}_1^1\right] = \widetilde{S}_0^1 \quad \text{and} \quad \text{ii)} \quad E_Q\left[\widetilde{S}_2^1\right] = \widetilde{S}_0^1.$$

A probability satisfying i) and ii) should satisfy

$$\begin{aligned} 6(q_1 + q_2) + 3(1 - q_1 - q_2) &= 5 \text{ and } 12q_1 + +8q_2 + 8(1 - q_1 - q_2) = 10 \\ &\Updownarrow \\ 3q_1 + 3q_2 &= 2 \text{ and } 4q_1 = 2 \Longleftrightarrow q_1 = \frac{1}{2},\ q_2 = \frac{1}{6} \text{ and } q_3 = \frac{1}{3}. \end{aligned}$$

So there exisist a unique RNPM $Q = (\frac{1}{2}; \frac{1}{6}; \frac{1}{3})$.

3. Consider the following model $\Omega := \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and that the volatility is given by

$$\sigma(\omega) = \begin{cases} h & \text{if } \omega \in \{\omega_1, \omega_2\} \\ \ell & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases}$$

where $0 < \ell < h < 1$ and l stands for low volatility whereas h stands for high volatility. The stock price S_1 is then modeled by:

$$S_1(\omega) = \begin{cases} S_0(1 + \sigma) & \text{if } \omega \in \{\omega_1, \omega_3\} \\ S_0(1 - \sigma) & \text{if } \omega \in \{\omega_2, \omega_4\} \end{cases}$$

where S_0 denotes the initial stock price.
The riskless asset is model by $S_0^0 = 1$ and $S_1^0 = 1 + r$.
Question: Is this model arbitrage free ?
Solution : If a RNPM $Q = (q_1, q_2, q_3, 1 - q_1 - q_2 - q_3)$ exists then it should satisfy

$$\begin{aligned} (1 + r) S_0 &= E_Q[S_1] \\ (1 + r) S_0 &= S_0(q_1(1 + h) + q_2(1 - h) + q_3(1 + \ell) + (1 - q_1 - q_2 - q_3)(1 - \ell)) \\ &\Updownarrow \\ \ell + r &= (h + \ell) q_1 - (h - \ell) q_2 + 2\ell q_3 \Longleftrightarrow q_1 = \frac{\ell + r + (h - \ell) q_2 - 2\ell q_3}{h + \ell} \end{aligned}$$

Therefore there infinitly many solutions under some conditions on r, h and ℓ. Then the mrket model may be free of arbitrage but not complete. See the notes of chapter 3.

Understanding questions (**See course notes for definitions**)

1. Give the definition of a portfolio in this market
2. Recall the self–financing property for this model
3. Give the definition of attainable payoffs for this model
4. Give the definition of a RNPM (risk neutral probability measure) in this setting.
5. Give the definition of a complete market
6. Give the definition of an incomplete market

Problem 1.

Assume that $T=1$ and let $(S_t^1)_{t\in\{0,1\}}$ be the price of a stock with initial price $S_0^1 = 100$ SAR and has two possible values a time $T=1$:

$$S_1^1 = \begin{cases} 200\ SAR \text{ with probability } p \\ 75\ SAR \text{ with probability } 1-p. \end{cases}$$

1. Denote by F the payoff of an European put option with strike price $K = 150\ SAR$. Give the value of F at time $T=1$.
   Solution :
   $$F = (K - S_T^1)^+ = (150 - S_1^1)^+ = \begin{cases} 0 & \text{if } S_1^1 = 200 \\ 75 & \text{if } S_1^1 = 75. \end{cases}$$

2. Find RNPM of the market $(S_t^0, S_t^1)_{t\in\{0,1\}}$ if it exists.
   Solution : If a RNPM $Q = (q, 1-q)$ exists then it should satisfy
   $$E_Q\left[\frac{S_1^1}{1+r}\right] = S_0^1 \Longleftrightarrow 200q + 75(1-q) = 100(1+r).$$
   Hence $125q = 25(1+4r)$, then $q = \frac{1+4r}{5}$ such that $0 < \frac{1+4r}{5} < 1$.

3. What are the values of r for which there is no arbitrages ?
   Solution: $-\frac{1}{4} < r < 1$.

4. Compute the price of the option at time 0 for a fixed r.
   Solution: For fixed r in $]-\frac{1}{4};1[$ a RNPM exists and the price of the put option at time 0 is given by
   $$P_0 = E_Q\left[\frac{(150 - S_1^1)^+}{1+r}\right] = \frac{1-q}{1+r}75 = 75 \times \frac{4}{5} \times \frac{1-r}{1+r} = 60\frac{1-r}{1+r}.$$

5. Is the option F attainable or not ? If yes find the replicating portfolio.
   Solution. The RNPM is unique hence the market model is complete, therefore the put option $(150 - S_1^1)^+$ is attainable. Then there exist a replicating portfolio $\phi = (\alpha, \Delta)$ such that $\alpha(1+r) + \Delta S_1^1 = (150 - S_1^1)^+$. This relationship leads to the following system
   $$\begin{cases} \alpha(1+r) + \Delta 200 = 0 \\ \alpha(1+r) + \Delta 75 = 75. \end{cases}$$
   Hence $\Delta = -\frac{75}{125} = -\frac{3}{5}$ and $\alpha = \frac{60}{1+r}$ the replicating portfolio is $\phi = (\frac{60}{1+r}, -\frac{3}{5})$.

Problem 2.

Now assume that $k=3$, $r=0$, $S_0^1 = 100$ SAR and assume that the stock S_1^1 can take the values 200 SAR, 150 SAR and 75 SAR.

1. Find RNPM if any for the model $(S_t^0, S_t^1)_{t\in\{0,1\}}$?
   Solution : If a RNPM $Q = (q_1, q_2, 1-q_1-q_2)$ exists then it should satisfy
   $$\begin{aligned} E_Q\left[S_1^1\right] &= S_0^1 \Longleftrightarrow 200q_1 + 150q_2 + 75(1-q_1-q_2) = 100 \\ &\Longleftrightarrow 125q_1 + 75q_2 = 25 \Longleftrightarrow 5q_1 + 3q_2 = 1 \Longleftrightarrow q_2 = \frac{1-5q_1}{3}. \end{aligned}$$
   and
   $$1 - q_1 - \frac{1-5q_1}{3} = \frac{2}{3}q_1 + \frac{2}{3}$$

then

$$\begin{aligned} q_3 &= 1 - q_1 - q_2 = 1 - q_1 - \frac{1-5q_1}{3} \\ &= \frac{2+2q_1}{3} \end{aligned}$$

hence $Q = (q, \frac{1-5q}{3}, \frac{2+2q}{3})$ such that $Q \in]0,1[^3$. Remark that there is infinitely many RNPM.

2. Is the market $(S_t^0, S_t^1)_{t\in\{0,1\}}$ arbitrage free ?
Solution: The market $(S_t^0, S_t^1)_{t\in\{0,1\}}$ arbitrage free because for risk neutral probability measures $Q = (q, \frac{1-5q}{3}, \frac{2+2q}{3})$ such that $0 < q < 1$, $0 < \frac{1-5q}{3} < 1$ and $0 < \frac{2+2q}{3} < 1$ which is equivalent $0 < q < \frac{1}{5}$ and $0 < q < \frac{1}{2}$, hence q should satisfy $0 < q < \frac{1}{5}$

3. Is the market $(S_t^0, S_t^1)_{t\in\{0,1\}}$ complete ?
Solution: The market is not complete because Q in not unique.

4. Find the set of attainable contingent claims.
Solution: Let F be a contingent claim on $\Omega = \{\omega_1, \omega_2, \omega_3\}$ with possible values x_1, x_2 and x_3. If F is attainable there there exist a replicating portfolio $\phi = (\alpha, \Delta)$ such that $\alpha + \Delta S_1^1 = F$ which leads to the following system

$$\begin{cases} \alpha + \Delta 200 = x_1 \\ \alpha + \Delta 150 = x_2 \\ \alpha + \Delta 75 = x_3. \end{cases}$$

Therefore

$$\begin{cases} \Delta = \dfrac{x_1 - x_2}{50} \\ \alpha = x_2 - \dfrac{x_1 - x_2}{50} 150 = 4x_2 - 3x_1 \\ 4x_2 - 3x_1 + \dfrac{x_1 - x_2}{50} 75 = x_3. \end{cases} \Longleftrightarrow \begin{cases} \Delta = \dfrac{x_1 - x_2}{50} \\ \alpha = 4x_2 - 3x_1 \\ \dfrac{5}{2}x_2 - \dfrac{3}{2}x_1 = x_3. \end{cases} \Longleftrightarrow \begin{cases} \Delta = \dfrac{x_1 - x_2}{50} \\ \alpha = 4x_2 - 3x_1 \\ 2x_3 = 5x_2 - 3x_1. \end{cases}$$

Consequently the set of all attainable contingent claim is given by $\{F = (x_1, x_2, \frac{5}{2}x_2 - \frac{3}{2}x_1);\ x_1, x_2 \in \mathbb{R}\}$.

5. Show that the value at time zero of an attainable claim is the same for all RNPM.
Solution: Let F be an attainable contingent claim then is of the $F = (x_1, x_2, \frac{5}{2}x_2 - \frac{3}{2}x_1)$. The value of F at time zero is given by

$$V_0 = E_Q[F] = qx_1 + x_2\left(\frac{1-5q}{3}\right) + \left(\frac{5}{2}x_2 - \frac{3}{2}x_1\right)\left(\frac{2+2q}{3}\right) = 2x_2 - x_1$$

which is independent from q.

Problem 3.

Consider now a second stock $(S_t^2)_{t\in\{0,1\}}$ with the values at time 1 are given by:

a) $S_0^2 = 50$ $S_1^2 = \begin{cases} 60 \text{ SAR with probability } p_1 \\ 60 \text{ SAR with probability } p_2 \\ 40 \text{ SAR with probability } p_3. \end{cases}$ and b) $S_0^2 = 5$ $S_1^2 = \begin{cases} 5 \text{ SAR with probability } p_1 \\ 3 \text{ SAR with probability } p_2 \\ 4 \text{ SAR with probability } p_3. \end{cases}$

1. Find a RNPM for the model $(S_t^0, S_t^1, S_t^2)_{t\in\{0,1\}}$.
   Solution: a) If a RNPM $Q = (q_1, q_2, 1 - q_1 - q_2)$ exists then it should satisfy

$$\text{i)}\quad E_Q\left[S_1^1\right] = S_0^1 \quad \text{and} \quad \text{ii)}\quad E_Q\left[S_1^2\right] = S_0^2.$$

   A probability satisfying i) is given by $Q = (q, \frac{1-5q}{3}, \frac{2+2q}{3})$ such that $0 < q < \frac{1}{5}$. But Q should satisfy also ii) then we get

$$60\left(q + \frac{1-5q}{3}\right) + 40\frac{2+2q}{3} = 50 \iff \frac{140}{3} - \frac{40}{3}q = 50$$

   which implies that $q = -\frac{1}{4} \notin]0, \frac{1}{5}[$. Hence there is no RNPM for the model $(S_t^0, S_t^1, S_t^2)_{t\in\{0,1\}}$.
   b) A probability satisfying i) is given by $Q = (q, \frac{1-5q}{3}, \frac{2+2q}{3})$ such that $0 < q < \frac{1}{5}$. But Q should satisfy also ii) then we get $\left(1 - \frac{5}{2}\right)\frac{1}{3} = -\frac{1}{2}$

$$5q + 3\left(\frac{1-5q}{3}\right) + 4\left(\frac{2+2q}{3}\right) = 5 \iff \frac{8}{3}q + \frac{11}{3} = 5 \iff q = \frac{1}{2}$$

   Therefore $Q = (\frac{1}{2}, -\frac{1}{2}, 1)$ impossible, there no RNPM.

2. Conclude:
   Solution: The market $(S_t^0, S_t^1, S_t^2)_{t\in\{0,1\}}$ presents arbitrage opportunities. Hence incomplete.