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Solutions of Exercises on Black–Scholes model and pricing financial derivatives

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Problem 1.

1. What is the price of a European call option on a non–dividend–paying stock when the stock follows the Black–Scholes model with current price is \$52, the strike price is \$50, the risk–free interest rate is 12% per annum, the volatility is 30% per annum, and the time to maturity is three months?
   Solution: Remember first that the important parameters d_1 and d_2 are given by
   $$d_1 = \frac{\ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \text{ you can also use } d_2 = d_1 - \sigma\sqrt{T}$$
   Therefore with our data we have
   $$d_1 = \frac{\ln(\frac{52}{50}) + (0.12 + \frac{0.3^2}{2})\frac{1}{4}}{0.3\sqrt{\frac{1}{4}}} = 0.5364 \;\text{ and }\; d_2 = 0.5364 - 0.15 = 0.3864$$
   and the price of the call option is given by
   $$\begin{aligned} C_0 &= S_0\mathbf{N}(d_1) - e^{-rT}K\mathbf{N}(d_2) \\ &= 52\mathbf{N}(0.5364) - e^{-0.12\times 0.25}50\mathbf{N}(0.3864) \\ &= 52\times 0.7041 - e^{-0.12\times 0.25}50\times 0.6504 = 5.0543 \end{aligned}$$

2. What is the price of a European put option on a non–dividend–paying stock when the stock price is \$69, the strike price is \$70, the risk–free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is six months?
   Solution: First recall the put price is given by
   $$P_0 = e^{-rT}K\mathbf{N}(-d_2) - S_0\mathbf{N}(-d_1)$$
   compute d_1 and d_2. We have
   $$d_1 = \frac{\ln(\frac{69}{70}) + (0.05 + \frac{0.35^2}{2})\frac{1}{2}}{0.35\sqrt{\frac{1}{2}}} = 0.16662 \;\text{ and }\; d_2 = 0.16662 - 0.35\sqrt{\frac{1}{2}} = -0.0809$$
   and the price of the put option is given by
   $$\begin{aligned} P_0 &= e^{-0.05\times 0.5}70\mathbf{N}(0.0809) - 69\mathbf{N}(-0.16662) \\ &= 70e^{-0.05\times 0.5} \times 0.5322 - 69\times 0.4338 = 6.4020. \end{aligned}$$

3. Show that the Black–Scholes formulas for call and put options satisfy call–put parity.
   Solution: We have
   $$\begin{aligned} C_0 - P_0 &= S_0\mathbf{N}(d_1) - e^{-rT}K\mathbf{N}(d_2) - e^{-rT}K\mathbf{N}(-d_2) + S_0\mathbf{N}(-d_1) \\ &= S_0\left(\mathbf{N}(d_1) + \mathbf{N}(-d_1)\right) - e^{-rT}K\left(\mathbf{N}(d_2) + \mathbf{N}(-d_2)\right). \end{aligned}$$
   But $\mathbf{N}(x) + \mathbf{N}(-x) = 1$ for all x. Hence
   $$C_0 - P_0 = S_0 - e^{-rT}K \;\Leftrightarrow\; C_0 + e^{-rT}K = S_0 + P_0.$$

Problem 2.

1. Consider an option on a non–dividend–paying stock when the stock price is \$30, the exercise price is \$29, the risk-free interest rate is 5% per annum, the volatility is 25% per annum, and the time to maturity is four months.

   (a) What is the price of the option if it is a European call?
   Solution:

   (b) What is the price of the option if it is a European put?
   Solution:

   (c) Verify that put–call parity holds.
   Solution:

2. The stock of XYZ currently sells for \$41 per share. The annual stock price volatility is 0.3, and the annual continuously compounded risk–free interest rate is 0.08. The stock pays no dividends.

   (a) Find the values of $\mathbf{N}(d_1)$ and $\mathbf{N}(d_2)$ in the Black–Scholes formula for the price of a call option on the stock with strike price \$40 and time to expiration of 3 months.
   Solution:

   (b) Find the Black–Scholes price of the call option.
   Solution:

3. Assume the Black–Scholes framework. For a dividend–paying stock and a European option on the stock, you are given the following information:
   The current stock price is \$58.96.
   The strike price of the option is \$60.00.
   The expected annual return on the stock is $\mu = 10\%$.
   The volatility is 20%.
   The continuously compounded risk–free rate is 6%.
   The continuously dividend yield is 5%.
   The expiration time is three months.

   (a) Calculate the price of the call.
   Solution: The Black–Scholes price of the call option is given by

$$C_0 = S_0 e^{-\delta T}\mathbf{N}\left(d_1\right) - K e^{-rT}\mathbf{N}\left(d_2\right),$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

then

$$d_1 = \frac{\ln\left(\frac{58.96}{60}\right) + \left(0.06 - 0.05 + \frac{0.2^2}{2}\right)\frac{1}{4}}{0.2 \times 0.5} = -0.0998$$

and

$$d_2 = -0.0998 - 0.2 \times 0.5 = -0.1998.$$

Then

$$C_0 = 58.96e^{-0.05\times 0.25} \times 0.46025 - 60e^{-0.06\times 0.25} \times 0.42082 = 1.9260.$$

For the put option

$$P_0 = K e^{-rT}\mathbf{N}\left(-d_2\right) - S_0 e^{-\delta T}\mathbf{N}\left(-d_1\right)$$

therefore

$$\begin{aligned} P_0 &= 60e^{-0.06\times 0.25}\mathbf{N}(0.1998) - 58.96e^{-0.05\times 0.25}\mathbf{N}(0.0998) \\ &= 60e^{-0.06\times 0.25} \times 0.57918 - 58.96e^{-0.05\times 0.25} \times 0.53975 \\ &= 2.8051 \end{aligned}$$

(b) Give the call–put parity:
Solution: It is given by

$$C_0 - P_0 = S_0 e^{-\delta T} - K e^{-rT}.$$

Numerically, we have $1.9260 - 2.8051 = -0.8791$ and $58.96e^{-0.05\times 0.25} - 60e^{-0.06\times 0.25} = -0.8791$ which are almost the same.
Remark that the pricing formulas do not depend on the expected return on the stock μ.

4. Assume the Black–Scholes framework. For a non–dividend–paying stock and a European option on the stock, you are given the following information:

- The current stock price is \$9.67.
- The strike price of the option is \$8.75.
- The volatility is 40%.
- The continuously compounded risk–free rate is 8%.
- The expiration time is three months.

(a) Calculate the price of the put.
Solution:

Problem 3.

1. One euro is currently trading for \$0.92. The dollar–denominated continuously compounded interest rate is 6% and the euro–denominated continuously compounded interest rate is 3.2%. Volatility is 10%.

(a) Find the Black–Scholes price of a 1–year dollar–denominated euro call with strike price of \$0.9/€.
Solution: The Black–Scholes pricing formula for a call on a currency is given by

$$C_0 = S_0 e^{-r_f T}\mathbf{N}(d_1) - e^{-rT} K \mathbf{N}(d_2)$$

$$d_1 = \frac{\ln(\frac{S_0}{K}) + (r - r_f + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \text{ you can also use } d_2 = d_1 - \sigma\sqrt{T}$$

Our input parameters are: $S_0 = \$0.92$, $K = \$0.9/€$, $r = 0.06$, $r_f = 0.032$ and $\sigma = 0.1$ and $T = 1$. Therefore substituting theses parameters in the Black–Scholes formula we get

$$d_1 = \frac{\ln(\frac{0.92}{0.90}) + (0.06 - 0.032 + \frac{0.1^2}{2})}{0.1} = 0.5498 \quad \text{and} \quad d_2 = 0.5498 - 0.1 = 0.4498$$

then the call price

$$\begin{aligned} C_0 &= 0.92e^{-0.032}\mathbf{N}(0.5498) - e^{-0.06}0.9\mathbf{N}(0.4498) \\ &= 0.92e^{-0.032} \times 0.7087 - 0.9e^{-0.06} \times 0.6735 = 0.0606. \end{aligned}$$

(b) Find the Black–Scholes price of a 1–year dollar–denominated euro put with strike price of \$0.9/€.
Solution: The Black–Scholes pricing formula for a put on a currency is given by

$$\begin{aligned} P_0 &= e^{-rT}K\mathbf{N}(-d_2) - S_0e^{-r_fT}\mathbf{N}(-d_1) \\ &= 0.9e^{-0.06} \times \mathbf{N}(-0.4498) - 0.92e^{-0.032}\mathbf{N}(-0.5498) \\ &= 0.9e^{-0.06} \times \mathbf{N}(-0.4498) - 0.92e^{-0.032} \times \mathbf{N}(-0.5498) \end{aligned}$$

2. Let $S = \$100$, $K = \$90$, $\sigma = 30\%$, $r = 8\%$, $\delta = 5\%$, and $T = 1$..

(a) What is the Black–Scholes call price?

(b) Now price a put where $S = \$90$, $K = \$100$, $\sigma = 30\%$, $r = 5\%$, $\delta = 8\%$, and $T = 1$.

(c) What is the link between your answers to (a) and (b)? Why?

3. The exchange rate is ¥95/€, the yen–denominated interest rate is 1.5%, the euro–denominated interest rate is 3.5%, and the exchange rate volatility is 10%.

(a) What is the price of a 90–strike yen–denominated euro put with 6 months to expiration?
Solution: The Black–Scholes pricing formula for a put on a currency is given by

$$P_0 = e^{-r_{€}T}K\mathbf{N}(-d_2) - S_0e^{-r_{¥}T}\mathbf{N}(-d_1)$$

where

$$d_1 = \frac{\ln(\frac{95}{90}) + (0.015 - 0.035 + \frac{0.1^2}{2})\frac{1}{2}}{0.1\sqrt{0.5}} = 0.659 \text{ and } d_2 = 0.659 - 0.1\sqrt{0.5} = 0.588$$

then the put

$$\begin{aligned} P_0 &= e^{-0.015\times0.5}90\mathbf{N}(-0.588) - 95e^{-0.035\times0.5}\mathbf{N}(-0.659) \\ &= e^{-0.015\times0.5}90 \times 0.278 - 95e^{-0.035\times0.5}\times 0.255 = 1.028. \end{aligned}$$

(b) What is the price of a 1/90–strike euro–denominated yen call with 6 months to expiration?
Solution:

$$C_0^{€} = \frac{1}{S_0}e^{-r_{€}T}\mathbf{N}(-d_2) - S_0e^{-r_{¥}T}\mathbf{N}(-d_1)$$

$$C_0 = \frac{1}{95}e^{-0.015\times0.5}\mathbf{N}(d_1') - \frac{1}{90}e^{-0.035\times0.5}\mathbf{N}(d_2')$$

where

$$d_1' = \frac{\ln(\frac{90}{95}) + (0.035 - 0.015 + \frac{0.1^2}{2})0.5}{0.1\sqrt{0.5}} = -0.588 \text{ and } d_2' = -0.588 - 0.1\sqrt{0.5} = -0.659$$

then the call price

$$\begin{aligned} C_0 &= \frac{1}{95}e^{-0.015\times0.5}\mathbf{N}(d_1) - \frac{1}{90}e^{-0.035\times0.5}\mathbf{N}(d_2) \\ &= \frac{1}{95}e^{-0.015\times0.5} \times 0.278 - \frac{1}{90}e^{-0.035\times0.5}\times 0.255 \\ &= 1.2027 \times 10^{-4}. \end{aligned}$$

(c) What is the link between your answer to (a) and your answer to (b), converted to yen? $1.2027 \times 10^{-4} \times 95 \times 90 = 1.028$

Problem 4.

1. Suppose $S = \$100$, $K = \$95$, $\sigma = 30\%$, $r = 0.08$, $\delta = 0.03$, and $T = 0.75$.

   (a) Compute the Black–Scholes price of a call.

   $$C_0 = 100 \times e^{-0.03\times 0.75}\mathbf{N}(d_1) - 95e^{-0.08\times 0.5}\mathbf{N}(d_2)$$

   where $d_1 = \frac{\ln(\frac{100}{95})+(0.08-0.03+\frac{0.3^2}{2})0.75}{0.3\sqrt{0.75}} = 0.4716$ and $d_2 = 0.4716 - 0.3\sqrt{0.75} = 0.2118$, then $C_0 = 100 \times e^{-0.03\times 0.75} \times 0.6814 - 95 \times e^{-0.08\times 0.75} \times 0.5838 = 14.392$.

   (b) Compute the Black–Scholes price of a call for which $S = \$100 \times e^{-0.03\times 0.75}$, $K = \$95 \times e^{-0.08\times 0.75}$, $\sigma = 0.3$, $T = 0.75$, $\delta = 0$, $r = 0$. How does your answer compare to that for (a)?

   $$C(S_0, K, \sigma, r, T, \delta) = C(S_0e^{-\delta T}, Ke^{-rT}, \sigma, 0, T, 0)$$

2. Make the same assumptions as in the question 1.

   (a) What is the 9–month forward price for the stock? $F_0 = 100 \times e^{0.05\times 0.75} = 103.82$

   (b) Compute the price of a 95–strike 9–month call option on a futures contract.

   $$C_0 = 103.82 \times e^{-0.08\times 0.75}\mathbf{N}(d_1) - 95e^{-0.08\times 0.5}\mathbf{N}(d_2)$$

   where $d_1 = \frac{\ln(\frac{103.82}{95})+(\frac{0.3^2}{2})0.75}{0.3\sqrt{0.75}} = 0.4716$ and $d_2 = 0.4716 - 0.3\sqrt{0.75} = 0.2118$

   $$C_0 = 103.82 \times e^{-0.08\times 0.75} \times 0.6814 - 95 \times e^{-0.08\times 0.75} \times 0.5838 = 14.392.$$

   (c) What is the relationship between your answer to (b) and the price you computed in the previous question? Why? We obtain the same value since $F_0 = S_0e^{(r-\delta)T}$ and then

   $$C(F_0, K, \sigma, r, T, r) = C(F_0e^{-rT}, Ke^{-rT}, \sigma, 0, T, 0)$$

3. Assume $K = \$40$, $\sigma = 30\%$, $r = 8\%$, T is six months, and the stock is to pay a single dividend of \$2 tomorrow, with no dividends thereafter.

   (a) Suppose $S_0 = \$50$. What is the price of a European call option?

   $$\begin{aligned} C_0 &= C(F_0, K, \sigma, r, T, 0) = C(F^p_{0,T}(S), F^p_{0,T}(K), \sigma, 0, T, 0) \\ &= F^p_{0,T}(S)\mathbf{N}(d_1) - F^p_{0,T}(K)\mathbf{N}(d_2) = \left(S_0 - De^{-rt}\right)\mathbf{N}(d_1) - Ke^{-rT}\mathbf{N}(d_2) \end{aligned}$$

   where t is the time where the dividend is paied and

   $$d_1 = \frac{\ln(\frac{F^p_{0,T}(S)}{F^p_{0,T}(K)}) + (\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

   We have $F^p_{0,T}(S) = 50 - 2e^{-0.08(1/365)} = 48.$, and

   $$d_1 = \frac{\ln(\frac{48.}{40e^{-0.08\times 0.5}}) + (\frac{0.3^2}{2})0.5}{0.3\sqrt{0.5}} = 1.1541 \quad \text{and} \quad d_2 = 1.1541 - 0.3\sqrt{0.5} = 0.9420.$$

   Hence

   $$C_0 = 48 \times 0.8757 - 40e^{-0.08\times 0.5} \times 0.8269 = 10.255.$$

(b) Repeat, only suppose $S_0 = \$60$. Now for $S_0 = \$60$ we have $F_{0,T}^{p}(S) = 60 - 2e^{-0.08(1/365)} = 58.$, and

$$d_1 = \frac{\ln(\frac{58}{40e^{-0.08\times 0.5}}) + (\frac{0.3^2}{2})0.5}{0.3\sqrt{0.5}} = 2.0462 \quad \text{and} \quad d_2 = 2.0462 - 0.3\sqrt{0.5} = 1.8341.$$

Hence

$$C_0 = 58 \times 0.9796 - 40e^{-0.08\times 0.5} \times 0.9667 = 19.665.$$

4. An index currently stands at $1,500$. European call and put options with a strike price of $1,400$ and time to maturity of six months have market prices of 154.00 and 34.25, respectively. The six–month risk–free rate is 5%.

(a) What is the implied dividend yield? From the call–put parity we can write

$$C_t - P_t = S_t e^{-\delta(T-t)} - Ke^{-r(T-t)}$$

then

$$e^{-\delta(T-t)} = \frac{C_t - P_t + Ke^{-r(T-t)}}{S_t} \iff e^{\delta(T-t)} = \frac{S_t}{C_t - P_t + Ke^{-r(T-t)}}$$

$$\iff \delta = \frac{1}{T-t}\ln\left(\frac{S_t}{C_t - P_t + Ke^{-r(T-t)}}\right),$$

hence

$$\delta = 2\ln\left(\frac{1500}{154 - 34.25 + 1400e^{-0.05\times 0.5}}\right) = 1.9853 \times 10^{-2} = 1.9853\% \simeq 2\%.$$

Problem 5.

1. Consider a six–month European call option on the spot price of gold, that is, an option to buy one ounce of gold in the spot market in six months. The strike price is $1,200, the six–month futures price of gold is $1,240, the risk–free rate of interest is 5% per annum, and the volatility of the futures price is 20%.

   (a) Compute the price of a six–month European option on the six–month futures price.

2. A futures price is currently 50. At the end of six months it will be either 56 or 46. The risk–free interest rate is 6% per annum.

   (a) What is the value of a six–month European call option on the futures with a strike price of 50?

   (b) How does the call–put parity formula for a futures option differ from call–put parity for an option on a non–dividend–paying stock?

3. Calculate the value of a five–month European put futures option when the futures price is $19, the strike price is $20, the risk–free interest rate is 12% per annum, and the volatility of the futures price is 20% per annum.

4. Suppose you sell a call option contract on April live cattle futures with a strike price of 90 cents per pound. Each contract is for the delivery of 40,000 pounds. What happens if the contract is exercised when the futures price is 95 cents?