King Saud University
College of Sciences
Mathematics Department

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Solution of the first midterm exam QMF: Actu. 468 (25%)

Problem 1. (8 marks)

1. Consider a call option on a stock. The call option will expire in 6 months. The current stock price is \$30, and the strike price of the call option is \$20. At expiration date, the stock price can either be \$35 or it can be \$25. The 6–month risk–free interest rate is 4%.

   (a) **(1 mark)** What is the value of the call option today?
   Solution: If we denote by C_0 value of the call option today, then

   $$C_0 = E_Q\left[\frac{\max(S_1 - 20, 0)}{1+r}\right]$$

   where $q = \frac{1+r-d}{u-d}$ and $u = \frac{35}{30} = \frac{7}{6}$ and $d = \frac{25}{30} = \frac{5}{6}$. The no arbitrage is satisfied since $d < 1+r < u$ that is $\frac{5}{6} < 1 + \frac{1}{25} < \frac{7}{6}$. So the RNPM $Q = (q, 1-q)$ where

   $$q = \frac{1 + \frac{1}{25} - \frac{5}{6}}{\frac{7}{6} - \frac{5}{6}} = \frac{31}{50}$$

   Therefore
   $$C_0 = \frac{31}{50}\frac{15}{1+\frac{1}{25}} + (1 - \frac{31}{50})\frac{5}{1+\frac{1}{25}} = \frac{140}{13} = 10.769.$$

   Remark that this call option presents arbitrage opportunity since $\mathbf{S_1 > 20}$.

   (b) **(1 mark)** Find the replicating portfolio of this call option.
   Solution: The replicating portfolio (α, Δ) should satisfies the following equations

   $$\alpha + \Delta S_0 = C_0 \quad \text{and } \alpha(1+r) + \Delta S_1 = C_1 = \max(S_1 - 20, 0)$$

   which lead to the system

   $$\begin{cases} \alpha\frac{26}{25} + 35\Delta = 15 \\ \alpha\frac{26}{25} + 25\Delta = 5 \end{cases} \Longleftrightarrow \begin{cases} \Delta = \frac{15-5}{35-25} = 1 \\ \alpha = -\frac{250}{13} \end{cases}$$

   An the solution is unique and is given by $(\alpha, \Delta) = (-\frac{250}{13}, 1)$. One can check that the value V_0 of this portfolio at time zero is

   $$V_0 = -\frac{250}{13} + 30 = \frac{140}{13} = C_0.$$

2. Consider a put option on a stock. The put option will expire in 9 months. The current stock price is \$40, and the strike price of the call option is \$42. In three months the stock price can either be \$50 or it can be \$32. The risk–free interest rate is 5% per annum. Assume that the stock follows a binomial model at each node of 3 months until 9 months.

(a) **(1 mark)** Build a three–step binomial tree for the stock.
Solution: We have $u = \frac{50}{40} = \frac{5}{4}$ and $d = \frac{32}{40} = \frac{4}{5}$. First remark that the no arbitrage is satisfied since $d < e^{rh} < u$ that is $\frac{4}{5} < e^{0.05\times 0.25} = 1.0126 = 1.01 < \frac{5}{4}$. So the RNPM $Q = (q, 1-q)$ where

$$q = \frac{1.0126 - \frac{4}{5}}{\frac{5}{4} - \frac{4}{5}} \simeq 0.4724.$$

The 3–month–step binomial tree for the 9 months stock is given by

3-month-step binomial tree for the 9 months stock

				78.13
			62.50	
		50.00		50.00
Stock tree	40.00		40.00	
		32.00		32.00
			25.60	
				20.48

Figure 1: 3-month-step binomial tree for the 9 months stock

(b) **(1 mark)** Build a three–step tree for the put option.
Solution: The 3–month–step binomial tree for the 9 months European put option is given by:

3-month-step binomial tree for the 9 months European put option

				0.00
			0.00	
		2.71		0.00
Option tree	6.84		5.21	
		10.70		10.00
			15.88	
				21.52

Figure 2: 3–month–step binomial tree for the 9 months European put option

(c) **(1 mark)** Find the riskless portfolio at times 0 and 1, when you short one call.
Solution: The riskless portfolio at times 0, 1 and 2 is given by the following tree

(d) **(1 mark)** Now, use call–put parity to find the price of the call option with the same strike price \$42. and expires in 9 months.

Alpha			Delta		
		0.00			0.00
	14.29			-0.23	
24.60		27.43	-0.44		-0.56
	34.41			-0.74	
		41.48			-1.00

Figure 3: 9–month riskless portfolio

Solution: The call–put parity of non–dividend–paying stock is

$$C_0 - P_0 = S_0 - Ke^{-rT}$$

hence $C_0 = P_0 + S_0 - Ke^{-rT} = 6.84 + 40 - 42e^{-0.05\times 9/12} = 6.38$.

3. Consider an American put option with the same characteristics as in question 2.

 (a) **(1 mark)** Build a three–step tree of the American put option.
 Solution:

3-month-step binomial tree for the 9 months American put option				
				0.00
			0.00	
		2.71		0.00
Option tree	6.99		5.21	
		10.98		10.00
			16.40	
				21.52

Figure 4: 9–month American put option

 (b) **(1 mark)** Find the replicating portfolio only at times 0 and 1, when you short one put.
 Solution:

Alpha			Delta		
		0.00			0.00
	11.58			-0.23	
18.36		22.22	-0.46		-0.56
	24.87			-0.78	
		25.60			-1.00

Figure 5: Hedging portfolio of 9–month American put option

Problem 2. **(6 marks)**

1. **(2 marks)** Calculate u, d, and the RNPM q when a binomial tree is constructed to value an option on a foreign currency. The tree step–size is 2 month, the domestic interest rate is 6% per annum, the foreign interest rate is 4% per annum, and the volatility is 12% per annum.

Solution: We have ($h = \frac{2}{12} = \frac{1}{6}$ and $\sigma = 0.25$) thus $u = e^{0.02\times\frac{1}{6}+0.12\sqrt{\frac{1}{6}}} = 1.0537$ and $d = e^{0.02\times\frac{1}{6}-0.12\sqrt{\frac{1}{6}}} = 0.9554$.

$$q = \frac{a-d}{u-d} \quad \text{where} \quad a = e^{(r-r_f)h}$$

Therefore

$$a = e^{(0.06-0.04)\frac{1}{6}} = 1.0033 \quad \text{and} \quad q = \frac{1.0033-0.9554}{1.0537-0.9554} \simeq 0.4873.$$

We can also use the formula for q

$$q = \frac{1-e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}}-e^{-\sigma\sqrt{h}}} = \frac{1}{1+e^{\sigma\sqrt{h}}}.$$

hence

$$q = \frac{1}{1+e^{0.12\sqrt{\frac{1}{6}}}} \simeq 0.4877.$$

2. **(2 mark)** Use the binomial tree to find the initial price (premium) of a European call option on a foreign currency with initial price 1.5000, maturity 4 months and strike price 1.4800.

Solution:

2-months steep 4-months stock tree				2-months steep 4-months European call option			
			1.6655				0.1855
		1.5806				0.1048	
Currency tree	1.5000		1.5100	Option tree	0.0580		0.0300
		1.4331				0.0145	
			1.3691				0.0000

Figure 6: 4–month European call option

3. **(2 mark)** Use the binomial tree to find the initial price (premium) of an American call option on the above foreign currency and compare it to that of the European call option.
Solution:

2-months steep 4-months stock tree				2-months steep 4-months American put option			
			1.6655				0.0000
		1.5806				0.0000	
Currency tree	1.5000		1.5100	Option tree	0.0285		0.0000
		1.4331				0.0562	
			1.3691				0.1109

Figure 7: 4–month American call option

<u>Problem 3.</u> (13 marks)

1. Consider a market where a bond (corresponding to a risk–free rate of 4% per year) and a stock are traded. The current price of the stock is $S_0^1 = \$5$ that, in one year, may move up to \$6 or to \$8, or down to \$3.

   (a) **(1 mark)** Complete the following table

n	S_n^0	S_n^1		
		ω_1	ω_2	ω_3
0	1	5	5	5
1	1.04	6	8	3

   (b) **(1 marks)** Is this model arbitrage free ? Is the market complete ? (Hind: write q_2 and q_3 in terms en $q_1 = q$).
   Solution: If a RNPM $Q = (q_1, q_2, 1 - q_1 - q_2)$ exists then it should satisfy,

$$E_Q\left[S_1^1\right] = (1+r)S_0^1 \iff 6q_1 + 8q_2 + 3(1 - q_1 - q_2) = 1.04 \times 5$$
$$\iff 3q_1 + 5q_2 + 3 = \frac{26}{5} \iff q_2 = \frac{11}{25} - \frac{3}{5}q.$$

$q_3 = 1 - q - \frac{11}{25} + \frac{3}{5}q = \frac{14}{25} - \frac{2}{5}q$, provided that $0 < q_i < 1$ for $i = 1, 2, 3$ that is $0 < q < 1$, $0 < \frac{11}{25} - \frac{3}{5}q < 1$ and $0 < \frac{14}{25} - \frac{2}{5}q < 1$, then $0 < q < 1$, $0 < q < \frac{55}{75}$ and $0 < q < \frac{14}{25}\frac{5}{2} = \frac{7}{5}$, which implies that $0 < q < \frac{55}{75} = \frac{11}{15} = 0.73$. Therefore the market is arbitrage free for any $0 < q < \frac{11}{15}$. But there is infinitely many solutions $(q, \frac{11}{25} - \frac{3}{5}q, \frac{14}{25} - \frac{2}{5}q)$, hence this model is not complete.

2. Consider a European Call option on the stock above, with strike of \$5 and with maturity of one year.

   (a) **(1 mark)** Give different values of this call option at time 1
   Solution : The payoff of the call option is given by $\max(S_1^1 - 5, 0) = (1, 3, 0)$.

   (b) **(2 marks)** Verify if it is possible to replicate such an option only by means of the underlying and of cash invested (or borrowed) at risk–free rate.
   Solution: If the call with payoff $(1, 3, 0)$ were attainable then there exists a replicating portfolio (α, Δ^1) such that t

$$(\alpha, \Delta^1) \cdot (S_1^0, S_1^1) = (1, 3, 0)\,.$$

   This formula gives the following system

$$\begin{cases} 1.04\alpha + 6\Delta^1 = 1 & (1) \\ 1.04\alpha + 8\Delta^1 = 3 & (2) \\ 1.04\alpha + 3\Delta^1 = 0 & (3) \end{cases}$$

   Remark that $(2) - (1)$ implies that $\Delta^1 = 1$ and $(2) - (3)$ gives $\Delta^1 = \frac{3}{5}$ which is impossible, hence the call with payoff $(1, 3, 0)$ is not attainable.

3. Consider another stock of current price $S_0^2 = 8$ dollars and that, in one year, may cost 12, 10 or 6 dollars. Now, consider the new market given by $(S_t^0, S_t^1, S_t^2)_{t \in \{0,1\}}$.

   (a) **(1 mark)** Complete the following table

n	S_n^0	S_n^1			S_n^2		
		ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
0	1	5	5	5	8	8	8
1	1.04	6	8	3	12	10	6

   (b) **(2 marks)** Is this new market model $(S_t^0, S_t^1, S_t^2)_{t \in \{0,1\}}$ arbitrage free ? Is the market complete ? (Hind: write q_2 and q_3 in terms en $q_1 = q$).
   Solution: If a RNPM $Q = (q, \frac{11}{25} - \frac{3}{5}q, \frac{14}{25} - \frac{2}{5}q)$, exists then it should satisfy

$$\begin{aligned} E_Q\left[S_1^2\right] &= 1.04 S_0^2 \iff 12q + 10\left(\frac{11}{25} - \frac{3}{5}q\right) + 6\left(\frac{14}{25} - \frac{2}{5}q\right) = \frac{104}{100}8 \\ &\iff \frac{18}{5}q + \frac{194}{25} = \frac{208}{25} \iff q = \frac{7}{45} < \frac{11}{15}. \end{aligned}$$

   Therefore the RNPM $Q = (\frac{7}{45}, \frac{26}{75}, \frac{112}{225}) = 0.155\,56 \quad 0.346\,67 \quad 0.497\,78$ which is unique then the market is arbitrage free and complete.

   (c) **(1 marks)** Is the call option in question 2 attainable. Explain your answer.
   Solution : The call option with payoff $(S_1^1 - 5)^+ = (1, 3, 0)$ is attainable since the new market is complete.

4. Consider a derivative with maturity one year and payoff

$$F = \left(\frac{S_1^1 + S_1^2}{2} - 7\right)^+ = \max\left(\frac{S_1^1 + S_1^2}{2} - 7; 0\right)$$

(a) **(1 mark)** Give different values of F at time 1.
Solution: The value of the call option on the mean of S^1 and S^2 at time 1 is

$$F = (2; 2; 0)$$

(b) **(1 mark)** Use the risk–neutral valuation to find its initial price (premium).
Solution: Denote by V_0 the risk–neutral price of the derivative F, then

$$V_0 = E_Q\left[\frac{F}{1+r}\right] = \frac{100}{104}\left(2 \times \frac{7}{45} + 2 \times \frac{26}{75}\right) = \frac{113}{117} \simeq 0.9658.$$

(c) **(3 marks)** Find the replicating strategy when you short the derivative F.
Solution: Since the $(S_t^0, S_t^1, S_t^2)_{t\in\{0,1\}}$ is complete, there exists a replicating portfolio $(\alpha, \Delta^1, \Delta^2)$ such that the value of this portfolio at time one matches F that is

$$V_1 = (\alpha, \Delta^1, \Delta^2) \cdot (S_1^0, S_1^1, S_1^2) = F.$$

This formula gives the following system

$$\begin{cases} 1.04\alpha + 6\Delta^1 + 12\Delta^2 = 2 \\ 1.04\alpha + 8\Delta^1 + 10\Delta^2 = 2 \\ 1.04\alpha + 3\Delta^1 + 6\Delta^2 = 0 \end{cases}$$

which is equivalent in matrix form to

$$\begin{pmatrix} 1.04 & 6 & 12 \\ 1.04 & 8 & 10 \\ 1.04 & 3 & 6 \end{pmatrix} \begin{pmatrix} \alpha \\ \Delta^1 \\ \Delta^2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix},$$

and the unique solution is then $(\alpha, \Delta^1, \Delta^2) = (-\frac{25}{13}, \frac{2}{9}, \frac{2}{9})$. The seller of this option will borrow $\frac{25}{13}$ from the bank and buy $\frac{2}{9}$ shares in the asset number 1 and $\frac{2}{9}$ shares in the asset number 2. This is possible since the two shares cost

$$5 \times \frac{2}{9} + 8 \times \frac{2}{9} = \frac{25}{13} + \frac{113}{117} = \frac{26}{9}$$

sum of the premium and the borrowed money.