King Saud University
College of Sciences
Mathematics Department

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Solution of the First Midterm exam: Stochastic Processes: Math. 380 (25%)

Sunday, March 19, 2017 (1 – 3) PM

Problem 1. (**5 marks**)

Let the random variable X have the p.m.f. $f(k) = \frac{1}{3}$, $k \in S_X = \{-1, 0, 1\}$. Let $Y = X^2$,

1. (**1 mark**) Find the distribution of Y,
   Solution: $S_Y = \{0, 1\}$, and the p.m.f. is given by
   $$f(0) = P(Y = 0) = P(X = 0) = \frac{1}{3}$$
   and
   $$\begin{aligned} f(1) &= P(Y = 1) = P\left(X^2 = 1\right) = P(X = -1 \text{ or } X = 1) \\ &= P(X = -1) + P(X = 1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}. \end{aligned}$$

2. (**1 mark**) Calculate the expectation of X and Y.
   Solution:
   $$E[X] = \sum_{k=-1}^{1} k\frac{1}{3} = \frac{1}{3}(-1 + 0 + 1) = 0 \text{ and } E[Y] = \sum_{k=0}^{1} kf(k) = \frac{1}{3} \times 0 + \frac{2}{3} \times 1 = \frac{2}{3}$$

3. (**1 mark**) Deduce the variance of X.
   Solution:
   $$Var(X) = E\left[X^2\right] - (E[X])^2 = E[Y] = \frac{2}{3}.$$

4. (**1 mark**) Find the covariance of X and Y.
   Solution:
   $$Cov(X, Y) = E[XY] - E[X]E[Y] = E\left[X^3\right] = \frac{1}{3}(-1 + 0 + 1) = 0.$$

5. (**1 mark**) Are X and Y independent ? Explain your answer.
   Solution: X and Y are dependent because $Y = X^2$.

Problem 2. (**5 marks**)

1. Let the joint p.m.f. of X and Y be defined by $f(i, j) = \frac{i+j}{32}$, $i = 1, 2$, and $j = 1, 2, 3, 4$.

   (a) (**1 mark**) Find $f_X(i)$ the marginal p.m.f. of X.
   $$f_X(i) = \sum_{j=1}^{4} f(i, j) = \sum_{j=1}^{4} \frac{i + j}{32} = \frac{5 + 2i}{16}$$

(b) (**1 mark**) Find $f_Y(j)$ the marginal p.m.f. of Y.

$$f_Y(j) = \sum_{i=1}^{2} f(i,j) = \sum_{i=1}^{2} \frac{i+j}{32} = \frac{3+2j}{32}$$

(c) (**1 mark**) Find $P(Y = 2X)$. Let us first remark that

$$\begin{aligned} \{Y = 2X\} &= \{Y = 2X\} \cap (\{X = 1\} \cup \{X = 1\}) \\ &= (\{Y = 2X\} \cap \{X = 1\}) \cup (\{Y = 2X\} \cap \{X = 2\}) \end{aligned}$$

hence

$$\begin{aligned} P(Y = 2X) &= P(\{Y = 2X\} \cap \{X = 1\}) + P(\{Y = 2X\} \cap \{X = 2\}) \\ &= P(Y = 2; X = 1) + P(Y = 4; X = 2) \\ &= \frac{1+2}{32} + \frac{2+4}{32} = \frac{9}{32}. \end{aligned}$$

(d) (**1 mark**) Find $P(X \le 3 - Y)$.

$$\begin{aligned} P(X \le 3 - Y) &= P(X \le 3 - Y; X = 1) + P(X \le 3 - Y; X = 2) \\ &= P(1 \le 3 - Y; X = 1) + P(2 \le 3 - Y; X = 2) \\ &= P(Y \le 2; X = 1) + P(Y \le 1; X = 2) \\ &= P(Y = 1; X = 1) + P(Y = 2; X = 1) + P(Y = 1; X = 2) \\ &= \frac{1+1}{32} + \frac{1+2}{32} + \frac{2+1}{32} = \frac{8}{32}. \end{aligned}$$

2. There are **eight** similar chips in a bowl: **three** marked $(0,0)$, **two** marked $(1,0)$, two marked $(0,1)$, and one marked $(1,1)$. A player selects a chip at random and is given the **sum** of the two coordinates in dollars. If X and Y represent those two coordinates, respectively, their joint p.m.f. is $f(i,j) = \frac{3-i-j}{8}$, $i = 0,1$ and $j = 0,1$.

(a) (**1 mark**) Calculate the expected given sum in dollars.

$$E[X+Y] = \sum_{i=0}^{1}\sum_{j=0}^{1} (i+j)\,\frac{3-i-j}{8} = \frac{3}{4}$$

Problem 3. (**5 marks**)

A **discrete** is a random variable whose range is either finite or countably in finite. A **continuous** random variable is a random variable whose range is an interval in $\mathbb{R}$: A **mixed random variable** is partially discrete and partially continuous.

State whether the random variables are discrete, continuous or mixed.

1. (**1 mark**) A coin is tossed ten times. The random variable X is the number of tails that are noted. Explain your answer.
Solution: The r.v. X follows a binomial with parameter 10 and $\frac{1}{2}$ since its has two alternative outcomes and can take the values $\{0, 1, \dots, 10\}$.

2. (**1 mark**) A light bulb is burned until it burns out. The random variable Y is its lifetime in hours. Explain your answer.
Solution: The life time a light bulb is a continuous random varibale since it can take any positive value.

3. (**1 mark**) $Z:]0;1[\longrightarrow \mathbb{R}$ where

$$Z(s) = \begin{cases} 1-s & \text{if } 0 < s < \frac{1}{2} \\ \frac{1}{2} & \text{if } \frac{1}{2} \leq s < 1 \end{cases} \quad \text{Explain your answer.}$$

Solution: The r.v. Z is a mixed distribution since it takes a continuous value between $0 < s \leq \frac{1}{2}$ and takes one single value on the internal $\frac{1}{2} \leq s < 1$.

4. (**1 mark**) Find the expectation of X and Z;
Solution: We know that $X \hookrightarrow \mathcal{B}(10, \frac{1}{2})$ the, $E[X] = 10\frac{1}{2} = 5$. and

$$E[Z] = \int_0^1 z(s)ds = \int_0^{\frac{1}{2}} (1-s)ds + \int_{\frac{1}{2}}^1 \frac{1}{2}ds = \frac{5}{8}.$$

5. (**1 mark**) Find the variance of X and Z **canceled.**

Problem 4. (**6 marks**)
Let X and Y be two random variables with the joint density

$$f(x,y) = \begin{cases} a(e^{-2x-3y}) & \text{if } 0 < x \text{ and } 0 < y \\ 0 & \text{elsewhere} \end{cases}$$

1. (**1 mark**) Find the value of a.
Solution: We have

$$\int_0^\infty \int_0^\infty a(e^{-2x-3y})dxdy = 1 \Longleftrightarrow a = 6$$

2. (**1 mark**) Calculate the marginal densities of X and Y.
Solution: We know

$$f_X(x) = \int f(x,y)dy = \int_0^\infty 6(e^{-2x-3y})dy = 2e^{-2x}$$

and

$$f_Y(y) = \int f(x,y)dx = \int_0^\infty 6(e^{-2x-3y})dx = 3e^{-3y}.$$

3. (**1 mark**) Are X and Y independent?
Solution: We have $f_X(x)f_Y(y) = f(x,y) = f_{(X,Y)}(x,y)$ then X and Y are independent.

4. (**1 mark**) Find $Cov(X,Y)$.
Solution: By definition

$$Cov(X,Y) = E[XY] - E[X]E[Y] = 0 \text{ since } X \text{ and } Y \text{ are independent.}$$

5. (**1 mark**) Set $Z = X+Y$, Calculate the c.d.f. $F_Z(z)$ of Z.
Solution: By definition

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P(X+Y \leq z) = \int\int_{\{x+y\leq z\}} f(x,y)dxdy \\ &= \int_0^z \left(\int_0^{z-x} f(x,y)dy\right)dx = \int_0^z 2e^{-2x}\left(\int_0^{z-x} 3e^{-3y}dy\right)dx \\ &= 1 + 2e^{-3z} - 3e^{-2z}. \end{aligned}$$

6. (**1 mark**) Find the p.d.f. $f_Z(z)$ of Z.
Solution: We know that $f_Z(z) = F'_Z(z) = 6\left(e^{-2z} - e^{-3z}\right)$.

Problem 5. (**6 mark**)

1. An insurance policy is written to cover a loss, X, where X has a uniform distribution on $[0, 1000]$. The policy has a deductible, d, and the expected payment under the policy is 25% of what it would be with no deductible.

   (a) (**2 mark**) Calculate d.
   Solution: Let Y be the payment under the deductible policy and X be the payment under policy with no deductible. So

$$E\left[Y\right] = E\left[\max(X-d, 0)\right] = \frac{25}{100} E\left[X\right] \tag{1}$$

   but we know that

$$E\left[\max(X-d, 0)\right] = \int_d^{1000} (x-d)\frac{1}{1000}dx = \frac{1}{2000}\left(d-1000\right)^2$$

   and

$$E\left[X\right] = \int_0^{1000} x\frac{1}{1000}dx = 500.$$

   Now substituting in the equation (**??**) we get

$$\begin{aligned} \frac{1}{2000}\left(d-1000\right)^2 &= \frac{25}{100}\times 500 = 125 \\ &\Longleftrightarrow \left(d-1000\right)^2 = 250000 \\ &\Longleftrightarrow |d-1000| = \sqrt{250000} = 500 \end{aligned}$$

   Then the possible solution are $d = 500$ and $d = 1500$. But since d shoul be less than the maximum value of X we finaly get as the unique solution $d = 500$.

2. A group insurance policy covers the medical claims of the employees of a small company. The value, V, of the claims made in one year is described by $V = 100{,}000Y$ where Y is a random variable with density function

$$f(y) = \begin{cases} c(1-y)^4 & \text{if } 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

   where c is a constant.

   (a) (**2 mark**) Calculate the conditional probability that V exceeds $40{,}000$, given that V exceeds $10{,}000$.
   Solution: First we have $\int_0^1 c(1-y)^4 dy = 1$ implies that $\frac{1}{5}c = 1$, then $c = 5$. And

$$\begin{aligned} P\left(V > 40{,}000 \mid V > 10{,}000\right) &= \frac{P(V > 40{,}000,\ V > 10{,}000)}{P(V > 10{,}000)} \\ &= \frac{P(V > 40{,}000,\ V > 10{,}000)}{P(V > 10{,}000)} \\ &= \frac{P(V > 40{,}000)}{P(V > 10{,}000)} = \frac{P(Y > \frac{4}{10})}{P(V > \frac{1}{10})} \\ &= \frac{\int_{\frac{2}{5}}^1 5(1-y)^4 dy}{\int_{\frac{1}{10}}^1 5(1-y)^4 dy} = \frac{\frac{243}{3125}}{\frac{59\,049}{100\,000}} = \frac{32}{243} \simeq 0.132 \end{aligned}$$

3. The stock prices of two companies at the end of any given year are modeled with random variables X and Y that follow a distribution with joint density function

$$f(x,y) = \begin{cases} 2x, & \text{if } 0 < x < 1 \text{ and } x < y < x+1, \\ 0, & \text{otherwise.} \end{cases}$$

Recall that

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} \text{ and } f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

where $f_X(x) = \int f(x,y)dy$ and $f_Y(y) = \int f(x,y)dx$. The condtional expectation of Y given $X = x$ is defined by

$$E[Y \mid X = x] = \int y f_{Y|X}(y|x)dy = h(x) \text{ and } E[X \mid Y = y] = \int x f_{X|Y}(x|y)dx = g(y)$$

and

$$Var(Y \mid X = x) = E\left[(Y - E[Y \mid X = x])^2 \mid X = x\right] = E[Y^2 \mid X = x] - (E[Y \mid X = x])^2.$$

(a) (**2 mark**) Determine the conditional variance of Y given that $X = x$.
Solution: We have

$$Var(Y \mid X = x) = E\left[(Y - E[Y \mid X = x])^2 \mid X = x\right] = E[Y^2 \mid X = x] - (E[Y \mid X = x])^2.$$

So we need first to calculate $f_{Y|X}(y|x)$. By definition we have

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} \text{ and } f_X(x) = \int_x^{x+1} 2x dy = 2x$$

hence

$$f_{Y|X}(y|x) = \begin{cases} 1, & \text{if } 0 < x < 1 \text{ and } x < y < x+1, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} E[Y|X = x] &= \int_{-\infty}^{\infty} y f_{Y|X}(y|x)dy = \int_x^{x+1} y dy \\ &= \left[\frac{y^2}{2}\right]_x^{x+1} = \frac{1}{2}\left[(x+1)^2 - x^2\right] \\ &= \frac{1}{2}[2x+1] = x + \frac{1}{2}. \end{aligned}$$

and

$$\begin{aligned} E\left[Y^2|X = x\right] &= \int_{-\infty}^{\infty} y^2 f_{Y|X}(y|x)dy = \int_x^{x+1} y^2 dy \\ &= \left[\frac{y3}{3}\right]_x^{x+1} = \frac{1}{3}\left[(x+1)^3 - x^3\right] = x^2 + x + \frac{1}{3} \end{aligned}$$

Therefore

$$Var(Y \mid X = x) = x^2 + x + \frac{1}{3} - \left(x + \frac{1}{2}\right)^2 = \frac{1}{12}.$$

Optional : Get marks as much as you can (bonus)

1. (**1 mark**) Let X, Y, Z be independent Poisson random variables with $E[X] = 3$; $E[Y] = 1$; and $E[Z] = 4$: Find $P(X + Y + Z \leq 1)$?
   Solution: We have

$$P(X+Y+Z \leq 1) = P(X+Y+Z=0) + P(X+Y+Z=1).$$

   But we know that

$$\begin{aligned} P(X+Y+Z=0) &= P(X=0, Y=0, Z=0) \\ &= P(X=0)\,P(Y=0)\,P(Z=0) \quad \text{since } X, Y, Z \text{ and independent} \\ &= e^{-3}e^{-1}e^{-4} = e^{-8} \end{aligned}$$

   and

$$\begin{aligned} P(X+Y+Z=1) &= P(X=0, Y=0, Z=1) \\ &\quad + P(X=0, Y=1, Z=0) \\ &\quad + P(X=1, Y=0, Z=0) \\ &= P(X=0)\,P(Y=0)\,P(Z=1) \\ &\quad + P(X=0)\,P(Y=1)\,P(Z=0) \\ &\quad + P(X=1)\,P(Y=0)\,P(Z=0) \\ &= e^{-3}e^{-1}e^{-4}4 + e^{-3}e^{-1}e^{-4} + 3e^{-3}e^{-1}e^{-4} \\ &= 8e^{-8}. \end{aligned}$$

   where we have used that fact that X, Y, Z are independent. Finally we get

$$P(X+Y+Z \leq 1) = e^{-8} + 8e^{-8} = 9e^{-8}.$$

2. Assume $X_1, \ldots, X_n$ are independent and identically distributed random variables with a common c.d.f. $F(x)$: Define U and L as follows $U = \max(X_1, \ldots, X_n)$ and $L = \min(X_1, \ldots, X_n)$.

   (a) (**1 mark**) Write in terms of the events $\{X_i \leq u\}$ the event $\{U \leq u\}$ for any $u \in \mathbb{R}$,
   Solution:

   (b) (**1 mark**) Deduce c.d.f. of U.
   Solution:

   (c) (**1 mark**) Find $P(L > \ell)$, for any $\ell \in \mathbb{R}$.
   Solution:

   (d) (**1 mark**) and Deduce c.d.f. of L.
   Solution:

   (e) (**1 mark**) Are U and L independent ?
   Solution: