SOLUTION TO M-107 FINAL EXAMINATION, DECEMBER 24, 2020

Solution to Q1. [**Marks: 4+3=7**]

(a)

$$\begin{pmatrix} 2 & 3 & \lambda+2 & 5 \\ 1 & 1 & 1 & 2 \\ 4\lambda & 3\lambda & 3 & 8\lambda-3 \end{pmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & \lambda+2 & 5 \\ 4\lambda & 3\lambda & 3 & 8\lambda-3 \end{pmatrix} \xrightarrow{-2R_1+R_2;\ -4\lambda R_1+R_3}$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & \lambda & 1 \\ 0 & 0 & \lambda^2-4\lambda+3 & \lambda-3 \end{pmatrix}$$

From the bottom line of the matrix we have: $(\lambda^2 - 4\lambda + 3)z = (\lambda - 3)$.

But $\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$.

Case I: If $\lambda = 3$, then the system has infinitely many solution;

Case II: If $\lambda = 1$, then the system has no solution;

Case II: If $\lambda \neq 3$ and $\lambda \neq 1$, then the system has unique solution.

(b) Given

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 4 \\ -1 & 6 & 2 \end{pmatrix}$$

Then $det(A) = 19$ and the matrix of the cofactors is

$$\begin{pmatrix} -22 & -8 & 13 \\ 24 & 7 & -9 \\ 7 & 6 & -5 \end{pmatrix}$$

and

$$adj(A) = \begin{pmatrix} -22 & 24 & 7 \\ -8 & 7 & 6 \\ 13 & -9 & -5 \end{pmatrix}$$

Therefore,

$$A^{-1} = \frac{1}{19} \begin{pmatrix} -22 & 24 & 7 \\ -8 & 7 & 6 \\ 13 & -9 & -5 \end{pmatrix}$$

Solution to Q2. [**Marks: 2+3+3=8**]

(a) $Comp_{\vec{b}}\ \vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|} = \dfrac{\langle -2,1,2\rangle \cdot \langle 1,-2,2\rangle}{\sqrt{1+4+4}} = \dfrac{-2-2+4}{3} = 0$. Similarly, $Comp_{\vec{c}}\ \vec{b} = 0$ and $Comp_{\vec{a}}\ \vec{c} = 0$.

(b) Consider the augmented matrix and apply Gaussian elimination:

$$\begin{pmatrix} 2 & 1 & 4 & 8 \\ 1 & 3 & -1 & -1 \end{pmatrix} \overset{R_1 \longleftrightarrow R_2}{\longrightarrow} \begin{pmatrix} 1 & 3 & -1 & -1 \\ 2 & 1 & 4 & 8 \end{pmatrix} \overset{-2R_1+R_2}{\longrightarrow} \begin{pmatrix} 1 & 3 & -1 & -1 \\ 0 & 1 & -\frac{6}{5} & -2 \end{pmatrix}$$

From the bottom line of the matrix, we have:

$$y - \frac{6}{5}z = -2 \Rightarrow y = -2 + \frac{6}{5}z \Rightarrow y = -2 + 6t; z = 5t$$

Therefore, the parametric equations for the line of intersection of the planes P_1 and P_2 are given by

$$x = 5 - 13t; \quad y = -2 + 6t; \quad z = 5t$$

(c) The given quadratic surface: $2x^2 + 3y^2 - 6z = 0$ is equivalent to $z = \dfrac{x^2}{3} + \dfrac{y^2}{2}$, which is a paraboloid.

- xy-trace: $(0,0,0)$ origin
- yz-trace: $z = \dfrac{y^2}{2}$, parabola
- zx-trace: $z = \dfrac{x^2}{3}$, parabola.

[**PS: Students need to draw the surface and show the traces**]

Solution to Q3. [**Marks: 3+3+3=9**]

(a) Given $C : x = a\cos t,\ y = a\sin t$. Let $f(t) = a\cos t$ and $g(t) = a\sin t$. then we have the curvature

$$\kappa = \frac{|f'(t)g''(t) - g'(t)f''(t)|}{|(f'(t))^2 + (g'(t))^2|^{\frac{3}{2}}} \Rightarrow$$

$$\kappa = \frac{|(-a\sin t)(-a\sin t) - (a\cos t)(-a\cos t)|}{|a^2\sin^2 t + a^2\cos^2 t|^{\frac{3}{2}}} = \frac{a^2}{(a^2)^{\frac{3}{2}}} = \frac{1}{a}$$

(b)

$$\vec{T}'(t) = \frac{\vec{r}(t)}{\|\vec{r'(t)}\|} = \frac{2t\vec{i} + 4\vec{j} + (4t-6)\vec{k}}{\sqrt{4t^2 + 16 + (4t-6)^2}} = \frac{t\vec{i} + 2\vec{j} + (2t-3)\vec{k}}{\sqrt{5t^2 - 12t + 13}}$$

Therefore, at $t = 2$, $\vec{T}(2) = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \rangle$.

(c) Given

$$\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}, \quad 1 \le t \le 4$$

We have

$$\vec{r}(t) = \vec{i} + 2t\vec{j} + 3t^2\vec{k},$$

$$\vec{r}''(t) = 2\vec{j} + 6t\vec{k},$$

and

$$\|\vec{r}'(t)\| = \sqrt{1 + 4t^2 + 9t^4}$$

Thus,

$$\vec{a}_T = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{\|\vec{r}'(t)\|} = \frac{4t + 18t^3}{\sqrt{1 + 4t^2 + 9t^4}}$$

Now

$$\vec{r}'(t) \times \vec{r}''(t) = 6t^2\vec{i} - 6t\vec{j} + 2\vec{k}$$

Therefore,

$$\vec{a}_N = \frac{\vec{r}'(t) \times \vec{r}''(t)}{\|\vec{r}'(t)\|} = \frac{\sqrt{36t^4 + 36t^2 + 4}}{\sqrt{1 + 4t^2 + 9t^4}} = 2\sqrt{\frac{9t^4 + 9t^2 + 1}{9t^4 + 4t^2 + 1}}$$

Solution to Q4. [**Marks: 2+2+4=8**]

(a) Given $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^6y + x^2y^3}{(x^4 + y^2)^2}$

- Along the path $y = x^2$, we have: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^8 + x^8}{4x^8} = \frac{1}{2}$
- Along the path $y = 2x^2$, we have: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^8 + 8x^8}{25x^8} = \frac{10}{25} = \frac{2}{5}$

Thus, limit does not exist.

(b) Given

$$w = \ln(u + v), \quad u = e^{-3t}, v = t^5 - t^2.$$

We have:

$$\frac{dw}{dt} = \frac{\partial w}{\partial u} \cdot \frac{du}{dt} + \frac{\partial w}{\partial v} \cdot \frac{dv}{dt}$$

$$= \left(\frac{1}{u + v}\right)(-3e^{-3t}) + \left(\frac{1}{u + v}\right)(5t^4 - 2t)$$

$$= \frac{5t^4 - 2t - 3e^{-3t}}{e^{-3t} + t^5 - t^2}$$

(c) Given $f(x, y) = \dfrac{x - y}{xy + 2}$, $P(1, 1)$ and $\vec{a} = 12\vec{i} + 5\vec{j}$.
We have the unit vector:

$$\vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{12\vec{i} + 5\vec{j}}{\sqrt{144 + 25}} = \frac{12\vec{i} + 5\vec{j}}{\sqrt{169}} \Rightarrow \vec{u} = \langle \frac{12}{13}, \frac{5}{13} \rangle$$

Now

$$\nabla f(x, y) = \frac{2 + y^2}{(xy + 2)^2}\vec{i} - \frac{2 + x^2}{(xy + 2)^2}\vec{j}$$

$$\Rightarrow \nabla f(1, 1) = \frac{1}{3}\vec{i} - \frac{1}{3}\vec{j} = \langle \frac{1}{3}, -\frac{1}{3} \rangle$$

Thus,

$$D_{\vec{u}}f(1,1) = \nabla f(1,1)\cdot\vec{u} = \frac{12}{39} - \frac{5}{39} = \frac{7}{39}.$$

Finally, maximum rate of change is $\|\nabla f(1,1)\| = \sqrt{\frac{1}{9}+\frac{1}{9}} = \sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{3}$.

Solution to Q5. Marks: 3+3+2=8]

Let $F(x,y,z) = x^3 - 12xy + 8y^3 - z = 0$. Then

$$\nabla F(x,y,z) = (3x^2-12y)\vec{i} + (-12x+24y^2)\vec{j} - \vec{k} \Rightarrow \nabla F(2,-1,24) = 24\vec{i} - \vec{k} = \langle 24, 0, -1\rangle$$

Thus, the equation of the tangent plane at $P(2,-1,24)$ is given by

$$24(x-2) + 0(y+1) - 1(z-24) = 0 \Rightarrow 24x - z = 24.$$

(b) Given $f(x,y) = x^3 - 12xy + 8y^3$. Then we have

$$\frac{\partial f}{\partial x} = 3x^2 - 12y;\quad \frac{\partial f}{\partial y} = -12x + 24y^2;\quad \frac{\partial^2 f}{\partial x^2} = 6x;\quad \frac{\partial^2 f}{\partial y^2} = 48y;\quad \frac{\partial^2 f}{\partial x\partial y} = -12$$

By setting $3x^2 - 12y = 0$ and $-12x + 24y^2 = 0$, we get the critical points: $(0,0)$ and $(2,1)$. Since

$$D(x,y) = \frac{\partial^2 f}{\partial x^2}\,\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\partial y}\right)^2 = 288xy - 144,$$

we get at $(0,0)$, $D(0,0) = -144 < 0$ gives saddle point, while at $(2,1)$, we get

$$D(2,1) = 288(2)(1) - 144 = 576 - 144 = 432 > 0$$

and

$$\frac{\partial^2 f}{\partial x^2}(2,1) = 6(2) = 12 > 0$$

yield that f has local minimum at $(2,1)$.

(c) Given $f(x,y,z) = x^2 + y^2 + z^2$. Let $g(x,y,z) = x + y + z - 36 = 0$.

By Lagrange multiplier we get $\nabla f(x,y,z) = \lambda\nabla g(x,y,z)$ which gives:

$2x = \lambda$, $2y = \lambda$, $2z = \lambda \Rightarrow x = y = z$. Putting all these in $x + y + z = 36$, we obtain: $x = 12$.

Hence $f(12,12,12) = 432$, a local minimum.