Problem Set 5 Solution Set

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1. (a) Let $g(z)$ be a holomorphic function in a neighbourhood of $z = a$. Suppose that $g(a) = 0$. Prove that $g(z)/(z - a)$ extends to a holomorphic function at $z = a$.

Solution. Define the function

$$f(z) = \begin{cases} g(z)/(z-a) & z \neq a, \\ g'(a) & z = a. \end{cases}$$

Clearly f is holomorphic in a neighborhood of a, though not necessarily at a. By the Riemann Removable Singularity Theorem, f is analytic at a if it is continuous at that point. We can then verify that

$$\lim_{z \to a} f(z) = \lim_{z \to a} \frac{g(z)}{z-a} = \frac{g(z) - g(a)}{z-a} = g'(a) = f(a).$$

Hence f is a holomorphic extension of $g(z)/(z - a)$ at $z = a$. □

(b) Let $f(z)$ be a holomorphic function in the neighborhood of $z = a$, except for a singularity at $z = a$. Suppose that the limit

$$\lim_{z \to a} (z-a)^n f(z)$$

exists for some integer n. Using part (a) show there exists an integer $n' \leq n$ such that

$$\lim_{z \to a} (z-a)^{n'} f(z)$$

exists and is non-zero.

Solution. Many people had lots of trouble with this question. Please read this solution carefully. I will point out the most common mistakes as we go along.
If the above limit exists and is non-zero we are done (take $n' = n$). So suppose that the above limit is zero. Let $g(z) = (z - a)^n f(z)$. Since $\lim_{z \to a} g(z)$ exists, the Riemann Removable Singularity theorem tells us that $g(z)$ can be extended to a function holomorphic at a. In fact, using part (a), the extension is given by

$$h_1(z) = \begin{cases} g(z)/(z-a) & z \neq a, \\ g'(a) & z = a. \end{cases}$$

Note that

$$\lim_{z\to a}(z-a)^{n-1}f(z) = \lim_{z\to a} h_1(z) = g'(a),$$

so if $g'(a)$ is non-zero we are done (take $n' = n-1$). In case $g'(a) = 0$, we may apply the above argument again to obtain a holomorphic function $h_2(z)$ given by

$$h_2(z) = \begin{cases} h_1(z)/(z-a) & z \neq a, \\ h_1'(a) & z = a. \end{cases}$$

As before,

$$\lim_{z\to a}(z-a)^{n-2}f(z) = \lim_{z\to a} h_2(z) = h_1'(a),$$

so if $h_1'(a)$ is non-zero we are done (take $n' = n-2$). Clearly we may keep repeating this process as necessary. The question is whether it terminates after a finite number of steps or not. This is the point where most proofs went awry.

Many of you said something like "the process will terminate after at most $n-1$ steps. Otherwise at the nth step you will consider

$$\lim_{z\to a} f(z) = \lim_{z\to a} h_(z) = h_{n-1}'(a),$$

but $\lim_{z\to a} f(z)$ does not exist because f has a singularity at a". This is INCORRECT. Here's a counterexample to that claim. Consider the function

$$f(z) = \frac{\sin z - z}{z^3}.$$

This function has a singularity at 0. It cannot be evaluated there. It is clear, however, that

$$\lim_{z\to a} z^3 f(z) = 0.$$

Moreover, using the power series expansion for $\sin z$ we easily see that

$$\begin{aligned}
\lim_{z\to a} z^2 f(z) &= \lim_{z\to a} z^2 \frac{(z - z^3/3! + z^5/5! - \cdots) - z}{z^3} &&= \lim_{z\to a} -z^2/3! + z^4/5! - \cdots = 0 \\
\lim_{z\to a} z f(z) &= \lim_{z\to a} z \frac{(z - z^3/3! + z^5/5! - \cdots) - z}{z^3} &&= \lim_{z\to a} -z/3! + z^3/5! - \cdots = 0 \\
\lim_{z\to a} f(z) &= \lim_{z\to a} \frac{(z - z^3/3! + z^5/5! - \cdots) - z}{z^3} &&= \lim_{z\to a} -1/3! + z^2/5! - \cdots = -1/6
\end{aligned}$$

So in this case $n = 3$ and $n' = 0$. The process terminates after 3 steps, and contrary to popular belief, $\lim_{z\to a} f(z)$ does exist.

So then why on Earth does the process above terminate? Suppose it does not. Then what happens? We are claiming that

$$\lim_{z\to a}(z-a)^N f(z) \quad \text{for } all\ N \leq n,$$

equivalently, in our notation,

$$h_1(a)(= g'(a)) = h_1'(a) = h_2'(a) = \cdots = h_m'(a) = \cdots = 0.$$

Now note that

$$0 = h_N'(a) = \lim_{z \to a} \frac{h_N(z) - h_1(a)}{z - a} = \frac{h_1(z) - h_1(a)}{(z-a)^N},$$

using L'Hopital's rule we compute this last quantity to be

$$\frac{h_1^{(N)}(a)}{N!}.$$

Hence $0 = h_1(a) = h_1'(a) = h_1''(a) = \cdots$. But h_1 is a holomorphic function by construction, therefore has a Taylor expansion that agrees with it at all points. We have consequently show that $h_1(z)$ is identically zero. This in turn means g and f are identically zero. But this is a contradiction because f has a singularity at a. Thus the above process must terminate finitely as claimed. □

(c) Let $f(z)$ be a holomorphic function in the neighborhood of $z = a$, except for a singularity at $z = a$. Show that either $f(z)$ has a pole of order n at a for some integer n or

$$\lim_{z \to a} (z-a)^n f(z)$$

does not exist for any n.

Solution. This is trivial after part (b). Either the said limit doesn't exist for any n, or if it exists for some N then from part (b) we know the limit exists and is non zero for some integer $n \leq N$. In this case f has a pole of order n at a. (Note that $n \leq 0$ because f has a singularity at a; if $n = 0$ the singularity is removable, as in the example we gave in part(b)). □

2. (a) If $f(z)$ is holomorphic inside and on the simple closed curve C containing $z = a$, prove that

$$f^{(n)}(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)^n}{(z-a)}\, dz.$$

Solution. Since f is analytic and products of analytic functions are analytic, f^n is also analytic and the above equality is a direct application of the general Cauchy Integral Formula. □

(b) Use (a) to prove that $|f(a)|^n \leq LM^n/2\pi D$, where is the minimum distance from a to the curve C, L is the length of C and M is the maximum value of $|f(z)|$ on C.

Solution.

$$|f(a)|^n = \left| \frac{1}{2\pi i} \oint_C \frac{f(z)^n}{(z-a)}\, dz \right| \leq \frac{1}{2\pi} \max_{z \in C} \left| \frac{f(z)^n}{(z-a)} \right| \cdot L$$
$$= \frac{L}{2\pi} \cdot \frac{\max_{z \in C} |f(z)|^n}{\min_{z \in C} |z-a|} = \frac{LM^n}{2\pi D}.$$

□

(c) Use (b) to prove that $|f(a)| \leq M$. In other words, the maximum value of $|f(z)|$ is obtained on its boundary (Maximum Modulus Principle).

Solution. Taking nth roots we see that

$$|f(a)| \leq \sqrt[n]{\frac{L}{2\pi D}} \cdot M \quad \text{for all } n.$$

Note that $L/2\pi D$ is a constant. Taking limits as $n \to \infty$ and using a standard result from real analysis,

$$|f(a)| = \lim_{n\to\infty} |f(a)| \leq \lim_{n\to\infty} \sqrt[n]{\frac{L}{2\pi D}} \cdot M = M.$$

□

(d) The maximal value of $1/z$ on the unit circle is 1, yet $|f(1/2)| = 2$. Explain why this does not contradict (c).

Solution. The function $1/z$ is not holomorphic on the unit disc (it has a simple pole at the origin). Hence it does not satisfy the hypothesis of (a) and consequently this phenomenon does not contradict (c). □

(e) (**Fundamental Theorem of Algebra**) Using the Maximum Modulus Principle prove the Fundamental Theorem of Algebra.

Solution. Let P be a polynomial of degree at least 1. If $P(z) \neq 0$, then $1/P(z)$ is analytic and its maximum modulus in the circle $|z| \leq R$ would have to occur on its boundary. We have seen, however that $P(z) \to \infty$ as $z \to \infty$, so we could choose an R so that $|1/P(z)| < |1/P(0)|$ for all $|z| = R$, and this is a contradiction. □

(f) Let f be holomorphic on and inside C. Let M be the maximal value of f on C. Suppose that $|f(a)| = M$ for some a *inside* C. Prove that $f(z)$ is constant.

Solution. By the Open Mapping Theorem, if f is not constant, then it takes a small neighborhood of a (which we can assume without loss of generality is contained in the region inside C) onto a neighborhood of $f(a)$ and this map is 1-1. This neighborhood of $f(a)$ must contain a point P such that $|P| > f(a)$, otherwise the open set would not be a neighborhood of a. But this point P is the image of some point b in the neighborhood of a under f. Hence $|f(b)| > |f(a)| = M$, and this is a contradiction. Therefore f is constant. □

Remark. Some people showed, using the maximum modulus principle that $|f(a)| = M$ for infinitely many a inside the region. This is not enough to show that $f(z) \equiv M$. For that you wouuld have needed to show $f(a) = M$ (no absolute value) for infinitely many a inside the region.

3. Let

$$f(z) = \frac{z}{e^z - 1} + \frac{z}{2} = 1 + \frac{z^2}{12} - \frac{z^4}{720} + \frac{z^6}{30240} - \cdots = \sum_{n=0}^{\infty} \frac{z^n B_n}{n!}.$$

(a) Prove that $f(-z) = f(z)$.

Solution.

$$\begin{aligned} f(-z) = \frac{-z}{e^{-z} - 1} - \frac{z}{2} &= \frac{-ze^z}{1 - e^z} - \frac{z}{2} \\ &= \frac{ze^z}{e^z - 1} - \frac{z}{2} = \frac{2ze^z - ze^z + z}{2(e^z - 1)} \\ &= \frac{ze^z - z + 2z}{2(e^z - 1)} = \frac{z(e^z - 1)}{e^z - 1} + \frac{2z}{2(e^z - 1)} \\ &= \frac{z}{2} + \frac{z}{e^z - 1} = f(z). \end{aligned}$$

□

(b) Show that $B_n =$ if n is odd.

Solution. Note that $f(z)$ extends to a holomorphic functions since the singularity at 0 is removable. It therefore agrees with the power expansion involving the B_n's. By part (a)

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-z)^n B_n}{n!} &= \sum_{n=0}^{\infty} \frac{z^n B_n}{n!} \\ \implies \sum_{n \text{ odd}} \frac{z^n B_n}{n!} &\equiv 0. \end{aligned}$$

This shows $B_n = 0$ for odd n. □

(c) Write $\tan z$ in terms of e^{iz} and use this to find the Taylor series of $\tan z$ around $z = 0$ in terms of the Bernoulli numbers B_n.

Solution.

$$\tan z = -i \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = -i \frac{e^{2iz} - 1}{e^{2iz} + 1} = -i + i \frac{2}{e^{2iz} + 1}.$$

From the definition of $f(z)$ we see that

$$e^{2iz} = \frac{2iz}{f(2iz) - iz} + 1.$$

Hence

$$\tan z = -i + \frac{2}{e^{2iz} + 1} = -i + i \frac{f(2iz) - iz}{f(2iz)} = \frac{z}{f(2iz)}.$$

We compute the first few terms of this series by inverting $f(2iz)$. The result is

$$\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \cdots .$$

Some of you successfully went through subtle manipulations to obtain the general expression

$$\tan z = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2(2^{2n} - 1)B_{2n}(2z)^{2n-1}}{(2n)!}.$$

□

(d) What is the radius of convergence of $\tan z$ around $z = 0$?

Solution. A Taylor expansion for a holomorphic function around a point agrees with the function until you hit a singularity. So the Radius of Convergence is $\pi/2$. □