PHYS 454
1st Midterm Exam
Wednesday 10th October 2012

Instructor: Dr. V. Lempesis

Student Name: ……………………………………………….

Student ID Number……………………………..

Student Grade: ……./15

Please answer all questions

1. The ground state for a particle in an infinite well has energy equal to 2 *eV*. If the state of the particle is $\psi = A\psi_1 + 2A\psi_2$, where *A* is a real constant.

a) What is *A*?

(2 marks)

b) What is the average energy?

(2 marks)

c) What is the uncertainty of energy?

(2 marks)

Solution: *From normalization condition we calculate the constant A.*

$$\int_0^a |\psi(x)|^2 dx = 1 \Rightarrow A^2 \underbrace{\int_0^a |\psi_1(x)|^2 dx}_{=1} + 4A^2 \underbrace{\int_0^a |\psi_2(x)|^2 dx}_{=1} + 2A^2 \underbrace{\int_0^a \psi_1(x)\psi_2(x) dx}_{=0} = 1$$

$$\Rightarrow 5A^2 = 1 \Rightarrow A = \pm 1/\sqrt{5}$$

Where we can keep the positive value.The probabilities for the particle to be in states 1 and 2 are:

$$P_1 = \left(\frac{1}{\sqrt{5}}\right)^2 = \frac{1}{5} \;\; and \;\; P_2 = \left(2\sqrt{\frac{1}{5}}\right)^2 = \frac{4}{5}$$

Thus the average energy is given by

$$\langle E \rangle = P_1 E_1 + P_2 E_2 = \frac{1}{4} E_1 + \frac{4}{5} E_2$$

but in an infinite well $E_n = E_1 n^2$ *thus* $E_2 = 4E_1$. *So*

$$\langle E\rangle = \frac{1}{5}E_1 + \frac{4}{5}4E_1 = \frac{17}{5}E_1 = \frac{34}{5}eV = 6.8eV$$

Similarly

$$\langle E^2\rangle = P_1E_1^2 + P_2E_2^2 = \frac{1}{5}E_1^2 + \frac{4}{5}E_2^2 = \frac{1}{5}E_1^2 + \frac{4}{5}(4E_1)^2 = 13E_1^2 = 52(eV)^2$$

Then

$$\Delta E = \sqrt{\langle E^2\rangle - \langle E\rangle^2} = 2.4eV\ .$$

2. An electron is inside an infinite square well of width equal to 1 Angstrom, and at time $t=0$ is at the state ψ_2.

a) What is the average position of the particle at t=0 s?
(2 marks)

b) What is the uncertainty in position at t=0 s?
(2 marks)

c) What is the average momentum?
(1 mark)

d) If we do not disturb the system what will be its state at t=1 s?
(1 mark)

Solution:

The wavefunction of the body is

$$\psi_2(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\right)$$

a) The average position is given by

$$\langle x\rangle = \int_0^a x\left|\psi_2(x)\right|^2 dx = \frac{2}{a}\int_0^a x\sin^2\left(\frac{n\pi x}{a}\right)dx = \frac{a}{2} = 0.5\overset{0}{\mathrm{A}}\ .$$

b) The uncertainty in the position is given by

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

$$\langle x^2\rangle = \int_0^a x^2\left|\psi_2(x)\right|^2 dx = \frac{2}{a}\int_0^a x^2\sin^2\left(\frac{n\pi x}{a}\right)dx = \frac{1}{24}a^2\left(\frac{8\pi^2-3}{\pi^2}\right) = 0.32a^2$$

thus

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{0.32a^2 - (0.5a)^2} = 0.26a = 0.26\,\overset{0}{A}$$

c) The average momentum is zero because the wavefunction is real.

d) The wavefunction after some time τ is given by

$$\psi_2(x,\tau) = \psi_2(x,0)e^{-iE_2\tau/\hbar} = \sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\right)\exp\left(-i\frac{\hbar\pi^2}{2ma^2}\tau\right)$$

3. A proton and an electron are inside identical infinite square potential wells. Which one has the smallest energy?
a) the proton b) the electron c) they have equal energies

(1 mark)

Solution: *From the energy formula $E_1 = \dfrac{\hbar^2\pi^2}{2ma^2}$ we see that the paertyicle with the largest mass has the smallest energy. Correct answer: (a)*

4. Which from the following functions could be acceptable as wavefunctions of a particle?

$$\psi_1(x) = Ne^{-\lambda x}, \quad \psi_2(x) = Ne^{-\lambda|x|}, \quad \psi_3(x) = \frac{Nx}{\sqrt{x^2 + a^2}}, \quad \psi_4(x) = Nxe^{-\lambda x^2}$$

a) ψ_2 and ψ_4 b) ψ_3 and ψ_4 c) only ψ_4 d) none of them

(1 mark)

Solution: *A physically acceptable wavefunction must satisfy the condition $\psi(\infty) = \psi(-\infty) = 0$. Correct answer: (a)*

5. The wave function for a particle in a one-dimensional box is $\psi = A\sin(n\pi x / a)$. Which statement is correct?

a. This wavefunction gives the probability of finding the particle at *x*.

b. $|\psi(x)|^2$ gives the probability of finding the particle at *x*.

c. $|\psi(x)|^2 dx$ gives the probability of finding the particle between *x* and *x* + *dx*.

d. $\int_0^a \psi(x)dx$ gives the probability of finding the particle at a particular value of x.

e. $\int_0^a |\psi(x)|^2 dx$ gives the probability of finding the particle between x and $x + dx$.

(1 mark)

Solution: *Correct answer: (c)*

$\hbar = 1.055\times10^{-34}\,J\cdot s$, $1\overset{o}{A} = 10^{-10}\,m$, $m_e = 9.1\times10^{-31}$ kg, $1eV = 1.6\times10^{-19}$ J

For an infinite square well:

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \qquad E_n = \frac{\hbar^2\pi^2}{2ma^2}n^2, \qquad n = 1,\ 2,\ \ldots,\infty$$