Math 316

Solutions of the First Midterm Exam 1434, 1st semester

Q1 Prove or disprove each of the following statements:

(a) If a set $\{x_1, x_2, ..., x_n\}$ is orthogonal in an inner product space X, then it is linearly independent.
Solution: (True)
If $\{x_1, x_2, ..., x_n\}$ is orthogonal in X, then for all $i, j \in \{1, 2, ..., n\}$

$$\begin{aligned} \langle x_i, x_j \rangle &= 0, \\ \|x_i\| &\neq 0. \end{aligned}$$

Now, let

$$\sum_{i=1}^{n} c_i x_i = 0$$

where c_i's are scalars. Then, for any $x_j \in \{x_1, x_2, ..., x_n\}$, we have

$$\left\langle \sum_{i=1}^{n} c_i x_i, x_j \right\rangle = \langle 0, x_j \rangle$$

$\Rightarrow$

$$\sum_{i=1}^{n} c_i \langle x_i, x_j \rangle = 0$$

$\Rightarrow$

$$c_j \langle x_j, x_j \rangle = 0$$

$\Rightarrow$

$$c_j \|x_j\|^2 = 0$$

$\Rightarrow$

$$c_j = 0$$

for all $j \in \{1, 2, ..., n\}$, which proves that the set $\{x_1, x_2, ..., x_n\}$ is linearly independent.

(b) If $f(x) = \ln x$ and $\rho(x) = \frac{1}{x}$, then $f \in \mathcal{L}^2_\rho(0,1)$.
Solution: (False)

$$\|\ln x\|^2_{\frac{1}{x}} = \int_0^1 |\ln x|^2 \frac{1}{x} dx$$

Using the substitution

$$\begin{aligned} u &= \ln x, \quad du = \frac{1}{x} dx, \\ x &= 0 \Rightarrow u = -\infty, \\ x &= 1 \Rightarrow u = 0, \end{aligned}$$

we get

$$\begin{aligned}\|\ln x\|_{\frac{1}{x}}^{2} &= \lim_{t\to-\infty}\int_{t}^{0}u^{2}du\\ &= \lim_{t\to-\infty}\left.\frac{u^{3}}{3}\right|_{u=t}^{u=0}\\ &= \lim_{t\to-\infty}-\frac{t^{3}}{3}\\ &= \infty\end{aligned}$$

Therefore, $f\notin\mathcal{L}_{\frac{1}{x}}^{2}(0,1)$.

Q2 Consider the sequence of functions

$$f_n(x)=x^n,\qquad x\in[0,1]$$

(a) Find the limit $f(x)$ of $f_n(x)$ as $n\to\infty$.
Solution:

$$f(x)=\lim_{n\to\infty}x^n=\begin{cases}0, & 0\le x<1\\ 1, & x=1\end{cases}$$

(b) Does $f_n(x)$ converge to $f(x)$ uniformly? Justify your answer.
Solution:
$f_n(x)$ does not converge uniformly to $f(x)$ because the function $f_n(x)=x^n$ is continuous on $[0,1]$ for all $n\in\mathbb{N}$, but $f(x)$ is not.

(c) Does $f_n(x)$ converge to $f(x)$ in $\mathcal{L}^2([0,1])$? Justify your answer.
Solution: yes

$$f_n\overset{\mathcal{L}^2}{\to}f\quad\Leftrightarrow\quad\lim_{n\to\infty}\|f_n-f\|=0$$

Now,

$$\begin{aligned}\lim_{n\to\infty}\|f_n-f\| &= \lim_{n\to\infty}\|x^n-0\|\\ &= \lim_{n\to\infty}\left(\int_0^1|x^n|^2\,dx\right)^{\frac{1}{2}}\\ &= \lim_{n\to\infty}\left(\int_0^1x^{2n}dx\right)^{\frac{1}{2}}\\ &= \lim_{n\to\infty}\left(\left.\frac{x^{2n+1}}{2n+1}\right|_0^1\right)^{\frac{1}{2}}\\ &= \lim_{n\to\infty}\frac{1}{\sqrt{2n+1}}\\ &= 0.\end{aligned}$$

Q3 Consider the eigenvalue problem

$$\begin{aligned} Lu + \lambda u = 0, &\quad x \in [a, b], \\ u(a) = 0, &\quad u(b) = 0 \end{aligned} \tag{1}$$

(a) Prove that if L is a self-adjoint operator, then $\lambda \in \mathbb{R}$.

Solution:

If L is a self-adjoint operator, then

$$\langle Lu, u \rangle = \langle u, Lu \rangle \tag{2}$$

for any $u \in \mathcal{L}^2([a, b])$. Now, let λ be an eigenvalue of $-L$ and let u be the corresponding eigenfunction, then

$$\langle Lu, u \rangle = \langle -\lambda u, u \rangle = -\lambda \langle u, u \rangle = -\lambda \|u\|^2,$$

On the other hand, we have

$$\langle u, Lu \rangle = \langle u, -\lambda u \rangle = -\overline{\lambda} \langle u, u \rangle = -\overline{\lambda} \|u\|^2,$$

Using (2), we get

$$-\lambda \|u\|^2 = -\overline{\lambda} \|u\|^2$$

but since $\|u\| \neq 0$ (because u is an eigenfunction), the above equation leads to

$$-\lambda = -\overline{\lambda}$$

i.e. $\lambda \in \mathbb{R}$.

(b) Show that if $L = \left(1 + 3x^2\right) \frac{d^2}{dx^2} + 6x \frac{d}{dx}$ in problem (1), then L is a self-adjoint operator.

Solution:

If $L = \left(1 + 3x^2\right) \frac{d^2}{dx^2} + 6x \frac{d}{dx}$, then we have

1)

$$p(x) = 1 + 3x^2, q(x) = 6x, r(x) = 0$$

are all real functions.

2)

$$p'(x) = 6x = q(x)$$

3) For any eigenfunctions u and v of (1), we have

$$\begin{aligned} p\left(u^{'}v - uv'\right)\Big|_a^b &= p(b)\left(u^{'}(b)\, v(b) - u(b)\, v'(b)\right) - p(a)\left(u^{'}(a)\, v(a) - u(a)\, v'(a)\right) \\ &= p(b)\left(u^{'}(b)(0) - (0)\, v'(b)\right) - p(a)\left(u^{'}(a)(0) - (0)\, v'(a)\right) \\ &= 0 \end{aligned}$$

i.e. L is a self-adjoint operator.

Q4 Consider the eigenvalue problem

$$u'' + 2u' + \lambda u = 0, \qquad x \in [0,1], \tag{3}$$
$$u(0) = 0, \qquad u(1) = 0$$

(a) Find the eigenvalues and eigenfunctions of problem (3).

Solution:

The auxiliary equation is

$$m^2 + 2m + \lambda = 0,$$

which have the solution

$$m = -1 \pm \sqrt{1-\lambda},$$

Thus, we have the following cases:

1) If $\lambda = 1$, then there is one root $m = -1$ and the general solution of (3) is given by

$$u(x) = c_1 e^{-x} + c_2 x e^{-x}$$

Using the boundary conditions, we have

$$\begin{aligned} u(0) &= c_1 e^0 \\ &\Rightarrow 0 = c_1 \end{aligned}$$

and

$$\begin{aligned} u(1) &= c_2 e^{-1} \\ &\Rightarrow 0 = c_2 e^{-1} \\ &\Rightarrow 0 = c_2 \end{aligned}$$

That is, $u(x) = 0$, which is not acceptable. Thus, $\lambda = 1$ is not an eigenvalue of (3).

2) If $\lambda < 1$, then we have two real roots $m_1 = -1 - \sqrt{1-\lambda}$ and $m_2 = -1 + \sqrt{1-\lambda}$, and the general solution is given by

$$u(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Using the boundary conditions, we have

$$\begin{aligned} u(0) &= c_1 e^0 + c_2 e^0 \\ &\Rightarrow 0 = c_1 + c_2 \\ &\Rightarrow c_2 = -c_1 \end{aligned}$$

and

$$\begin{aligned} u(1) &= c_1 e^{m_1} + c_2 e^{m_2} \\ &\Rightarrow 0 = c_1 e^{m_1} - c_1 e^{m_2} \\ &\Rightarrow 0 = c_1 \left(e^{m_1} - e^{m_2}\right) \end{aligned}$$

but, $e^{m_1} - e^{m_2} \neq 0$ because the exponential function is one-to-one. Therefore, c_1 must be zero, and consequently $u(x) = 0$. Thus, there are no eigenvalue of (3) in $(-\infty, 1)$.

3) If $\lambda > 1$, we have two complex roots $m_1 = -1 - \sqrt{\lambda - 1}i$ and $m_2 = -1 + \sqrt{\lambda - 1}i$. The general solution is given by

$$u(x) = e^{-x}\left(c_1 \cos\sqrt{\lambda - 1}x + c_2 \sin\sqrt{\lambda - 1}x\right)$$

Using the boundary conditions, we have

$$\begin{aligned} u(0) &= e^0 (c_1 \cos 0 + c_2 \sin 0) \\ &\Rightarrow 0 = c_1 \end{aligned}$$

and

$$\begin{aligned} u(1) &= e^{-1} c_2 \sin\sqrt{\lambda - 1} \\ &\Rightarrow 0 = c_2 \sin\sqrt{\lambda - 1} \end{aligned}$$

but $c_2 \neq 0$, otherwise we would have a zero eigenfunction. Thus,

$$\sin\sqrt{\lambda - 1} = 0$$
$$\Rightarrow \sqrt{\lambda - 1} = n\pi, \quad n \in \mathbb{N}$$

The eigenvalues of (3) are thus given by

$$\lambda_n = n^2\pi^2 + 1, \quad n \in \mathbb{N}$$

and the corresponding eigenfunctions are

$$u_n(x) = e^{-x} \sin n\pi x, \quad n \in \mathbb{N}$$

(b) Show that L is not a self-adjoint operator.

Solution:

In (3), L is given by

$$L = \frac{d^2}{dx^2} + 2\frac{d}{dx} \tag{4}$$

and since

$$p'(x) = 0 \neq 2 = q(x)$$

L is not a self-adjoint operator.

(c) Transform L into a self-adjoint operator.

Solution:

We first find the function $\rho(x) > 0$ that produces a formally self-adjoint operator ρL.

$$\begin{aligned} \rho(x) &= \frac{1}{p(x)} e^{\int \frac{q(x)}{p(x)} dx} \\ &= \frac{1}{1} e^{\int \frac{2}{1} dx} \\ &= e^{2x} \end{aligned}$$

Therefore,

$$\rho L = e^{2x} \frac{d^2}{dx^2} + 2e^{2x} \frac{d}{dx}$$

is a formally self-adjoint operator. It is a self-adjoint operator for the above specific problem since the boundary conditions are separated and homogenous.

(d) Write the orthogonality relation between the eigenfunctions of problem (3).

Solution:

The eigenfunction of the operator $-L$ are eigenfunction of the self-adjoint operator $-\rho L$. Therefore, these eigenfunctions are orthogonal in $\mathcal{L}^2_\rho(0,1)$. Namely, for $n \neq m$ we have

$$\left\langle e^{-x}\sin n\pi, e^{-x}\sin m\pi\right\rangle_{e^{2x}} = \int_0^1 \left(e^{-x}\sin n\pi\right)\left(e^{-x}\sin m\pi\right) e^{2x} dx = 0$$

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