PHYS 404
2nd MIDTERM EXAM Exam
Sunday 17th November 2019

Instructor: Dr. V. Lempesis

Student Name: ……………………………………………….

Student ID Number……………………………..

Student Grade: ……/15

Please answer all the following questions

1. Show that: $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$.

Hint: You are given that,

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x), \quad J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$$

(5 marks)

Solution:

$$\left.\begin{array}{l} J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x) \\ J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x) \end{array}\right\} \underset{+}{\Rightarrow} J_{n-1}(x) + J_{n+1}(x) + J_{n-1}(x) - J_{n+1}(x) = \frac{2n}{x}J_n(x) + 2J_n'(x) \Rightarrow$$

$$2J_{n-1}(x) = \frac{2n}{x}J_n(x) + 2J_n'(x) \Rightarrow J_{n-1}(x) = \frac{n}{x}J_n(x) + J_n'(x) \Rightarrow$$

$$x^n J_{n-1}(x) = \frac{n}{x}x^n J_n(x) + x^n J_n'(x) \Rightarrow x^n J_{n-1}(x) = nx^{n-1}J_n(x) + x^n J_n'(x) \Rightarrow$$

$$x^n J_{n-1}(x) = \left(x^n\right)' J_n(x) + x^n J_n'(x) \Rightarrow x^n J_{n-1}(x) = \frac{d}{dx}\left[x^n J_n(x)\right]$$

2. Calculate the integral $\int x^{n+1} J_n\left(x\right)dx$. You are given that $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$.

(5 marks)

Solution:

We know that:

$$\frac{d}{dx}\left[x^m J_m(x)\right] = x^m J_{m-1}(x) \underset{m=n+1}{\Rightarrow} \frac{d}{dx}\left[x^{n+1} J_{n+1}(x)\right] = x^{n+1} J_{n+1-1}(x) \Rightarrow$$
$$\frac{d}{dx}\left[x^{n+1} J_{n+1}(x)\right] = x^{n+1} J_n(x)$$

Thus

$$\int x^{n+1} J_n\left(x\right) dx = \int \frac{d}{dx}\left[x^{n+1} J_{n+1}(x)\right] dx = x^{n+1} J_{n+1}(x) + c$$

3. Prove that:

$$e^x = I_0(x) + 2\sum_{n=1}^{\infty} I_n(x)$$

You are given that:

$$I_{-n}(x) = I_n(x)$$
$$e^{\left(\frac{x}{2}\right)\left(t+\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} I_n(x) t^n$$

(5 marks)

Solution:

$$e^{\left(\frac{x}{2}\right)\left(t+\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} I_n(x) t^n \underset{t=1}{\Rightarrow} e^{\left(\frac{x}{2}\right)\left(1+\frac{1}{1}\right)} = \sum_{n=-\infty}^{\infty} I_n(x) \Rightarrow e^{\left(\frac{x}{2}\right)2} = \sum_{n=-\infty}^{\infty} I_n(x) \Rightarrow$$
$$e^x = \sum_{n=-\infty}^{\infty} I_n(x) \Rightarrow e^x = \sum_{n=-\infty}^{-1} I_n(x) + I_0(x) + \sum_{n=1}^{\infty} I_n(x) \Rightarrow$$
$$e^x = \sum_{n=1}^{\infty} I_{-n}(x) + I_0(x) + \sum_{n=1}^{\infty} I_n(x) \underset{I_{-n}(x)=I_n(x)}{\Rightarrow} e^x = I_0(x) + 2\sum_{n=1}^{\infty} I_n(x)$$

4. (i) Find the general solution of the following differential equation:

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(1-\frac{4}{x^2}\right)u=0.$$

(ii)Find the general solution of the following differential equation:

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(1-\frac{1}{9x^2}\right)u=0.$$

(iii) Find the general solution of the following differential equation:

$$x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}+\left(9x^2-4\right)u=0$$

(iv) Find the general solution of the differential equation:

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(1+\frac{4}{x^2}\right)u=0$$

You are given:

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(1-\frac{v^2}{x^2}\right)u=0,\quad \text{Bessel Diff. Equation}$$

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(1+\frac{v^2}{x^2}\right)u=0,\quad \text{Modified Bessel Diff. Equation}$$

Solution:

Vasileios Lembessis 17/11/2019 18:54
Comment [1]: Some of you found the value of v but they did not give the solution!

(i) $\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(1-\frac{4}{x^2}\right)u=0 \Rightarrow \frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(1-\frac{2^2}{x^2}\right)u=0$

But 3 is an integer thus the general solution is given as

$$u(x)=AJ_2(x)+BN_2(x)$$

(ii)

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(1-\frac{1}{9x^2}\right)u=0 \Rightarrow \frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(1-\frac{(1/3)^2}{x^2}\right)u=0$$

$$u(x)=AJ_{1/3}(x)+BJ_{-1/3}(x)$$

(iii) As we have discussed the general solution is

$$x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}+\left(9x^2-4\right)u=0\Rightarrow x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}+\left(3^2x^2-2^2\right)u=0$$

$$u(x)=AJ_2(3x)+BN_2(3x)$$

(iv) Find the general solution of the differential equation:

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(1+\frac{4}{x^2}\right)u=0\Rightarrow\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(1+\frac{2^2}{x^2}\right)u=0$$

$$u(x)=AI_2(x)+BK_2(x)$$