PHYS 454
FINAL EXAM
Monday 7th January 2013

Instructor: Dr. V. Lempesis

Student Name: ……………………………………………….

Student ID Number…………………………..

Student Grade: ……./40

Please answer all questions

1. A quantum SHO at t=0 is at the

$$\Psi(0) = A(6\psi_0 + 8\psi_1)$$

a) Calculate A?

(2 marks)

b) What is the average energy?

(2 marks)

c) What is the uncertainty in energy?

(2 marks)

d) What is the state of the SHO at time $t = \frac{\pi}{\omega}$?

(3 marks)

e) If at t=0 we do a measurement and we find that the SHO is at state ψ_1, what will be the energy of a particle at time $t = \frac{\pi}{3\omega}$?

(1 mark)

Solution: *a) From normalization condition we calculate the constant A.*

$$\int_0^a |\psi(x)|^2 dx = 1 \Rightarrow 36A^2 \underbrace{\int_0^a |\psi_0(x)|^2 dx}_{=1} + 64A^2 \underbrace{\int_0^a |\psi_1(x)|^2 dx}_{=1} + 96A^2 \underbrace{\int_0^a \psi_0(x)\psi_1(x)dx}_{=0} = 1$$

$$\Rightarrow 100A^2 = 1 \Rightarrow A = 0.1$$

The probabilities for the particle to be in states 0 and 1 are:

$$P_0 = (6A)^2 = 0.36 \quad and \quad P_1 = (8A)^2 = 0.64$$

b) Thus the average energy is given by

$$\langle E\rangle = P_0E_0 + P_1E_1 = 0.36E_0 + 0.64E_1$$

but in a quantum SHO $E_n = \left(n+\frac{1}{2}\right)\hbar\omega$ *thus* $E_0 = \frac{\hbar\omega}{2}$ *and* $E_1 = \frac{3\hbar\omega}{2}$. *So*

$$\langle E\rangle = 1.14\hbar\omega$$

c) Similarly

$$\langle E^2\rangle = P_0E_0^2 + P_1E_1^2 = 0.36E_0^2 + 0.64E_1^2 = 1.53(\hbar\omega)^2$$

Then

$$\Delta E = \sqrt{\langle E^2\rangle - \langle E\rangle^2} = 0.48\hbar\omega.$$

d) The wavefunction after some time t is given by

$$\Psi(x,t) = 0.6\psi_0(x)e^{-iE_0t/\hbar} + 0.8\psi_1(x)e^{-iE_1t/\hbar} = 0.6\psi_0(x)e^{-i\omega t/2} + 0.8\psi_1(x)e^{-i3\omega t/2} \overset{t=\pi/\omega}{=}$$
$$0.6\psi_0(x)e^{-i\pi/2} + 0.8\psi_1(x)e^{-i3\pi/2} = -0.6i\psi_0(x)e^{-i\pi/2} + 0.8i\psi_1(x)$$

e) After the measurement the system collapses to the state ψ_1 *so any repeated measurement of the energy will give us the value* E_1

2. An electron is confined in an infinite potential well. When this electron makes a transition from the state n=2 to the state n=1 it emits a photon with a wavelength $\lambda = 700$ nm.

a) Calculate the width a of the well.

(2 marks)

If the electron is prepared, at t=0, at the state $\Psi = A(3\psi_1 + 4\psi_2)$:

b) Calculate A.

(2 marks)

c) Calculate the average energy of the electron at this state.

(2 points)

d) What is the probability of finding the particle in the region $(0,\ a/6)$?

(4 points)

Solution: *a) The energy of the photon is equal to the difference of the energies of the two levels:*

$$E_2 - E_1 = hf \Rightarrow E_2 - E_1 = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{E_2 - E_1} \Rightarrow \lambda = \frac{hc}{\frac{\hbar^2\pi^2}{2ma^2}\left(2^2 - 1^2\right)} \Rightarrow$$

$$a^2 = \frac{3\lambda\hbar^2\pi^2}{2mhc} \Rightarrow a = \sqrt{\frac{3h\lambda}{8cm}} = \sqrt{\frac{3\times 6.64\times 10^{-34}\times 700\times 10^{-9}}{8\times 3\times 10^{8}\times 9.1\times 10^{-31}}} = 0.8\times 10^{-9}m = 0.8 \text{ nm}$$

b) From normalization condition we calculate the constant A.

$$\int_0^a \left|\psi(x)\right|^2 dx = 1 \Rightarrow 9A^2 \underbrace{\int_0^a \left|\psi_1(x)\right|^2 dx}_{=1} + 16A^2 \underbrace{\int_0^a \left|\psi_2(x)\right|^2 dx}_{=1} + 24A^2 \underbrace{\int_0^a \psi_1(x)\psi_2(x)dx}_{=0} = 1$$

$$\Rightarrow 25A^2 = 1 \Rightarrow A = 0.2$$

The probabilities for the particle to be in states 1 and 2 are:

$$P_1 = \left(3A\right)^2 = 0.36 \quad and \; P_1 = \left(16A\right)^2 = 0.64$$

c) Thus the average energy is given by

$$\left\langle E\right\rangle = P_1E_1 + P_2E_2 = 0.36E_0 + 0.64E_1$$

but in an infinite well $E_n = E_1 n^2$ *thus* $E_2 = 4E_1$. *So*

$$\left\langle E\right\rangle = 0.36E_1 + 0.64E_2 = 0.36E_1 + 0.64\times 4E_1 = 2.92E_1 = 2.92\frac{\hbar^2\pi^2}{2ma^2} =$$

$$2.92\frac{\left(1.055\times 10^{-34}\right)^2\times 3.14^2}{2\times 9.1\times 10^{-31}\times\left(0.8\times 10^{-9}\right)^2} = 2.75\times 10^{-19}J = 1.72\ eV$$

d) The probability is given by

$$P(0 < x < a/6) = 9A^2 \int_0^{a/6} \left|\psi_1(x)\right|^2 dx + 16A^2 \int_0^{a/6} \left|\psi_2(x)\right|^2 dx + 24A^2 \int_0^{a/6} \psi_1(x)\psi_2(x)dx$$

$$= \frac{9}{25}\left(\frac{2}{a}\right)\int_0^{a/6} \sin\left(\frac{\pi x}{a}\right)^2 dx + \frac{16}{25}\left(\frac{2}{a}\right)\int_0^{a/6} \sin\left(\frac{2\pi x}{a}\right)^2 dx + \frac{24}{25}\left(\frac{2}{a}\right)\int_0^{a/6} \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)dx$$

$$= 0.124$$

3. For a quantum SHO show the following:

a) $\left[N,\, a^2\right] = -2a^2$.

(2 marks)

b) $\left[N,\, aa^\dagger a\right] = -aa^\dagger a$.

(2 marks)

c) Assume that the SHO is at the state $\left|n\right\rangle$. Using algebraic techniques show that $\left\langle n\right|px\left|n\right\rangle = \frac{ih}{2}$

(4 marks)

Solution:

a) $\left[N,\, a^2\right] = \left[N,\, aa\right] = \left[N,\, a\right]a + a\left[N,\, a\right]$

But $\left[N,\, a\right] = -a$, *so*

$$\left[N,\, a\right] = -aa - aa = -2a^2.$$

b)

$$\left[N,\, aa^\dagger a\right] = \left[N,\, a\left(a^\dagger a\right)\right] = a\left[N,\, \left(a^\dagger a\right)\right] + \left[N,\, a\right]\left(a^\dagger a\right) = 0 - a\left(a^\dagger a\right) = -aa^\dagger a$$

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c) From the relations

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right), \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$$

we solve for x and p and we get

$$x = \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}\left(a + a^\dagger\right), \quad p = \frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}\left(a - a^\dagger\right)$$

so we have

$$px = \frac{\hbar}{2i}\left(a + a^{\dagger}\right)\left(a - a^{\dagger}\right) = \frac{\hbar}{2i}\left(a^2 + a^{\dagger 2} - aa^{\dagger} + a^{\dagger}a\right)$$

$$\langle n|px|n\rangle = \frac{\hbar}{2i}\langle n|\left(a^2 + a^{\dagger 2} - aa^{\dagger} + a^{\dagger}a\right)|n\rangle =$$

$$\frac{\hbar}{2i}\left\{\underbrace{\langle n|a^2|n\rangle}_{=0} + \underbrace{\langle n|a^{\dagger 2}|n\rangle}_{=0} - \underbrace{\langle n|aa^{\dagger}|n\rangle}_{1+n} + \underbrace{\langle n|a^{\dagger}a|n\rangle}_{=n}\right\} =$$

$$-\frac{\hbar}{2i} = \frac{\hbar}{2}i$$

4. (a) An electron with kinetic energy E = 16.0 eV is incident on a "step" of positive potential and is reflected with a probability equal to R=1/9. What is the "height" V_0 of the potential barrier?

(3 marks)

(b) An electron is trapped in a one-dimensional finite potential well of depth $V_0 = 6$ eV and of width $a = 10\,\overset{0}{A}$. Calculate the number of bound states.

(2 marks)

(c) A particle of mass m is moving in the potential $V = Ax^2e^{-ax^2}$. Use the parabolic approximation to calculate the first two energy levels of this potential.

(3 marks)

Solution:
(a) *Solution: The reflection probability is*

$$R = \left(\frac{\sqrt{E} - \sqrt{E - V_0}}{\sqrt{E} + \sqrt{E - V_0}}\right)^2 \Rightarrow \frac{1}{9} = \left(\frac{\sqrt{16} - \sqrt{16 - V_0}}{\sqrt{16} + \sqrt{16 - V_0}}\right)^2 \Rightarrow \frac{1}{9} = \left(\frac{4 - \sqrt{16 - V_0}}{4 + \sqrt{16 - V_0}}\right)^2$$

Put $x = \sqrt{16 - V_0}$ *and remember that x > 0.*

$$\frac{1}{9} = \left(\frac{4 - x}{4 + x}\right)^2 \Rightarrow \left(4 + x\right)^2 = 9\left(4 - x\right)^2 \Rightarrow \left(x^2 + 8x + 16\right) = 9\left(x^2 - 8x + 16\right)$$

$$\Rightarrow 8x^2 - 80x + 128 = 0 \Rightarrow x^2 - 10x + 16 = 0$$

Solving this you get

$x_1 = 8, \quad x_2 = 2$

Thus

$$8=\sqrt{16-V_0}\Rightarrow 16-V_0=64\Rightarrow V_0=-48eV \ \ impossible$$
$$2=\sqrt{16-V_0}\Rightarrow 16-V_0=4\Rightarrow V_0=12\ eV$$

(b) *Solution: The number of bound states in a finite well is given by*

$$N=\left[\frac{\lambda}{\pi/2}\right]+1 \qquad U_0=\frac{2mV_0}{\hbar^2} \qquad \lambda=a\sqrt{U_0}$$

$$\lambda=\frac{a\sqrt{2mV_0}}{\hbar}=\frac{10\times10^{-10}\sqrt{2\times9.1\times10^{-31}\times6\times1.6\times10^{-19}}}{1.055\times10^{-34}}=12.5$$

$$N=\left[\frac{\lambda}{\pi/2}\right]+1=\left[\frac{12.5}{1.57}\right]+1=\left[7.96\right]+1=8$$

c) *Solution: The parabolic approximation for any potential is given by*

$$V(x)=V(0)+V^{'}(0)x+\frac{1}{2}V^{''}(0)x^2+....$$

For the given potential we have

$$V(0)=0$$
$$V^{'}(0)=2ax(1-ax^2)\exp(-ax^2)\Big|_{x=0}=0$$
$$V^{''}(0)=2A\exp(-ax^2)\left(1-5x^2a+2a^2x^4\right)\Big|_{x=0}=2A$$

Thus

$$V(x)=\frac{1}{2}\left(2A\right)x^2$$

As we see this a SHO potential with $k=2A\Rightarrow m\omega^2=2A\Rightarrow\omega=\sqrt{\frac{2A}{m}}$

Thus the energies of the first two states are

$$E_0=\hbar\omega/2=\frac{\hbar}{2}\sqrt{\frac{2A}{m}}, \quad E_1=3\hbar\omega/2=\frac{3\hbar}{2}\sqrt{\frac{2A}{m}}$$

Multiple Choice Section

Each question gets 1 mark for correct answer

5 . Which of the following functions could represent the wave function of a free particle?

a. Ae^{ikx^2}
b. Ae^{ikx}
c. Ae^{-x^2}
d. $A\sin(kx)$

6. Particles have a total energy that is greater than the "potential step." What is the probability that the particles will be reflected?

a. more information is needed
b. zero
c. 100%
d. greater than zero

7. Particles have a total energy that is less than that of a potential barrier. When a particle penetrates a barrier, its wave function is

a. exponentially increasing
b. oscillatory
c. exponentially decreasing
d. none of the above

8. The ground state energy of a harmonic oscillator is

a. $E = \hbar\omega$
b. $\boldsymbol{E = \hbar\omega/2}$
c. $E = (2/3)\hbar\omega$
d. $E = 0$
e. $E = \hbar\omega/4$

Physical constants and formulas

$h = 6.63\times10^{-34}\, J\cdot s$, $\hbar = h/2\pi = 1.055\times10^{-34}\, J\cdot s$, $1\overset{o}{A} = 10^{-10}\, m$, $m_e = 9.1\times10^{-31}$ kg, $1eV = 1.6\times10^{-19}$J

For an infinite square well:

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \qquad E_n = \frac{\hbar^2\pi^2}{2ma^2}n^2, \qquad n = 1,\ 2,\ \ldots,\infty$$

For a particle wave:

$\lambda = h/p$ and $k = 2\pi/\lambda$.

For an finite square well:

$$N = \left[\frac{\lambda}{\pi/2}\right]+1 \qquad U_0 = \frac{2mV_0}{\hbar^2} \qquad \lambda = a\sqrt{U_0}$$

For a potential step: $E > V_0$

$$R = \left(\frac{k-k'}{k+k'}\right)^2 = \left(\frac{\sqrt{E}-\sqrt{E-V_0}}{\sqrt{E}+\sqrt{E-V_0}}\right)^2, \quad T = \frac{4kk'}{\left(k+k'\right)^2} = \frac{4\sqrt{E\left(E-V_0\right)}}{\left(\sqrt{E}+\sqrt{E-V_0}\right)^2}$$

For a potential barrier of finite width: $E > V_0$

$$T = \frac{4E\left(E-V_0\right)}{V_0^2\sin^2\left(L\left(2m/\hbar^2\right)^{0.5}\sqrt{E-V_0}\right)+4E\left(E-V_0\right)}$$

For a potential barrier of finite width: $E < V_0$

$$T = \frac{4\left(V_0-E\right)E}{V_0^2\sinh^2\left(L\left(2m/\hbar^2\right)^{0.5}\sqrt{V_0-E}\right)+4\left(V_0-E\right)E}$$

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$

For the SHO:

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega\,, \quad \psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}\,, \quad \xi \equiv \sqrt{\frac{m\omega}{\hbar}}x$$

$$H_0 = 1$$
$$H_1 = 2\xi$$
$$H_2 = 4\xi^2 - 2$$
$$H_3 = 8\xi^3 - 12\xi$$
$$H_4 = 16\xi^4 - 48\xi^2 + 12$$
$$H_5 = 32\xi^5 - 160\xi^3 + 120\xi$$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right), \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$$

$$a\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle, \quad a^\dagger\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle$$

$$N = a^\dagger a, \quad [a,\, a^\dagger] = 1$$

Useful mathematics:

$$\int_{-\infty}^{+\infty} x^n e^{-\lambda x} dx = \frac{n!}{\lambda^{n+1}}\,, \qquad \int_{-\infty}^{+\infty} e^{-\lambda x^2} dx = \sqrt{\frac{\pi}{\lambda}}\,, \qquad \int_{-\infty}^{+\infty} x^2 e^{-\lambda x^2} dx = \frac{1}{2\lambda}\sqrt{\frac{\pi}{\lambda}}\,,$$

$$\int_{-\infty}^{+\infty} x^{2v} e^{-\lambda x^2} dx = \frac{(2n)!}{n!(4\lambda)^n}\sqrt{\frac{\pi}{\lambda}}$$

$$\int \sin^2(kx)\,dx = \frac{x}{2} - \frac{1}{4k}\sin(2kx)$$

$$\int \sin(kx)\sin(2kx)\,dx = \frac{1}{6k}\big(3\sin(kx) - \sin(3kx)\big)$$

$$V(x) = V(0) + V^{'}(0)x + \frac{1}{2}V^{"}(0)x^2 + ...$$
$$(\sin x)^{'} = \cos x, \quad (\cos x)^{'} = -\sin x$$

$$\left[A,B\right] = AB - BA, \quad \left[A,B\right] = -\left[B,A\right], \quad \left[A,B+C\right] = \left[A,B\right] + \left[A,C\right],$$
$$\left[A,BC\right] = B\left[A,C\right] + \left[A,B\right]C$$