Q1 $E(X_1) = 4$, $Var(X_1) = 3$

$E(X_2) = 9$, $Var(X_2) = 7$

$E(X_3) = 3$, $Var(X_3) = 5$

(a) $E(Y) = E(2X_1 - 3X_2 + 4X_3) = 2E(X_1) - 3E(X_2) + 4E(X_3)$

$= 2(4) - 3(9) + 4(3)$

$= 8 - 27 + 12 = -7$

$E(Z) = E(X_1 + 2X_2 - X_3) = E(X_1) + 2E(X_2) - E(X_3)$

$= 4 + (2)9 - 3$

$= 4 + 18 - 3 = 19$

(b) Since X_1, X_2 and X_3 are indep.

$Var(Y) = Var(2X_1 - 3X_2 + 4X_3) = 4V(X_1) + 9V(X_2) + 16V(X_3)$

$= 4(3) + 9(7) + 16(5)$

$= 12 + 63 + 8 = 155$

$Var(Z) = V(X_1 + 2X_2 - X_3) = V(X_1) + 4V(X_2) + V(X_3)$

$= 3 + 4(7) + 5$

$= 3 + 28 + 5 = 36$

Q2 X and Y are indep.

(a) $E(XY) = E(X)E(Y) = 3(5) = 15$

(b) $E(X^2Y) = E(X^2)E(Y) = 11(5) = 55$

where $Var(X) = E(X^2) - [E(X)]^2$

$\Leftrightarrow 2 = E(X^2) - 3^2$

$\Leftrightarrow E(X^2) = 2 + 9 = 21$

Q3

ⓐ $E(X) = \int_0^\infty x f(x)\,dx = \int_0^\infty x\, e^{-x}\,dx$

بالقانون

$= \frac{\Gamma 2}{1^2} = \frac{1!}{1} = 1$

بالتجزئ

$u = x \Rightarrow du = dx \quad , \quad dv = e^{-x}dx \Rightarrow v = -e^{-x}$

$= -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 1$

$Var(X) = E(X^2) - [E(X)]^2 = 2 - 1^2 = 2-1 = 1$

$E(X^2) = \int_0^\infty x^2 e^{-x}\,dx$

بالقانون

$= \frac{\Gamma 3}{1^3} = 2! = 2$

بالتجزئ

$u = x^2 \Rightarrow du = 2x\,dx \quad , \quad dv = e^{-x}dx \Rightarrow v = -e^{-x}$

$= -x^2 e^{-x}\Big|_0^\infty + 2\int_0^\infty x e^{-x}\,dx$

$= 2\int_0^\infty x e^{-x}\,dx$ نفس E(X)

$= 2(1) = 2$

ⓑ نفس طريقة حل a وبالتالي

$E(Y) = 1$ and $V(Y) = 1$

ⓒ as X and Y are indep. $\Rightarrow E(XY) = E(X)\,E(Y) = 1$

ⓓ as X and Y are indep. $\Rightarrow E(X^2Y^3) = E(X^2)\,E(Y^3) = 2(6) = 12$

$E(X^2) = 2$ من فقرة a أعلاه

بالتجزئ

بالقانون

$u = y^3 \Rightarrow du = 3y^2dy, \quad dv = e^{-y}dy \Rightarrow v = -e^{-y}$

$\int_0^\infty y^3 e^{-y}\,dy = \frac{\Gamma 4}{1^4} = 3! = 6$

$\int_0^\infty y^3 e^{-y}\,dy = -y^3e^{-y}\Big|_0^\infty + 3\int_0^\infty y^2 e^{-y}\,dy$ نفس $E(Y^2)$

$= 0 + 3(2)$

$= 6$

(١٢) 4

$$|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases} \Rightarrow f(x) = \begin{cases} \frac{1}{2} e^{x}, & x < 0 \\ \frac{1}{2} e^{-x}, & x > 0 \end{cases}$$

$$E(X) = \frac{1}{2}\left[\int_{-\infty}^{0} x e^{x}\,dx + \int_{0}^{\infty} x e^{-x}\,dx\right] = \frac{1}{2}\left[I_1 + I_2\right] = \frac{1}{2}\left[(-1) + 1\right] = 0$$

I_1 :

بالقانون

$$= \left[\frac{x}{1} - \frac{1}{1}\right] e^{x}\Big|_{-\infty}^{0}$$

$$= \left[x e^{x} - e^{x}\right]\Big|_{-\infty}^{0}$$

$$= \left[0 e^{0} - e^{0}\right] - \left[(-\infty) e^{-\infty} - e^{-\infty}\right]$$

$$= -1$$

ممكن حلة بالتعويض وذلك بأخذ $z = -x$

بالتجزئ

$$u = x \Rightarrow du = dx, \quad dV = e^{x} dx \Rightarrow V = e^{x}$$

$$= x e^{x}\Big|_{-\infty}^{0} - \int_{-\infty}^{0} e^{x} dx = 0 - e^{x}\Big|_{-\infty}^{0} = -1$$

I_2 :

بالقانون

$$= \frac{\Gamma 2}{1^2} = 1! = 1$$

بالتجزئ

$$u = x \Rightarrow du = dx, \quad dV = e^{-x} dx \Rightarrow V = -e^{-x}$$

$$= -x e^{-x}\Big|_{0}^{\infty} + \int_{0}^{\infty} e^{-x} dx$$

$$= 0 - e^{-x}\Big|_{0}^{\infty} = 1$$

$$V(X) = E(X^2) - \left[E(X)\right]^2 = 2 - 0 = 2$$

where

$$E(X^2) = \frac{1}{2}\left[\int_{-\infty}^{0} x^2 e^{x} dx + \int_{0}^{\infty} x^2 e^{-x} dx\right] = \frac{1}{2}\left[I_3 + I_4\right] = \frac{1}{2}\left[2 + 2\right] = \frac{4}{2} = 2$$

I_3 :

بالقانون

$$= \left[\frac{x^2}{1} - \frac{2x}{1^2} + \frac{2}{1^3}\right] e^{x}\Big|_{-\infty}^{0}$$

$$= \left[x^2 e^{x} - 2x e^{x} + 2 e^{x}\right]\Big|_{-\infty}^{0}$$

$$= [2] - [0] = 2$$

ممكن حلة كذلك بالتعويض وذلك بأخذ $z = -x$

بالتجزئ

$$u = x^2 \Rightarrow du = 2x\,dx, \quad dV = e^{x} dx \Rightarrow V = e^{x}$$

$$= x^2 e^{x}\Big|_{-\infty}^{0} - 2\int_{-\infty}^{0} x e^{x} dx \longrightarrow \text{نفس } I_1$$

$$= 0 - 2(-1) = 2$$

I_4 :

بالقانون

$$= \frac{\Gamma 3}{1^3} = 2! = 2$$

بالتجزئ

$$u = x^2 \Rightarrow du = 2x\,dx, \quad dV = e^{-x} dx \Rightarrow V = -e^{-x}$$

$$= -x^2 e^{-x}\Big|_{0}^{\infty} + 2\int_{0}^{\infty} x e^{-x} dx \longrightarrow \text{نفس } I_2$$

$$= 0 + 2(1) = 2$$

Q5 $X_1, X_2, \dots, X_n \sim f(x)$ iid and $f(x)$ has $\mu = E(X)$, $\sigma^2 = Var(X)$

Note:

for any distribution has μ and σ^2. Then $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ has

$$* \quad E(\overline{X}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{n}{n} E(X) = E(X) = \mu$$

$$** \quad Var(\overline{X}) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{n}{n^2} Var(X) = \frac{1}{n} Var(X)$$

$$= \frac{1}{n}\sigma^2$$

Claim: $E\left[\sum_{i=1}^{n}(X_i - \overline{X})^2\right] = (n-1)\sigma^2$

$$\sum_{i=1}^{n}(X_i - \overline{X})^2 = \sum_{i=1}^{n}\left(X_i^2 - 2X_i\overline{X} + \overline{X}^2\right)$$

$$= \sum_{i=1}^{n} X_i^2 - 2\overline{X}\sum_{i=1}^{n} X_i + n\overline{X}^2$$

$$= \sum_{i=1}^{n} X_i^2 - 2\overline{X}(n\overline{X}) + n\overline{X}^2 \qquad \left(\text{as } \sum_{i=1}^{n} X_i = n\overline{X}\right)$$

$$= \sum_{i=1}^{n} X_i^2 - 2n\overline{X}^2 + n\overline{X}^2$$

$$= \sum_{i=1}^{n} X_i^2 - n\overline{X}^2$$

$$\therefore \text{L.H.S} = E\left[\sum_{i=1}^{n}(X_i - \overline{X})^2\right] = E\left[\sum_{i=1}^{n} X_i^2 - n\overline{X}^2\right]$$

$$= E\left(\sum_{i=1}^{n} X_i^2\right) - E(n\overline{X}^2)$$

$$= \sum_{i=1}^{n} E(X_i^2) - nE(\overline{X}^2)$$

$$= nE(X^2) - nE(\overline{X}^2)$$

$Var(X) = E(X^2) - [E(X)]^2$

$\Rightarrow \sigma = E(\overline{X}) - \mu^2$

$\Rightarrow E(X^2) = \sigma^2 + \mu^2$

$Var(\overline{X}) = E(\overline{X}^2) - [E(\overline{X})]^2$

$\Rightarrow \frac{\sigma^2}{n} = E(\overline{X}^2) - \mu^2$

$\Rightarrow E(\overline{X}^2) = \frac{\sigma^2}{n} + \mu^2$

$$= n\left[\sigma^2 + \mu^2\right] - n\left[\frac{\sigma^2}{n} + \mu^2\right]$$

$$= n\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2$$

$$= \sigma^2(n-1) = \text{R.H.S}$$

Or $\sum_{i=1}^{n}(X_i - \overline{X})^2 = (n-1)s^2 \Rightarrow E\left[\sum_{i=1}^{n}(X_i - \overline{X})^2\right] = E\left[(n-1)s^2\right]$

$$= (n-1)E(s^2) = (n-1)\sigma^2$$

because s^2 is unbaised estemator for σ^2

6

ⓐ $E(X) = \int_a^b x f(x)\,dx = \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{2(b-a)} x^2 \Big|_a^b$

$= \frac{1}{2(b-a)}(b^2-a^2) = \frac{1}{2(b-a)}(b+a)(b-a) = \frac{a+b}{2}$

$E(X^2) = \int_a^b x^2 f(x)\,dx = \frac{1}{b-a}\int_a^b x^2\,dx = \frac{1}{3(b-a)} x^3 \Big|_a^b$

$= \frac{1}{3(b-a)}(b^3-a^3) = \frac{(b-a)\,(b^2+ab+a^2)}{3(b-a)} = \frac{a^2+ab+b^2}{3}$

$\therefore Var(X) = E(X^2) - [E(X)]^2 = \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4}$

$= \frac{1}{3}(a^2+ab+b^2) - \frac{1}{4}(a^2+2ab+b^2)$

$= \frac{1}{12}(4a^2+4ab+4b^2-3a^2-6ab-3b^2)$

$= \frac{1}{12}(a^2-2ab+b^2) = \frac{1}{12}(a-b)^2$ or $\frac{1}{12}(b-a)^2$

ⓑ $E(X) = \int_0^\infty x f(x)\,dx = \lambda\int_0^\infty x e^{-\lambda x}\,dx = \lambda \frac{1}{\lambda^2} = \frac{1}{\lambda}$

بالقانون

$= \frac{\Gamma 2}{\lambda^2} = \frac{1}{\lambda^2}$

بالتجزئة

$u = x \Rightarrow du = dx \,,\; dV = e^{-\lambda x}\,dx \Rightarrow V = \frac{-1}{\lambda} e^{-\lambda x}$

$= -\frac{1}{\lambda} x e^{-\lambda x}\Big|_0^\infty + \frac{1}{\lambda}\int_0^\infty e^{-\lambda x}\,dx$

$= 0 - \frac{1}{\lambda^2} e^{-\lambda x}\Big|_0^\infty = \frac{1}{\lambda^2}$

$E(X^2) = \int_0^\infty x^2 f(x)\,dx = \lambda\int_0^\infty x^2 e^{-\lambda x}\,dx = \lambda\frac{2}{\lambda^3} = \frac{2}{\lambda^2}$

بالقانون

$= \frac{\Gamma 3}{\lambda^3} = \frac{2}{\lambda^3}$

بالتجزئة

$u = x^2 \Rightarrow du = 2x\,du \,,\; dV = e^{-\lambda x}\,dx \Rightarrow V = \frac{-1}{\lambda} e^{-\lambda x}$

$= -\frac{1}{\lambda} x^2 e^{-\lambda x}\Big|_0^\infty + \frac{2}{\lambda}\int_0^\infty x\, e^{-\lambda x}\,du$ موجودة عند $E(X)$

$= 0 + \frac{2}{\lambda}\left(\frac{1}{\lambda^2}\right) = \frac{2}{\lambda^3}$

$\therefore Var(X) = E(X^2) - [E(X)]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

(c) Note. * $f(x)$ even function $\Leftrightarrow f(-x) = f(x)$

$$\Leftrightarrow \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

** $f(x)$ odd function $\Leftrightarrow f(-x) = -f(x)$

$$\Leftrightarrow \int_{-a}^{a} f(x)\,dx = 0$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} x\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (\sigma u + \mu)\, e^{-\frac{1}{2}u^2}\,du$$

$$= \underbrace{\frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u\, e^{-\frac{1}{2}u^2}\,du}_{=0 \text{ because it is odd function}} + \underbrace{\frac{\mu}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\,du}_{=1}$$

$$= \mu$$

بالتعويض

$u = \frac{x-\mu}{\sigma} \rightarrow du = \frac{1}{\sigma}dx \Rightarrow \sigma\,du = dx$

$\sigma\left(u + \frac{\mu}{\sigma}\right) = x \Rightarrow x = \sigma u + \mu$

$x = -\infty \Rightarrow u = -\infty$

$x = \infty \Rightarrow u = \infty$

بالتعويض

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (\sigma u + \mu)^2 e^{-\frac{1}{2}u^2}\,du = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (\sigma^2 u^2 + 2\sigma u\mu + \mu^2)\, e^{-\frac{1}{2}u^2}\,du$$

$$= \frac{\sigma^2}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty} u^2 e^{-\frac{1}{2}u^2}\,du}_{=2\int_0^\infty u^2 e^{-\frac{1}{2}u^2}du \text{ because it's even function}} + \frac{2\sigma\mu}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty} u\, e^{-\frac{1}{2}u^2}\,du}_{=0 \text{ because it's odd function}} + \underbrace{\frac{\mu^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\,du}_{=1}$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}}\int_{0}^{\infty} u^2 e^{-\frac{1}{2}u^2}\,du + \mu^2 = \frac{\sigma^2}{\sqrt{2\pi}}\int_{0}^{\infty} z^{\frac{1}{2}} e^{-\frac{1}{2}z}\,dz + \mu^2$$

$$= \frac{\sigma^2}{\sqrt{2\pi}}\,\frac{\Gamma\left(\frac{1}{2}+1\right)}{\left(\frac{1}{2}\right)^{\frac{1}{2}+1}} + \mu^2$$

$$= \frac{\sigma^2}{\sqrt{\pi}}\,2\,\Gamma\left(\tfrac{3}{2}\right) + \mu^2, \text{ as } \Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\sqrt{\pi}$$

$$= \sigma^2 + \mu^2$$

بالتعويض

$z = u^2 \rightarrow dz = 2u\,du \Rightarrow \frac{1}{2}\frac{1}{u}dz = du$

$\Rightarrow \frac{1}{2} z^{-\frac{1}{2}} dz = du$

$u = z^{\frac{1}{2}}$

$u = 0 \Rightarrow z = 0$

$u = \infty \Rightarrow z = \infty$

$\therefore Var(X) = E(X^2) - [E(X)]^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$

Q7

$X \sim Exp(2=\lambda) \Rightarrow E(X) = \frac{1}{\lambda} = \frac{1}{2}, \quad V(X) = \frac{1}{\lambda^2} = \frac{1}{4}$

$Y \sim Gamma(3=\alpha, 4=\beta) \Rightarrow E(Y) = \frac{\alpha}{\beta} = \frac{3}{4}, \quad V(Y) = \frac{\alpha}{\beta^2} = \frac{3}{16}$

ⓐ $E(XY) = E(X)\,E(Y) = \frac{1}{2}\,\frac{3}{4} = \frac{3}{8}$

ⓑ $E(X^2 Y^3) = E(X^2)\,E(Y^3) = \frac{1}{2}\,\frac{15}{16} = \frac{15}{32}$

$V(X) = E(X^2) - [E(X)]^2$

$\Rightarrow \frac{1}{4} = E(X^2) - \frac{1}{4}$

$\Rightarrow E(X^2) = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$

$= \int_0^\infty y^3 f(y)\,dy$

$= \frac{4^3}{\Gamma 3} \int_0^\infty y^3 y^{3-1} e^{-4y}\,dy$

$= \frac{4^3}{2} \int_0^\infty y^5 e^{-4y}\,dy$

$= \frac{4^3}{2}\,\frac{\Gamma 6}{4^6} = \frac{15}{16}$

ⓒ $V(X-Y) = V(X) + V(Y) = \frac{1}{4} + \frac{3}{16} = \frac{7}{16}$

ⓓ $V(3X+2Y) = 9\,V(X) + 4\,V(Y) = 9\,\frac{1}{4} + 4\,\frac{3}{16} = 3$

Q8 $f(x) = 2e^{-2x}$, $x > 0$

$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx}(2e^{-2x})\,dx = 2\int_0^\infty e^{-(2-t)x}\,dx$

or

$= \frac{2}{-(2-t)} e^{-(2-t)x}\Big|_0^\infty = \frac{-2}{2-t}(-1) = \frac{2}{2-t}$

$= 2\int_0^\infty x^{1-1} e^{-(2-t)x}\,dx = 2\frac{\Gamma 1}{(2-t)^1} = \frac{2}{2-t}$

$\therefore M_X(t) = \frac{2}{2-t}$, $2-t > 0 \Rightarrow 2 > t$

Q9 X and Y are indep. , $M_{X+Y}(t) = \frac{e^{2t}-1}{2t-t^2}$, $f(x) = \lambda e^{-\lambda x}$; $x > 0$

find the dis. of Y ?

as X and Y are indep. , then

$M_{X+Y}(t) = M_X(t)\,M_Y(t) \Rightarrow M_Y(t) = \frac{M_{X+Y}(t)}{M_X(t)} = M_{X+Y}(t) \cdot \frac{1}{M_X(t)}$

So, we need to find Mgf of X which is

$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx}(\lambda e^{-\lambda x})\,dx = \lambda\int_0^\infty e^{-(\lambda-t)x}\,dx$

or

$= \frac{\lambda}{-(\lambda-t)} e^{-(\lambda-t)x}\Big|_0^\infty = \frac{\lambda}{\lambda-t}$

$= \lambda\int_0^\infty x^{1-1} e^{-(\lambda-t)x}\,dx = \lambda\frac{\Gamma 1}{(\lambda-t)} = \frac{\lambda}{\lambda-t}$

$\therefore M_X(t) = \frac{\lambda}{\lambda-t}$, $\lambda - t > 0 \Rightarrow \lambda > t$

$\therefore M_Y(t) = \frac{e^{2t}-1}{2t-t^2}\,\frac{1}{\frac{\lambda}{\lambda-t}} = \frac{e^{2t}-1}{2t-t}\,\frac{\lambda-t}{\lambda}$

Q 10

$f(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < \infty$

$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \Rightarrow f(x) = \begin{cases} \frac{1}{2} e^{-x}, & x \geq 0 \\ \frac{1}{2} e^{x}, & x < 0 \end{cases}$

a) $M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} (\frac{1}{2} e^{-x}) dx + \int_{-\infty}^{0} e^{tx} (\frac{1}{2} e^{x}) dx$

$= \frac{1}{2} \int_0^{\infty} e^{-(1-t)x} dx + \frac{1}{2} \int_{-\infty}^{0} e^{x(t+1)} dx$

or

$= \frac{1}{2} \int_0^{\infty} x^{1-1} e^{-(1-t)x} dx = \frac{1}{2} \frac{\Gamma 1}{1-t} = \frac{1}{2(1-t)}$

$= \frac{1}{2} \frac{1}{-(1-t)} e^{-(1-t)x} \Big|_0^{\infty}$

$= \frac{1}{2} \frac{1}{-(1-t)} (-1) = \frac{1}{2(1-t)}$

or or

$= \frac{1}{2} \int_{-\infty}^{0} x^0 e^{(t+1)x} dx$

$= \frac{1}{2} \left[\frac{x^0}{t+1} \right] e^{(t+1)x} \Big|_{-\infty}^{0}$

$= \frac{1}{2} \left[\frac{1}{t+1} \right] e^{(t+1)x} \Big|_{-\infty}^{0}$

$= \frac{1}{2} \left[\frac{1}{t+1} \right] (1-0) = \frac{1}{2(1+t)}$

take $u = -x \rightarrow du = -dx \Rightarrow -du = dx$

$x = -u$

$x = -\infty \Rightarrow u = \infty$
$x = 0 \Rightarrow u = 0$

$= \frac{1}{2} \frac{1}{(t+1)} e^{(t+1)x} \Big|_{-\infty}^{0}$

$= \frac{1}{2(t+1)} (1 - 0) = \frac{1}{2(t+1)}$

$\therefore \frac{1}{2} \int_{-\infty}^{0} e^{x(t+1)} dx = \frac{1}{2} \int_{\infty}^{0} e^{-(t+1)u} (-du)$

$= \frac{1}{2} \int_0^{\infty} e^{-(t+1)u} du = \frac{1}{2(t+1)}$

$\therefore M_X(t) = \frac{1}{2(1-t)} + \frac{1}{2(1+t)} = \frac{1}{2} \left[\frac{1}{1-t} + \frac{1}{1+t} \right] = \frac{1}{2} \left[\frac{\cancel{1+t} + \cancel{1-t}}{(1-t)(1+t)} \right]$

$= \frac{1}{\cancel{2}} \frac{\cancel{2}}{1-t^2} = \frac{1}{1-t^2}, \quad 1-t^2 > 0 \Rightarrow 1 > t^2 \Rightarrow 1 > |t| \Rightarrow -1 < t < 1$

b) $M_X(t) = (1-t^2)^{-1} \Rightarrow M_X'(t) = (-1)(1-t^2)^{-2}(-2t) = 2t(1-t^2)^{-2} \Rightarrow M_X'(0) = E(X) = 0$

$\Rightarrow M_X''(t) = 2t(-2)(1-t^2)^{-3}(-2t) + (1-t^2)^{-2}(2) \Rightarrow M_X''(0) = E(X^2) = 2$

$\therefore E(X) = 0, \quad E(X^2) = 2, \quad var(X) = E(X^2) - E(X)^2 = 2$

Q 11 $f(x) = \frac{3}{2}x^2, \quad -1<x<1$

بالتجزئة مرتين

① المرة الأولى: $u = x^2$, $dv = e^{tx}dx$

② المرة الثانية: $u = x$, $dv = e^{tx}dx$

لا ينفع استخدام الدالة الأسية في التكامل لأن t مجهولة

فائدة

a)

$$M_X(t) = E(e^{tX}) = \int_{-1}^{1} e^{tx}\left(\frac{3}{2}x^2\right)dx = \frac{3}{2}\int_{-1}^{1} x^2 e^{tx}dx = \frac{3}{2}\left[\frac{x^2}{t} - \frac{2x}{t^2} + \frac{2}{t^3}\right]e^{tx}\Big|_{-1}^{1}$$

$$= \frac{3}{2}\left[\left(\frac{1}{t} - \frac{2}{t^2} + \frac{2}{t^3}\right)e^{t} - \left(\frac{1}{t} + \frac{2}{t^2} + \frac{2}{t^3}\right)e^{-t}\right]$$

$$= \frac{3}{2}\left[\frac{1}{t}(e^{t} - e^{-t}) + \frac{2}{t^2}(-e^{t} - e^{-t}) + \frac{2}{t^3}(e^{t} - e^{-t})\right]$$

b,c) Mgf expanded form $M_X(t) = 1 + \frac{t}{1!}E(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \dots$

في الخلف

$$M_X(t) = \frac{3}{2}\Big[\frac{1}{t}\left(\cancel{1} + \frac{t}{1!} + \cancel{\frac{t^2}{2!}} + \frac{t^3}{3!} + \cancel{\frac{t^4}{4!}} + \dots - \cancel{1} + t - \cancel{\frac{t^2}{2!}} + \frac{t^3}{3!} - \cancel{\frac{t^4}{4!}} + \dots\right)$$
$$+ \frac{2}{t^2}\left(-1 - \cancel{t} - \frac{t^2}{2!} - \cancel{\frac{t^3}{3!}} - \frac{t^4}{4!} - \dots - 1 + \cancel{t} - \frac{t^2}{2!} + \cancel{\frac{t^3}{3!}} - \frac{t^4}{4!} + \dots\right)$$
$$+ \frac{2}{t^3}\left(\cancel{1} + \frac{t}{1!} + \cancel{\frac{t^2}{2!}} + \frac{t^3}{3!} + \cancel{\frac{t^4}{4!}} + \dots - \cancel{1} + t - \cancel{\frac{t^2}{2!}} + \frac{t^3}{3!} - \cancel{\frac{t^4}{4!}} + \dots\right)\Big]$$

$$= \frac{3}{2}\Big[\frac{1}{t}\left(2t + 2\frac{t^3}{3!} + 2\frac{t^5}{5!} + 2\frac{t^7}{7!} + \dots\right)$$
$$+ \frac{2}{t^2}\left(-2 - 2\frac{t^2}{2!} - 2\frac{t^4}{4!} - 2\frac{t^6}{6!} - \dots\right)$$
$$+ \frac{2}{t^3}\left(2t + 2\frac{t^3}{3!} + 2\frac{t^5}{5!} + 2\frac{t^7}{7!} + \dots\right)\Big]$$

$$= \frac{3}{2}\Big[2 + 2\frac{t^2}{3!} + 2\frac{t^4}{5!} + 2\frac{t^6}{7!} + \dots$$
$$- 2\frac{2}{t^2} - (2)(2)\frac{1}{2!} - (2)(2)\frac{t^2}{4!} - (2)(2)\frac{t^4}{6!} - \dots$$
$$+ (2)(2)\frac{1}{t^2} + (2)(2)\frac{1}{3!} + (2)(2)\frac{t^2}{5!} + (2)(2)\frac{t^4}{7!} + \dots\Big]$$

$$= 3\Big[\cancel{1} + \frac{t^2}{3!} + \frac{t^4}{5!} + \frac{t^6}{7!} + \dots$$
$$- \cancel{\frac{2}{t^2}} - 2\cancel{\frac{1}{2!}} - 2\frac{t^2}{4!} - 2\frac{t^4}{6!} - \dots$$
$$+ \cancel{\frac{2}{t^2}} + 2\frac{1}{3!} + 2\frac{t^2}{5!} + 2\frac{t^4}{7!} + \dots\Big]$$

$$= 3\left[\frac{1}{3} + \left(\frac{1}{3!} - \frac{2}{4!} + \frac{2}{5!}\right)t^2 + \left(\frac{1}{5!} - \frac{2}{6!} + \frac{2}{7!}\right)t^4 + \left(\frac{1}{7!} - \frac{2}{8!} + \frac{2}{9!}\right)t^6 + \dots\right]$$

$$= 1 + 3\left(\frac{1}{3!} - \frac{2}{4!} + \frac{2}{5!}\right)t^2 + 3\left(\frac{1}{5!} - \frac{2}{6!} + \frac{2}{7!}\right)t^4 + 3\left(\frac{1}{7!} - \frac{2}{8!} + \frac{2}{9!}\right)t^6 + \dots$$

$$= 1 + 3\left(\frac{1}{3!} - \frac{2}{4!} + \frac{2}{5!}\right)2!\frac{t^2}{2!} + 3\left(\frac{1}{5!} - \frac{2}{6!} + \frac{2}{7!}\right)4!\frac{t^4}{4!} + 3\left(\frac{1}{7!} - \frac{2}{8!} + \frac{2}{9!}\right)6!\frac{t^6}{6!} + \dots$$

$$= 1 + \frac{3}{5}\frac{t^2}{2!} + \frac{3}{7}\frac{t^4}{4!} + \frac{3}{9}\frac{t^6}{6!} + \dots = \sum_{n=0}^{\infty}\frac{3}{2n+3}\frac{t^{2n}}{(2n)!}$$

$$\therefore E(X^n) = \begin{cases} 0, & n \text{ odd} \\ \frac{3}{n+3}, & n \text{ even}\end{cases}$$

$E(X^n) = \int x^n f(x)\,dx$

$E(X) = \frac{3}{2}\int_{-1}^{1} x^3\,dx = 0$

$E(X^2) = \frac{3}{2}\int_{-1}^{1} x^4\,dx = \frac{3}{5}$

$E(X^3) = \frac{3}{2}\int_{-1}^{1} x^5\,dx = 0$

$E(X^4) = \frac{3}{2}\int_{-1}^{1} x^6\,dx = \frac{3}{7}$

$E(X^5) = \frac{3}{2}\int_{-1}^{1} x^7\,dx = 0$

$E(X^6) = \frac{3}{2}\int_{-1}^{1} x^8\,dx = \frac{3}{9}$

$\therefore M_X(t) = 1 + \frac{t}{1!}(0) + \frac{t^2}{2!}\left(\frac{3}{5}\right) + \frac{t^3}{3!}(0) + \frac{t^4}{4!}\left(\frac{3}{7}\right) + \cdots$

where we can see that

$$E(X^n) = \begin{cases} 0 & , n \text{ odd} \\ \frac{3}{3+n} & , n \text{ even} \end{cases}$$

Q12 X and Y are iid (i.e. متغيران مستقلان وكلاهما من نفس التوزيع)

with $M(t) = e^{3t+t^2}$ (i.e. $M_X(t) = M_Y(t) = e^{3t+t^2}$)

mgf of $Z = 2X - 3Y + 4$?

as X and Y are from the same dis., then

$Z = -X + 4$ where $M_X(t) = e^{3t+t^2}$

$\therefore M_Z(t) = e^{4t} M_X(-t) = e^{4t} e^{3(-t)+t^2} = e^{t+t^2}$

هذا الحل خاطئ

والحل الصحيح هو

or

$M_{2X-3Y+4}(t) = M_{2X}(t)\, M_{-3Y+4}(t)$ as X, Y are indep.

$= M_X(2t)\left[e^{4t} M_Y(-3t)\right]$

$= e^{4t}\left(e^{6t+4t^2}\right)\left(e^{-9t+9t^2}\right)$

$= e^{t+13t^2}$

$M_{2X-3Y+4}(t) = E\left(e^{t(2X-3Y+4)}\right)$

$= E\left(e^{2tX} e^{-3tY} e^{4t}\right)$

$= e^{4t} E\left(e^{2tX} e^{-3tY}\right)$ since e^{4t} is constant

$= e^{4t} E\left(e^{2tX}\right) E\left(e^{-3tY}\right)$ since X, Y are indep.

$= e^{4t} M_X(2t)\, M_Y(-3Y) = e^{t+13t^2}$

and so, $E(Z) = 1$, $Var(Z) = 26$

Q13 (b) $M_X(t) = e^{3t+t^2}$

$$M_Z(t) = M_{\frac{1}{4}(X-3)}(t) = M_{\frac{1}{4}X-\frac{3}{4}}(t) = e^{-\frac{3}{4}t} M_X(\tfrac{1}{4}\,t) = e^{-\frac{3}{4}t}\, e^{3(\frac{1}{4}t)+(\frac{1}{4}t)^2}$$

$$= e^{-\frac{3}{4}t}\, e^{\frac{3}{4}t+\frac{1}{16}t^2} = e^{\frac{1}{16}t^2}$$

$$E(Z) = M_Z'(t)\Big|_{t=0} = \frac{1}{16}2t\, e^{\frac{1}{16}t^2}\Big|_{t=0} = 0 = \text{mean}$$

$$E(Z^2) = M_Z''(t)\Big|_{t=0} = \frac{2}{16}\left[t\,\frac{2}{16}\,t\, e^{\frac{1}{16}t^2} + e^{\frac{1}{16}t^2}\right]\Big|_{t=0} = \frac{2}{16} = \frac{1}{8}$$

$$\therefore Var(Z) = E(Z^2) - E(Z)^2 = \frac{1}{8} = \text{Variance}$$

Q14 (b) $M_X(t) = \frac{1}{4}(3e^t + e^{-t})$

a) $\text{mean} = E(X) = M_X'(t)\Big|_{t=0} = \frac{1}{4}(3e^t - e^{-t})\Big|_{t=0} = \frac{1}{4}(3-1) = \frac{2}{4} = \frac{1}{2}$

$E(X^2) = M_X''(t)\Big|_{t=0} = \frac{1}{4}(3e^t + e^{-t})\Big|_{t=0} = \frac{1}{4}(3+1) = \frac{4}{4} = 1$

$\therefore \text{Variance} = Var(X) = E(X^2) - E(X)^2 = 1 - (\frac{1}{2})^2 = 1 - \frac{1}{4} = \frac{4-1}{4} = \frac{3}{4}$

b) $M_X(t) = 1 + t\,E(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \cdots$

os

$$M_X(t) = \frac{1}{4}(3e^t + e^{-t}) = \frac{1}{4}\left[3\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right) + \left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\cdots\right)\right]$$

$$= \frac{1}{4}\left[4 + 2t + 4\frac{t^2}{2!} + 2\frac{t^3}{3!} + 4\frac{t^4}{4!} + \cdots\right]$$

$$= \frac{1}{2}\left[2 + t + 2\frac{t^2}{2!} + \frac{t^3}{3!} + 2\frac{t^4}{4!} + \cdots\right] = 1 + \frac{t}{2} + \frac{t^2}{2!} + \frac{1}{2}\frac{t^3}{3!} + \frac{t^4}{4!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} E(X^n)$$

where

$$E(X^n) = \begin{cases} 1, & n \text{ even} \\ \frac{1}{2}, & n \text{ odd} \end{cases}$$

we have

$E(X) = M_X'(0) = \frac{1}{2}$, $E(X^2) = M_X''(0) = 1$, $E(X^3) = M_X^{(3)}(0) = \frac{1}{2}$

$E(X^4) = M_X^{(4)}(0) = 1$, $E(X^5) = M_X^{(5)}(0) = \frac{1}{2}$

$$\therefore E(X^n) = \begin{cases} 1, & n \text{ even} \\ \frac{1}{2}, & n \text{ odd} \end{cases}$$

Q 15

$f(x) = 1 \qquad 0 \le x \le 1$

$$M_x(t) = \int_0^1 e^{tx}\,dx = \left.\frac{e^{tx}}{t}\right]_0^1 = \frac{e^t}{t} - \frac{1}{t}$$

$$\boxed{M_x(t) = \frac{e^t - 1}{t}}$$

but if $a<0$

$$M_y(t) = \frac{-(e^{bt} - e^{(b+a)t})}{at} = \frac{e^{bt} - e^{(b+a)t}}{-at}$$

$\Rightarrow -a>0$

$Y \sim unif(a+b, b)$

$Y = aX + b$

$$M_Y(t) = e^{bt} M_x(at)$$

$$= e^{bt} \cdot \frac{e^{at} - 1}{at} = \frac{e^{(b+a)t} - e^{bt}}{at}, \quad at>0 \Leftrightarrow a>0$$

$Y \sim unif(b, a+b)$

$\underline{\underline{Q_{16}}}$ $f(x) = e^{-x}$ $x > 0$

$Z = 3 - 2X$

$$M_X(t) = \int_0^\infty e^{tx} \cdot e^{-x}\, dx$$

$$= \int_0^\infty e^{-x(1-t)}\, dx = \left. \frac{e^{-x(1-t)}}{-(1-t)} \right|_0^\infty$$

or $\frac{1}{(1-t)}$

$$= \frac{-1}{-(1-t)} = \frac{1}{1-t}$$

$$\boxed{\therefore M_X(t) = \frac{1}{1-t}}$$

$Z = 3 - 2X$

$$M_Z(t) = e^{3t} M_X(-2t)$$

$$= e^{3t} \frac{1}{1+2t} = \frac{e^{3t}}{1+2t} , \quad 1+2t>0$$

$2t > -1$

$t > \frac{-1}{2}$

Q 17

note: if $X \sim N(\mu, \sigma^2)$ i.e X has normal dis. with mean μ and Variance σ^2, then

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

X, Y and Z are indep.

$X \sim N(1,3) \Rightarrow M_X(t) = M^{t + \frac{3}{2}t^2}$

$Y \sim N(5,2) \Rightarrow M_Y(t) = M^{5t + t^2}$

$M_{X+Y+Z}(t) = e^{13t + 3t^2}$

dis. of Z ?!

as X, Y and Z are indep,

$$M_{X+Y+Z}(t) = M_X(t)\, M_Y(t)\, M_Z(t) \Rightarrow M_Z(t) = \frac{M_{X+Y+Z}(t)}{M_X(t)\, M_Y(t)}$$

$$\Rightarrow M_Z(t) = \frac{e^{13t+3t^2}}{(e^{t+\frac{3}{2}t^2})(e^{5t+t^2})} = \frac{e^{13t+3t^2}}{e^{6t+\frac{5}{2}t^2}} = e^{7t+\frac{1}{2}t^2} = e^{7t+\frac{1}{2}(1)^2 t^2}$$

$\therefore Y \sim N(7,1)$