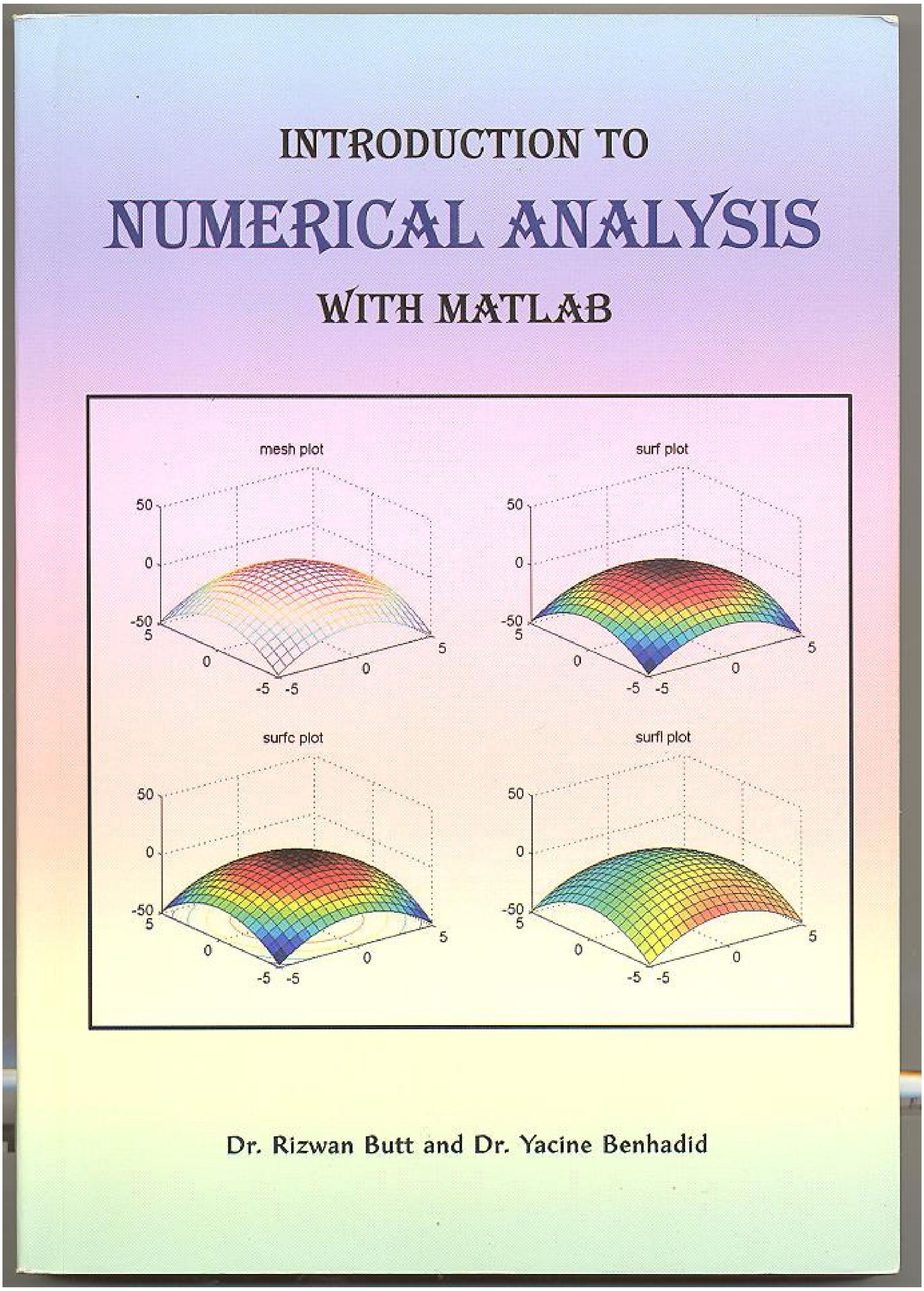
Solutions of the exercises of :



Chapter 1 Number Systems and Error

1. Convert the following binary numbers to decimal form

$$(1010)_2,\ (100101)_2,\ (.1100011)_2$$

Solution:

$$\begin{aligned}
(1010)_2 &= (0\times 2^0)+(1\times 2^1)+(0\times 2^2)+(1\times 2^3)\\
&= 0+2+0+8=(10)_{10}\\
(100101)_2 &= (1\times 2^0)+(0\times 2^1)+(1\times 2^2)+(0\times 2^3)\\
&= (0\times 2^4)+(1\times 2^5)\\
&= 1+0+4+0+0+32=(37)_{10}\\
(.1100011)_2 &= (1\times 2^{-7})+(1\times 2^{-6})+(0\times 2^{-5})+(0\times 2^{-4})\\
&= (0\times 2^{-3})+(1\times 2^{-2})+(1\times 2^{-1})\\
&= 1/2^7+1/2^6+0+0+0+1/2^2+1/2=(99/128)_{10}
\end{aligned}$$

2. Convert the following binary numbers to decimal form.

$$(101101)_2,\ (10110)_2,\ (100111)_2,\ (10000001)_2$$

Solution:

$$\begin{aligned}
(101101)_2 &= (1\times 2^0)+(0\times 2^1)+(1\times 2^2)+(1\times 2^3)\\
&= (0\times 2^4)+(1\times 2^5)\\
&= 1+0+4+8+0+32=(45)_{10}\\
(101111)_2 &= (1\times 2^0)+(1\times 2^1)+(1\times 2^2)+(1\times 2^3)\\
&= (0\times 2^4)+(1\times 2^5)\\
&= 1+2+4+8+0+32=(47)_{10}\\
(100111)_2 &= (1\times 2^0)+(1\times 2^1)+(1\times 2^2)+(0\times 2^3)\\
&= (0\times 2^4)+(1\times 2^5)\\
&= 1+2+4+0+0+32=(39)_{10}\\
(10000001)_2 &= (1\times 2^0)+(0\times 2^1)+(0\times 2^2)+(0\times 2^3)\\
&= (0\times 2^4)+(0\times 2^5)+(0\times 2^6)+(1\times 2^7)\\
&= 1+0+0+0+0+0+0+128=(129)_{10}
\end{aligned}$$

3. Find the first five binary digits of $(0.1)_{10}$. Obtain values for the absolute and relative errors in yours results.

Solution: Apply the conversion procedure as follows:

$$\begin{array}{ll} 0.1 & \\ \times 2 & \\ \hline 0.2 & + \text{ integer part } 0 \ (d_{-1}) \\ \times 2 & \\ \hline 0.4 & + \text{ integer part } 0 \ (d_{-2}) \\ \times 2 & \\ \hline 0.8 & + \text{ integer part } 0 \ (d_{-3}) \\ \times 2 & \\ \hline 0.6 & + \text{ integer part } 1 \ (d_{-4}) \\ \times 2 & \\ \hline 0.2 & + \text{ integer part } 1 \ (d_{-5}) \end{array}$$

Thus

$$(0.1)_{10} = (.00011)_2$$

4. Convert the following:

(a) decimal numbers to binary numbers form.

$$165, \ 3433, \ 111, \ 2345, \ 278.5, \ 347.45$$

(b) decimal numbers to hexadecimal decimal numbers.

$$1025, \ 278.5, \ 14.09375, \ 1445, \ 347.45$$

(c) hexadecimal numbers to both decimal and binary.

$$1F.C, \ FFF.118, \ 1A4.C, \ 1023, \ 11.1$$

Solution: (a)

$$\begin{array}{lcll} 165 & = & 2 \times 82 + 1, & b_0 = 1 \\ 82 & = & 2 \times 41 + 0, & b_1 = 0 \\ 41 & = & 2 \times 20 + 1, & b_2 = 1 \\ 20 & = & 2 \times 10 + 0, & b_3 = 0 \\ 10 & = & 2 \times 5 + 0, & b_4 = 0 \\ 5 & = & 2 \times 2 + 1, & b_5 = 1 \\ 2 & = & 2 \times 1 + 0, & b_6 = 0 \\ 1 & = & 2 \times 0 + 1, & b_7 = 1 \end{array}$$

Thus the binary representation for 165 is

$$165 = (b_7b_6b_5b_4b_3b_2b_1b_0)_2 = (10100101)_2$$

$$\begin{array}{lcll} 111 & = & 2 \times 55 + 1, & b_0 = 1 \\ 55 & = & 2 \times 27 + 1, & b_1 = 1 \\ 27 & = & 2 \times 13 + 1, & b_2 = 1 \\ 13 & = & 2 \times 6 + 1, & b_3 = 1 \\ 6 & = & 2 \times 3 + 0, & b_4 = 0 \\ 3 & = & 2 \times 1 + 1, & b_5 = 1 \\ 1 & = & 2 \times 0 + 1, & b_6 = 1 \end{array}$$

Thus the binary representation for 111 is

$$111 = (b_6b_5b_4b_3b_2b_1b_0)_2 = (1001111)_2$$

$$\begin{array}{lcll} 278 & = & 2 \times 139 + 0, & b_0 = 0 \\ 139 & = & 2 \times 69 + 1, & b_1 = 1 \\ 69 & = & 2 \times 34 + 1, & b_2 = 1 \\ 34 & = & 2 \times 17 + 0, & b_3 = 0 \\ 17 & = & 2 \times 8 + 1, & b_4 = 1 \\ 8 & = & 2 \times 4 + 0, & b_5 = 0 \\ 4 & = & 2 \times 2 + 0, & b_6 = 0 \\ 2 & = & 2 \times 1 + 0, & b_7 = 0 \\ 1 & = & 2 \times 0 + 1, & b_8 = 1 \end{array}$$

Also

$$\begin{array}{l} 0.5 \\ \times 2 \\ \overline{} \\ 0.0 \quad + \text{ integer part } 1 \ (d_{-1}) \end{array}$$

Thus the binary representation for 278.5 is

$$278.5 = (100010110.1)_2$$

(c)

$$\begin{aligned} (1F.C)_{16} &= C(16)^{-1} + F(16)^0 + 1(16)^1 \\ &= 12/16 + 15 + 16 = (31.75)_{10} \end{aligned}$$

Thus the decimal and binary representation for $(1F.C)_{16}$ is

$$(1F.C)_{16} = (31.75)_{10} = (11111.11)_2$$

$$\begin{aligned} (1A4.C)_{16} &= C(16)^{-1} + 4(16)^0 + A(16)^1 + 1(16)^2 \\ &= 12/16 + 4 + 160 + 256 = (420.75)_{10} \end{aligned}$$

Thus the decimal and binary representation for $(1A4.C)_{16}$ is

$$(1A4.C)_{16} = (420.75)_{10} = (110100100.11)_2$$

$$\begin{aligned}(11.1)_{16} &= 1(16)^{-1} + 1(16)^0 + 1(16)^1 \\ &= 1/16 + 1 + 16 = (17.0625)_{10}\end{aligned}$$

Thus the decimal and binary representation for $(11.1)_{16}$ is

$$(11.1)_{16} = (17.0625)_{10} = (10001.0001)_2$$

5. What is the absolute error in approximating 1/3 by 0.3333 ? What is the corresponding relative error ?

Solution: Let $x = 1/3 = 0.33333$ and $\hat{x} = 0.3333$, then the absolute error is

$$Abs.\ Error = |0.33333 - 0.3333| = 3 \times 10^{-5}$$

and the relative error is

$$Rel.\ Error = \frac{3 \times 10^{-5}}{0.33333} = 9.0001 \times 10^{-5}$$

Since $9.0001 \times 10^{-5} < \frac{1}{2} \times 10^{-4}$, therefore, $\hat{x}$ approximates x to 4 significant digits.

6. Evaluate the absolute error in each of the following calculations and hence give the answer to a suitable degree of accuracy.

(**a**) $9.01 + 9.96$, (**b**) $4.65 - 3.429$, (**c**) 0.7425×0.7199, (**d**) $0.7078 \div 0.87$

Solution: (a) Let $x_1 = 9.01, x_2 = 9.96$, and $S = x_1 + x_2 = 18.97$. Also, let e_1 and e_2 be the errors in x_1 and x_2 respectively. Then the respective absolute errors are as follows:

$$|e_1| \leq \frac{1}{2} \times 10^{-2}, \quad |e_2| \leq \frac{1}{2} \times 10^{-2}$$

and

$$Abs.\ Error \leq |e_1| + |e_2| = 1 \times 10^{-2}$$

The relative error is

$$Rel.\ Error = \frac{Abs.\ Error}{S} = \frac{(1 \times 10^{-2})}{18.97} = 5.2715 \times 10^{-4}$$

Thus, the result lies in the range $S \pm Abs.\ Error$

$$18.97 \pm 1 \times 10^{-2}, \quad \text{or} \quad 18.96 \leq S \leq 18.98$$

The answer may be rounded meaningfully to 18.9 which is correct to 3 significant digits (sd)(1 decimal place (dp)).

(c) Let $x_1 = 0.7425, x_2 = 0.7199$, and $P = x_1 x_2 = 0.5345$. Let e_1 and e_2 be the errors in x_1 and x_2 respectively and

$$|e_1| \leq \frac{1}{2} \times 10^{-4}, \quad |e_2| \leq \frac{1}{2} \times 10^{-4}$$

Then the relative error is

$$Rel.\ Error = \frac{|e_1|}{|x_1|} + \frac{|e_2|}{|x_2|} \leq \frac{\frac{1}{2} \times 10^{-4}}{0.7425} + \frac{\frac{1}{2} \times 10^{-4}}{0.7425} = 0.1368 \times 10^{-3}$$

and the absolute error is

$$Abs.\ Error = Rel.\ Error \times P = 7.3116 \times 10^{-5}$$

Thus the product lies in the range

$$P \pm Abs.\ Error = 0.5345 \pm 7.3116 \times 10^{-5}, \quad \text{or} \quad 0.5344 \leq P \leq 0.5346$$

which is correct to 3 sd (3 dp).

7. Find the absolute and relative errors in approximating π by 3.1416. What are the corresponding errors in the approximation $100\pi \approx 314.16$

Solution: Let $x = \pi = 3.14159654$ and $\hat{x} = 3.1416$, then the absolute error is

$$Abs.\ Error = |3.14159654 - 3.1416| = 7.3464 \times 10^{-6}$$

and the relative error is

$$Rel.\ Error = \frac{7.3464 \times 10^{-6}}{3.141592654} = 2.3384 \times 10^{-6}$$

Since $2.3384 \times 10^{-6} < \dfrac{1}{2} \times 10^{-5}$, therefore, $\hat{x}$ approximates x to 5 significant digits. Similarly, the absolute error in the approximation $100\pi \approx 314.16$ is 7.3464×10^{-4} and the relative error is 2.3384×10^{-6}. Since $2.3384 \times 10^{-6} < \dfrac{1}{2} \times 10^{-5}$, therefore, $\hat{x}$ approximates x to 5 significant digits.

8. Calculate the absolute error, relative error, and number of significant digits in the following approximations, with $p \approx x$:

(a) $x = 25.234, \quad p = 25.255$
(b) $x = e, \quad p = 19/7$
(c) $x = \sqrt{2}, \quad p = 1.414$

Solution: (a) Let $x = 25.234$ and $p = 25.255$, then the absolute error is define as

$$Abs.\ Error = |25.234 - 25.255| = 0.021$$

Since $0.021 < \dfrac{1}{2} \times 10^{-1}$, therefore, p approximates x to 1 significant digit.
Now we define the relative error as follows

$$Rel.\ Error = \frac{0.021}{25.234} = 8.3221 \times 10^{-4}$$

Since $8.3221 \times 10^{-4} < \dfrac{1}{2} \times 10^{-3}$, therefore, p approximates x to 3 significant digits.

(b) Let $x = e = 2.71828$ and $p = 19/7 = 2.71429$, then the absolute error is define as

$$Abs.\ Error = |2.71828 - 2.71429| = 3.9900 \times 10^{-3}$$

Since $3.9900 \times 10^{-3} < \dfrac{1}{2} \times 10^{-1}$, therefore, p approximates x to 1 significant digit. The relative error is define as

$$Rel.\ Error = \frac{(3.9900 \times 10^{-3})}{2.71828} = 1.4678 \times 10^{-3}$$

Since $1.4678 \times 10^{-3} < \dfrac{1}{2} \times 10^{-1}$, therefore, p approximates x to 1 significant digit.

(c) Let $x = \sqrt{2} = 1.4142$ and $p = 1.414$, then the absolute error is define as

$$Abs.\ Error = |1.4142 - 1.414| = 2.0 \times 10^{-4}$$

Since $2.0 \times 10^{-4} < \dfrac{1}{2} \times 10^{-3}$, therefore, p approximates x to 3 significant digits. Now we define the relative error as follows

$$Rel.\ Error = \frac{(2.0 \times 10^{-4})}{1.4142} = 1.4142 \times 10^{-4}$$

Since $1.4142 \times 10^{-4} < \dfrac{1}{2} \times 10^{-3}$, therefore, p approximates x to 3 significant digits.

9. Write each of the following numbers in (decimal) floating point form, starting the word length m and the exponent e

$$13.2, \quad -12.532, \quad 2/125$$

Solution: The floating decimal point representation of 13.2 is

$$13.2 \quad \text{has representation} \quad 0.132 \times 10^2 \quad (\mathbf{M = 0.132, e = 2})$$

Similarly, writing -12.532 in decimal floating point form

$$-12.532 \quad \text{has representation} \quad -0.12532 \times 10^2 \quad (\mathbf{M = 0.12532, e = 2})$$

and the number $2/125$ in decimal floating point form

$$\frac{2}{125} \quad \text{has representation} \quad (\frac{2}{125} \times 10) \times 10^{-1} \quad (\mathbf{M = \frac{20}{125}, e = -1})$$

10. Find absolute error in each of the following calculations (all numbers are rounded):

(a) $187.2 + 93.5$
(b) 0.281×3.7148
(c) $\sqrt{28.315}$

(d) $\sqrt{(6.2342 \times 0.82137)/27.268}$

Solution: (a) Let $x_1 = 187.2, x_2 = 93.5$, and $S = x_1 + x_2 = 280.7$. Also, let e_1 and e_2 be the errors in x_1 and x_2 respectively, then the respective absolute errors are as follows:

$$|e_1| \leq \frac{1}{2} \times 10^{-1}, \quad |e_2| \leq \frac{1}{2} \times 10^{-1}$$

and

$$Abs.\ Error \leq |e_1| + |e_2| = 1 \times 10^{-1}$$

Thus, the result lies in the range $S \pm Abs.\ Error$

$$280.7 \pm 1 \times 10^{-1}, \quad \text{or} \quad 280.6 \leq S \leq 280.8$$

which is correct to 3 significant digits.

(b) Let $x_1 = 0.281, x_2 = 3.7148$, and $P = x_1 x_2 = 1.0439$. Let e_1 and e_2 be the errors in x_1 and x_2 respectively, define as follows

$$|e_1| \leq \frac{1}{2} \times 10^{-3}, \quad |e_2| \leq \frac{1}{2} \times 10^{-4}$$

Then the relative error is

$$Rel.\ Error = \frac{|e1|}{|x_1|} + \frac{|e_2|}{|x_2|} \leq \frac{\frac{1}{2} \times 10^{-3}}{0.281} + \frac{\frac{1}{2} \times 10^{-4}}{3.7148} = 1.7929 \times 10^{-3}$$

and the absolute error is

$$Abs.\ Error = Rel.\ Error \times P = 1.8716 \times 10^{-3}$$

Thus the product lies in the range

$$P \pm Abs.\ Error = 1.0439 \pm 1.1.8716 \times 10^{-3}, \quad \text{or} \quad 1.0420 \leq P \leq 1.0458$$

or 1.04 which is correct to 3 sd (2 dp).

(c) Let $x = 28.315, n = 1/2$, and $R = \sqrt{x} = 5.3212$. Let e be the error in x and define as follows

$$|e| \leq \frac{1}{2} \times 10^{-3}$$

Then the relative error is

$$Rel.\ Error = \frac{|e|}{|x|} \leq \frac{\frac{1}{2} \cdot \frac{1}{2} \times 10^{-3}}{28.315} = 8.8292 \times 10^{-6}$$

and the absolute error is

$$Abs.\ Error = Rel.\ Error \times R = (8.8292 \times 10^{-6}) \times 5.3212 = 4.6982 \times 10^{-5}$$

The correct value of R lies in the range $R \pm Abs.\ Error$, that is

$$5.3212 \leq R \leq 5.3213$$

or 5.321 which is correct to 4 sd (3 dp).

(d) Let $x_1 = 6.2342, x_2 = 0.82137, x_3 = 27.2680, n = 1/2$, and $D = \sqrt{\dfrac{x_1 x_2}{x_3}} = 0.4333$. Let e_1, e_2, and e_3 be the errors in x_1, x_2, and x_3 respectively, and are define as follows

$$|e_1| \leq \frac{1}{2} \times 10^{-4}, \quad |e_2| \leq \frac{1}{2} \times 10^{-5}, \quad |e_3| \leq \frac{1}{2} \times 10^{-3}$$

Then the relative error is

$$Rel.\ Error \leq \frac{1}{2}\left(\frac{12 \times 10^{-4}}{6.2342} + \frac{12 \times 10^{-5}}{0.82137} + \frac{12 \times 10^{-3}}{27.2680}\right)$$

gives

$$Rel.\ Error \leq 1.6222 \times 10^{-5}$$

Now the absolute error is define as

$$Abs.\ Error = Rel.\ Error \times D = (1.6222 \times 10^{-5}) \times 0.43333 = 7.0291 \times 10^{-6}$$

So the answer lies in the range $D \pm Abs.\ Error$, that is

$$0.43329 \leq D \leq 0.4333$$

or 0.433 which is correct to 3 sd (3 dp).

11. Express the base of natural logarithms e as a normalized floating point number, using both chopping and symmetric rounding for each of the following system.

Solution: (a)
Chopping: $e = 2.718 \times 10^0$
Rounding: $e = 2.718 \times 10^0$

(b)
Chopping: $e = 2.718281 \times 10^0$
Rounding: $e = 2.718282 \times 10^0$

(c)
Chopping: $e = 1.010110111 \times 2^1$
Rounding: $e = 1.010111000 \times 2^1$

12. Write down the normalized binary floating point representations of $1/3, 1/5, 1/7, 1/9$ and $1/10$. Use enough bits in the mantissa to see the recurring patterns.

Solution: The normalized binary floating point representation of $1/5$ can be written as

$$\frac{1}{5} = (\frac{1}{10} \times 4)2^{-2} \quad (M = \frac{4}{5}, e = -2)$$

Similarly, the normalized binary floating point representation of $1/10$ can be written as

$$\frac{1}{10} = (\frac{1}{10} \times 8)2^{-3} \quad (M = \frac{4}{5}, e = -3)$$

Chapter 2 Solution of Nonlinear Equations

1. Find the root of $f(x) = e^x - 2 - x$ in the interval $[-2.4, -1.6]$ accurate to 10^{-4} using the bisection method.

Solution: Using the bisection method gives $a_1 = -2.4$ and $b_1 = -1.6$, so $f(-2.4) = 0.4907 > 0$ and $f(-1.6) = -0.1981 < 0$. Thus $\alpha \in [-2.4, -1.6]$ we have $x_1 = \frac{1}{2}(a_1+b_1) = -2.0$ and $f(x_1) = 0.1353 > 0$. Since $f(x_1)f(b_1) < 0$ so $\alpha \in [-2.0, -1.6]$ we have $x_2 = \frac{1}{2}(x_1 + b_1) = -1.8$ and $f(x_2) = -0.0347 < 0$. Continuing in this manner, the bisection gives $x_{10} = -1.84141$, accurate to within 10^{-4}.

2. Use the bisection method to find solutions accurate to within 10^{-4} on the interval $[-5, 5]$ of the following functions:

(a) $f(x) = x^5 - 10x^3 - 4$
(b) $f(x) = 2x^2 + \ln(x+6) - 3$
(c) $f(x) = \ln(x+1) + 30e^{-x} - 3$

Solution: **(a)** The bisection method gives $x_{17} = 3.1818$, accurate to within 10^{-4}.
(b) The bisection method fails because $f(-5) = 47$ and $f(5) = 49.3979$, which gives $f(-5)f(5) > 1$.
(c) The bisection method gives $x_{17} = 2.9084$, accurate to within 10^{-4}.

3. The following equations have a root in the interval $[0, 1.6]$. Determine these with an error less than 10^{-4} using the bisection method.

$$\textbf{(a)}\quad 2x - e^{-x} = 0 \qquad \textbf{(b)}\quad e^{-3x} + 2x - 2 = 0.$$

Solution: **(a)** The bisection method gives $x_{16} = 0.35173$, accurate to within 10^{-4}.
(b) The bisection method gives $x_{15} = 0.9730$, accurate to within 10^{-4}.

4. Estimate the number of iterations needed to achieve an approximation with accuracy 10^{-4} to the solution of $f(x) = x^3 + 4x^2 + 4x - 4$ lying in the interval $[0, 1]$ using the bisection method.

Solution: To find an approximation of that is accurate to within 10^{-4}, we need to determine the number of iterations n so that

$$|\alpha - x_n| < \frac{b-a}{2^n} < 10^{-4}$$

gives

$$2^n > 10^4, \quad \text{or } n \ln 2 > \ln\left(10^4\right)$$

Thus a bound for the number of iterations is

$$n \geq \frac{9.2103}{0.96312} = 13.2882, \quad \text{gives} \quad n = 14$$

So using the bisection method, we have $x_{14} = 0.5943$

5. Use the bisection method for $f(x) = x^3 - 3x + 1$ in $[1, 3]$ to find:
(a) The first eight approximation to the root of the given equation.
(b) Find an error estimate $|\alpha - x_8|$.

Solution: **(a)** Using the bisection method, we have the first eight approximations as

$$x_1 = 1.5000, \quad x_2 = 1.7500, \quad x_3 = 1.6250, \quad x_4 = 1.5625$$

$$x_5 = 1.5313, \quad x_6 = 1.5469, \quad x_7 = 1.5391, \quad x_8 = 1.5352$$

(b) The error bound for the approximation in part (a), we have

$$|\alpha - x_8| \leq \frac{3-1}{2^8} = 0.00781$$

6. Solve the Problem 1 by the false position method.

Solution: Since $f(x) = e^x - 2 - x$, using the false position method gives $a_1 = -2.4$ and $b_1 = -1.6$, so $f(-2.4) = 0.4907 > 0$ and $f(-1.6) = -0.1981 < 0$. Thus $\alpha \in [-2.4, -1.6]$ we have

$$x_1 = \frac{b_1 f(a_1) - a_1 f(b_1)}{f(a_1) - f(b_1)} = \frac{(-1.6)(0.4907) - (-2.4)(-0.1981)}{(0.4907) - (-0.1981)} = -1.8301$$

Since $f(x_1) = -0.0095 < 0$ and $f(a_1)f(x_1) < 0$ so $\alpha \in [-2.4, -1.8301]$ we have

$$x_2 = \frac{x_1 f(a_1) - a_1 f(x_1)}{f(a_1) - f(x_1)} = \frac{(-1.,8301)(0.4907) - (-2.4)(-0.0095)}{(0.4907) - (-0.0095)} = -1.8409$$

Since $f(x_2) = -0.0004 < 0$ and $f(a_1)f(x_2) < 0$ so $\alpha \in [-2.4, -1.8409]$, therefore, the false position method gives $x_3 = -1.8414$, accurate to within 10^{-4}.

7. Use the false position method to find the root of $f(x) = x^3 + 4x^2 + 4x - 4$ on the interval $[0, 1]$ accurate to 10^{-4}.

Solution: The false position method gives $x_8 = 0.59431$, accurate to within 10^{-4}.

8. Use the false position method to find solution accurate to within 10^{-4} on the interval $[1, 1.5]$ of the equation $2x^3 + 4x^2 - 2x - 5 = 0$.

Solution: The false position method gives $x_7 = 1.0782$, accurate to within 10^{-4}.

9. Use the false position method to find solution accurate to within 10^{-4} on the interval $[3, 4]$ of the equation $e^x - 3x^2 = 0$.

Solution: The false position method gives $x_9 = 3.73308$, accurate to within 10^{-4}.

10. The cubic equation $x^3 - 3x - 20 = 0$ can be written as

(a) $x = \dfrac{(x^3 - 20)}{3}$.

(b) $x = \dfrac{20}{(x^2 - 3)}$.

(c) $x = (3x + 20)^{1/3}$.
Choose the form which satisfies the condition $|g'(x)| < 1$ on $[1, 4]$ and then find third approximation x_3 when $x_0 = 1.5$.

Solution: For (a) and (b), $|g'(x)| > 0$ on $[1, 4]$, but for (c), we have $g'(x) = \dfrac{1}{(3x + 20)^{2/3}} < 1$ on $[1, 4]$. So by using (c), the third approximation by the fixed-point method is, $x_3 = 3.0789$.

11. Consider the nonlinear equation $g(x) = \dfrac{1}{2}e^{0.5x}$ defined on the interval $[0, 1]$. Then

(a) Show that there exists a unique fixed-point for g in $[0, 1]$.
(b) Use the fixed-point iterative method to compute x_3, set $x_0 = 0$.
(c) Compute an error bound for your approximation in part (b).

Solution: **(a)** Since given g is continuous in $[0, 1]$ and $g(0) = 0.5 \in [0, 1]$ and $g(1) = 0.8243 \in [0, 1]$. Also, $g'(x) = \frac{1}{4}e^{0.5x}$ and $g'(0) = 0.25$, $g'(1) = 0.41218$, so $|g'(x)| < 1 \quad \text{for} \quad x \in [0, 1]$.
(b) The fixed-point iterative method using $x_0 = 0$ gives

$$x_1 = g(x_0) = 0.5, \quad x_2 = g(x_1) = 0.64201, \quad x_3 = g(x_2) = 0.58705$$

(c) The error bound for the approximation is

$$|\alpha - x_3| \leq \frac{k^3}{1 - k}|x - 1 - x_0|$$

where $k = \max_{0 \leq x \leq 1} |g'(x)| = g'(1) = 0.41218$. Thus

$$|\alpha - x_3| \leq \frac{(0.41218)^3}{1 - 0.41218}|0.5 - 0| = 0.05957$$

12. An equation $x^3 - 2 = 0$ can be written in form $x = g(x)$ in two ways:
(a) $x = g_1(x) = x^3 + x - 2$
(b) $x = g_2(x) = \dfrac{(2 + 5x - x^3)}{5}$
Generate first four approximations from $x_{n+1} = g_i(x_n)$, $i = 1, 2$ by using $x_0 = 1.2$. Show which sequence converges to $2^{1/3}$ and why ?

Solution: **(a)** The fixed-point iterative method using $x_0 = 0$ gives $x_4 = -16.3514$.
(b) The fixed-point iterative method using $x_0 = 0$ gives $x_4 = 1.2599$.
The second sequence converges to $2^{1/3}$ because $|g_2'(2^{1/3})| = 0.0476 < 1$ whereas the first sequence does not converge to $2^{1/3}$ because $|g_1'(2^{1/3})| = 4.7622 > 1$.

13. Find value of k such that the iterative scheme $x_{n+1} = \dfrac{x_n^2 - 4kx_n + 7}{4}$, $n \geq 0$ converges to 1. Also, find the rate of convergence of the iterative scheme.

Solution: Since $g(x) = \dfrac{x^2 - 4kx + 7}{4}$ and $\alpha = 1$ is the fixed-point, then

$$g(1) = 1 = \frac{1 - 4k + 7}{4}, \quad \text{gives} \quad k = 1$$

Also

$$g'(x) = \frac{2x - 4k}{4}, \quad \text{gives} \quad g'(1) = -\frac{1}{2} \neq 0$$

a linear convergence.

14. Write the equation $x^2 - 6x + 5 = 0$ in the form $x = g(x)$, where $x \in [0, 2]$, so that the iteration $x_{n+1} = g(x_n)$ will converge to the root of the given equation for any initial approximation $x_0 \in [0, 2]$.

Solution: Taking $g(x) = \dfrac{x^2 + 5}{6}$, then

$$g'(x) = \frac{x}{3} < 1 \quad \text{for all} \quad x \in [0, 2]$$

Thus the iteration $x_{n+1} = g(x_n) = \dfrac{x_n^2 + 5}{6}$ will converge to the root of the given equation for any initial approximation $x_0 \in [0, 2]$.

15. Which of the following iterations

(a) $\quad x_{n+1} = \dfrac{1}{4}\left(x_n^2 + \dfrac{6}{x_n}\right)$

(b) $\quad x_{n+1} = \left(4 - \dfrac{6}{x_n^2}\right)$

is suitable to find a root of the equation $x^3 = 4x^2 - 6$ in the interval $[3, 4]$? Estimate the number of iterations required to achieve 10^{-3} accuracy, starting from $x_0 = 3$.

Solution: **(a)** Let $g_1(x) = \dfrac{1}{4}\left(x^2 + \dfrac{6}{x}\right)$ which is continuous in $[3, 4]$, but $g_1'(x) > 1$ for all $x \in (3, 4)$. So $g_1(x)$ is not suitable.

(b) $g_2(x) = \left(4 - \dfrac{6}{x_n^2}\right)$ which is continuous in $[3, 4]$ and $g(x) \in [3, 4]$ for all $x \in [3, 4]$. Also, $|g_2'(x)| = |12/x^3| < 1$ for all $x \in (3, 4)$. Then from the Theorem **??** implies that a unique fixed-point exists in $[3, 4]$. To find an approximation of that is accurate to within 10^{-3}, we need to determine the number of iterations n so that

$$|\alpha - x_n| \leq \frac{k^n}{1 - k}|x_1 - x_0| < 10^{-3}$$

With $k = \max_{3 \leq x \leq 4} |g'(x)| = 4/9$ and using the fixed-point method by taking $x_0 = 3$, we have $x_1 = 10/3$, we have and

$$|\alpha - x_n| \leq \frac{(4/9)^n}{(1 - 4/9)}|10/3 - 3| < 10^{-3}$$

Thus a bound for the number of iterations is

$$\frac{(4/9)^n}{(1-4/9)}|10/3-3| < 10^{-3}$$

and solving for n, we get, $n = 8$.

16. An equation $e^x = 4x^2$ has a root in $[4,5]$. Show that we cannot find that root using $x = g(x) = \frac{1}{2}e^{x/2}$ for the fixed-point iteration method. Can you find another iterative formula which will locate that root ? If yes, then find third iterations with $x_0 = 4.5$. Also find the error bound.

Solution: Since $g'(x) = \frac{1}{4}e^{x/2} > 0$ for all $x \in (4,5)$, therefore, the fixed-point iteration fails to converge to the root in $[4,5]$. Consider $x = g(x) = \ln(4x^2)$ and its derivative can be found as, $g'(x) = \frac{2}{x}$. Note that $|g'(x)| < 1$ for all $x \in (4,5)$ and the fixed-point iteration converges to the root in $[4,5]$. Using the fixed-point iteration method gives the third iteration as $x_3 = 4.3253$. With $k = \max_{4\leq x\leq 5} |g'(x)| = g'(4) = 0.5$ and taking $x_0 = 4.5$, we have $x_1 = 4.3944$. Thus the error bound for the above approximation, gives

$$|\alpha - x_3| \leq \frac{(0.5)^3}{(1-0.5)}|4.3944 - 4.5| = 0.0264$$

17. Solve the Problem 1 by the Newton's method by taking initial approximation $x_0 = -2$.

Solution: Since $f(x) = e^x - 2 - x$ and its derivative is $f'(x) = e^x - 1$. Using the Newton's iterative formula, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -2 - \frac{(0.1353)}{(-0.8647)} = -1.8435$$

and

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -1.8435 - \frac{(0.0017)}{(-0.8417)} = -1.8414$$

Thus, the Newton's method gives $x_2 = -1.8414$, accurate to within 10^{-4}.

18. Let $f(x) = e^x + 3x^2$. **(a)** Find the Newton's formula $g(x_k)$.
(b) Start with $x_0 = 4$ and compute x_4. **(c)** Start with $x_0 = -0.5$ and compute x_4.

Solution: **(a)** Using the Newton's formula gives

$$x_{k+1} = g(x_k) = x_k - \frac{e^{x_k} + 3x_k^2}{e^{x_k} + 6x_k} = \frac{(x_k - 1)e^{x_k} + 3x_k^2}{e^{x_k} + 6x_k}$$

(b) For $x_0 = 4, \; x_4 = 0.1215$
(c) For $x_0 = -0.5, \; x_4 = -32.1077$

19. Use the Newton's formula for the reciprocal of square root of a number 15 and then find the 3rd approximation of number, with $x_0 = 0.05$.

Solution: Let N be a positive number and $x = 1/\sqrt{N}$. If $f(x) = 0$, then $x = \alpha = 1/\sqrt{N}$ is the exact zero of the function

$$f(x) = 1/x^2 - N, \quad \text{gives } f'(x) = -2/x^3$$

Hence, assuming an initial estimate to the root, say, $x = x_0$ and by using the Newton's iterative formula, we get

$$x_1 = x_0 - \frac{(1/x_0^2 - N)}{(-2/x_0^3)} = x_0(3 - Nx_0^2)/2.$$

In general, we have

$$x_{n+1} = x_n(3 - Nx_n^2)/2, \qquad n = 0, 1, \ldots,$$

Now to find the reciprocal of square root of a number $N = 15$, using an initial gauss of say $x_0 = 0.05$, we have

$$\begin{array}{rcll} n &=& 0, & x_1 = x_0(3 - Nx_0^2)/2 = 0.05(3 - (15)(0.05)^2)/2 = 0.0741 \\ n &=& 1, & x_2 = x_1(3 - Nx_1^2)/2 = 0.0741(3 - (15)(0.0741)^2)/2 = 0.1081 \\ n &=& 0, & x_3 = x_2(3 - Nx_2^2)/2 = 0.1081(3 - (15)(0.1081)^2)/2 = 0.1527 \end{array}$$

20. Use the Newton's method to find solution accurate to within 10^{-4} of the equation $\tan(x) - 7x = 0$, with initial approximation $x_0 = 4$.

Solution: The Newton's method gives $x_6 = -4.1231e - 014$, accurate to within 10^{-4}.

21. Find the Newton's formula for $f(x) = x^3 - 3x + 1$ in $[1, 3]$ to calculate x_3, if $x_0 = 1.5$. Also, find the rate of convergence of the method.

Solution: The Newton's method gives $x_3 = 1.5321$. To find the order of convergence, we do the following:

$$g(x) = x - \frac{x^3 - 3x + 1}{3x^2 - 3} = \frac{2x^3 - 1}{3x^2 - 3}$$

and its derivative is

$$g'(x) = \frac{6x^4 - 18x^2 + 6x}{(3x^2 - 3)^2} = \frac{6x(x^3 - 3x + 1)}{(3x^2 - 3)^2} = 0$$

Thus $g'(\alpha) = 0$, gives at least quadratic convergence.

22. Rewrite the nonlinear equation $g(x) = \frac{1}{2}e^{0.5x}$ which defined in the interval $[0, 1]$ in the equivalent form $f(x) = 0$ and then use the Newton's method with $x_0 = 0.5$ to find third approximation x_3.

Solution: Writing $x = g(x)$ or $x - g(x) = 0$ gives $f(x) = 2x - e^{0.5x}$. Using the Newton's method gives $x_3 = 0.7148$.

23. Given the iterative scheme $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, $n \geq 0$ with $f(\alpha) = f'(\alpha) = 0$ and $f''(\alpha) \neq 0$. Find the rate of convergence for this scheme.

Solution: Since the Newton's iteration function is

$$\begin{aligned} g(x) &= x - \frac{f(x)}{f'(x)} \\ g'(x) &= 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2} \end{aligned}$$

Since $f'(\alpha) = 0$, so by using the Hopi'tal rule, it gives us

$$\begin{aligned} g'(x) &= \frac{f'(x)f''(x) + f(x)f'''(x)}{2f'(x)f''(x)} \\ g'(x) &= \frac{[f''(x)]^2 + f(x)f'''(x) + f'(x)f'''(x) + f(x)f^{(4)(x)}}{2[f'(x)]^2 + 2f'(x)f'''(x)} \\ g'(\alpha) &= \frac{1}{2} \neq 0. \end{aligned}$$

Thus the rate of convergence for the given scheme is linear.

24. The Halley's iterative method

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)} \left[1 - \frac{f(x_n)f''(x_n)}{2[f'(x_n)]^2}\right]^{-1}, \quad n \geq 0$$

has a order of convergence 3 for a simple root of $f(x) = 0$. Use it to find a iterative procedure to approximate the square root of a number N. Then use it to approximate $\sqrt{8}$, using $x_0 = 2.5$ and compute first three iterations.

Solution: We shall compute $x = N^{1/2}$ by finding a positive root for the nonlinear equation

$$x^2 - N = 0$$

where $N > 0$ is the number whose square root is to be found. Therefore, if $f(x) = 0$, then $x = N^{1/2}$ is the exact root. Let

$$f(x) = x^2 - N \quad \text{then} \quad f'(x) = 2x, \quad \text{and} \quad f''(x) = 2$$

Hence, taking $N = 8$ and the initial estimate to the root, say, $x = x_0 = 2.5$ and by using the Halley's iterative formula, we get first approximation

$$x_1 = \frac{(2x_0^3 + 2x_0N)}{(3x_0^2 + N)} = 2.6636$$

second approximation

$$x_2 = \frac{(2x_1^3 + 2x_1N)}{(3x_1^2 + N)} = 2.7459$$

and third approximation is

$$x_3 = \frac{(2x_2^3 + 2x_2N)}{(3x_2^2 + N)} = 2.7872$$

25. Find the positive root of $f(x) = x^{10} - 1$ by the secant method by using starting values $x_0 = 1.2$ and $x_1 = 1.1$ accurate to within 10^{-4}.

Solution: Using the secant method gives $x_0 = 1.2$ and $x_1 = 1.1$, so $f(1.2) = 5.1917$ and $f(1.1) = 1.5937$, and the new approximation

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(1.2)(1.5937) - (1.1)(5.1917)}{1.5937 - 5.1917} = 1.0557$$

Similarly, the secant method gives other approximations

$$x_3 = 1.0192, \quad x_4 = 1.0042, \quad x_5 = 1.0004, \quad x_6 = 1.0000, \quad x_7 = 1.0000$$

accurate to within 10^{-4}.

26. Find the first three estimates for the equation $x^3 - 2x - 5 = 0$ by the secant method using $x_0 = 2$ and $x_1 = 3$.

Solution: Using the secant method gives $x_0 = 2$ and $x_1 = 3$, so $f(2) = -1$ and $f(3) = 16$, and the first approximation

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(2)(16) - (3)(-1)}{16 + 1} = 2.0588$$

second approximation using $x_1 = 3, x_2 = 2.0588, f(3) = 16$, and $f(2.0588) = -0.3908$, gives

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(3)(-0.3908) - (2.0588)(16)}{-0.3908 - 16} = 2.0813$$

and third approximation using $x_2 = 2.0588, x_3 = 2.0813, f(2.0588) = -0.3908$, and $f(2.0813) = -0.1472$ is

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = \frac{(2.0588)(-0.1472) - (2.0813)(-0.3908)}{-0.1472 + 0.3908} = 2.0948$$

27. Solve the equation $e^{-x} - x = 0$ by using the secant method, starting with $x_0 = 0$ and $x_1 = 1$, accurate to 10^{-4}.

Solution: The secant method gives $x_4 = 0.56714$, accurate to within 10^{-4}.

28. Use the secant method to find a solution accurate to within 10^{-4} for $\ln(x) + x - 5 = 0$ on $[3, 4]$.

Solution: The secant method gives $x_3 = 3.6934$, accurate to within 10^{-4}.

29. Find the root of multiplicity of the function $f(x) = (x-1)^2 \ln(x)$ at $\alpha = 1$.

Solution: From the function $f(x) = (x-1)^2 \ln(x)$ we have $f(1) = 0$ and

$$\begin{array}{lllll} f' &=& 2(x-1)\ln x + \dfrac{(x-1)^2}{x} : & f'(1) &= 0 \\ f'' &=& 2\ln x + 4\dfrac{(x-1)}{x} - \dfrac{(x-1)^2}{x^2} : & f''(1) &= 0 \\ f''' &=& \dfrac{6}{x} - \dfrac{6(x-1)}{x^2} + \dfrac{(x-1)^2}{x^3} : & f'''(1) &= 6 \neq 0. \end{array}$$

Thus the root of multiplicity of the given function is 3, that is $m = 3$.

31. Show that the root of multiplicity of the function $f(x) = x^4 - x^3 - 3x^2 + 5x - 2$ is 3 at $\alpha = 1$. Estimate the number of iterations required to solve the problem with accuracy 10^{-4}, start with the starting value $x_0 = 0.5$ by using:
(a) the Newton's method
(b) the first modified Newton's method
(c) the second modified Newton's method.

Solution: From the function $f(x) = x^4 - x^3 - 3x^2 + 5x - 2$ we have $f(1) = 0$ and

$$\begin{array}{lllll} f' &=& 4x^3 - 3x^2 - 6x + 5 : & f'(1) &= 0 \\ f'' &=& 12x^2 - 6x - 6 : & f''(1) &= 0 \\ f''' &=& 24x - 6 : & f'''(1) &= 18 \neq 0. \end{array}$$

Thus the root of multiplicity of the given function is 3, that is $m = 3$.

(a) Using the Newton's method gives the first approximation as follows

$$x_1 = x_0 - \frac{x_0^4 - x_0^3 - 3x_0^2 + 5x_0 - 2}{4x_0^3 - 3x_0^2 - 6x_0 + 5} = 0.6785714$$

Thus the Newton's method gives $x_{21} = 0.999863$, accurate to within 10^{-4}.

(b) Using the first modified Newton's method gives the first approximation as follows

$$x_1 = x_0 - 3\frac{x_0^4 - x_0^3 - 3x_0^2 + 5x_0 - 2}{4x_0^3 - 3x_0^2 - 6x_0 + 5} = 1.0357143$$

Thus the first modified Newton's method gives $x_4 = 0.9999998$, accurate to within 10^{-4}.

(c) Using the second modified Newton's method gives the first approximation as follows

$$x_1 = x_0 - \frac{(x_0^4 - x_0^3 - 3x_0^2 + 5x_0 - 2)(4x_0^3 - 3x_0^2 - 6x_0 + 5)}{(4x_0^3 - 3x_0^2 - 6x_0 + 5)^2 - (x_0^4 - x_0^3 - 3x_0^2 + 5x_0 - 2)(12x_0^2 - 6x_0 - 6)}$$

that is, $x_1 = 0.960526$. Thus the second modified Newton's method gives $x_4 = 1.000001$, accurate to within 10^{-4}.

32. If $f(x), f'(x)$ and $f''(x)$ are continuous and bounded on a certain interval containing $x = \alpha$ and if both $f(\alpha) = 0$ and $f'(\alpha) = 0$ but $f''(\alpha) \neq 0$, show that

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)}$$

will converge quadratically if x_n is in the interval.

Solution: Let $f(x) = (x-\alpha)^2 h(x),\ h(\alpha) \neq 0$. Then

$$g(x) = x - 2\frac{(x-\alpha)^2 h(x)}{(2(x-\alpha)h(x) + (x-\alpha)^2 h'(x))}$$

and

$$g'(\alpha) = 1 - \frac{2}{2} = 0, \quad g''(\alpha) \neq 0$$

33. Show that iterative scheme $x_{n+1} = 1 + x_n - \frac{x_n^2}{2},\ n \geq 0$ converges to $\sqrt{2}$. Find the rate of convergence of the sequence.

Solution: Since

$$\begin{aligned} g(x) &= 1 + x - \frac{x^2}{2} \\ g(\sqrt{2}) &= 1 + \sqrt{2} - 1 = \sqrt{2} \end{aligned}$$

So $\sqrt{2}$ is a fixed-point of $g(x)$. Also,

$$g'(x) = 1 - \frac{2x}{2}, \quad \text{and} \quad |g'(\sqrt{2})| = |-0.414214| < 1.$$

Hence the given sequence converges to $\sqrt{2}$. Also, since $g'(\sqrt{2}) = -0.414214 \neq 0$, therefore, the rate of convergence of the sequence is linear.

35. The following sequence is linearly convergent. Use the Aitken's method to generate the first four terms of the sequence $\{x_n'\}$:

$$x_0 = 0.5, \quad x_{n+1} = \cos x_n, \qquad n \geq 0$$

Solution: Starting with $x_0 = 0.5$, the first two approximations by using the given iterative scheme $x_{n+1} = \cos x_n, \quad n \geq 0$, gives $x_1 = 0.8776$ and $x_1 = 0.6390$. Using the Aitken's method

$$\hat{x}_0 = x_0 - \frac{(x_1 - x_0)^2}{x_2 - 2x_1 + x_0} = 0.7314$$

Similarly the other three approximations by the Aitken's method are:

$$\hat{x}_1 = 0.73609, \quad \hat{x}_2 = 0.73765, \quad \hat{x}_3 = 0.73847$$

37. Use the fixed-point iteration to find the first four approximations to the root of the $x_{n+1} = \frac{1}{2}\ln(1 + x_n), \quad n \geq 0$, using $x_0 = 0.5$. Use the Aitken accelerated method to speed up the convergence of the iteration method.

Solution: Using the fixed-point method and the Aitken's method, we have the desired approximations as follows:

Fixed-point method	Aitken Accelerated method
$x_1 = 0.20270$	$\hat{x}_0 = 0.02702$
$x_2 = 0.09230$	$\hat{x}_1 = 0.00691$
$x_3 = 0.04414$	$\hat{x}_2 = 0.00175$
$x_4 = 0.02160$	$\hat{x}_3 = 0.00044$

39. Solve the following system using the Newton's method:

$$\begin{array}{rcl} 4x^3 \; + y & = & 6 \\ x^2 y & = & 1 \end{array}$$

Start with initial approximation $x_0 = y_0 = 1$. Stop when successive iterates differ by less than 10^{-7}.

Solution: To solve the given nonlinear system using the Newton's method, we do the following:

$$\begin{array}{rcl} f_1(x,y) & = & 4x^3 + y - 6, \quad f_{1x} = 12x^2, \quad f_{1y} = 1 \\ f_2(x,y) & = & x^2 y - 1, \qquad f_{2x} = 2xy, \quad f_{2y} = x^2 \end{array}$$

At the given initial approximation $x_0 = 1.0$ and $y_0 = 1.0$, we get

$$f_1(1.0, 1.0) = -1, \quad \frac{\partial f_1}{\partial x} = f_{1x} = 12.0, \quad \frac{\partial f_1}{\partial y} = f_{1y} = 1$$

$$f_2(1.0, 1.0) = 0, \quad \frac{\partial f_1}{\partial x} = f_{2x} = 2, \quad \frac{\partial f_2}{\partial y} = f_{2y} = 1.0$$

The Jacobian matrix J and its inverse J^{-1} at the given initial approximation can be calculated as

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 12.0 & 1.0 \\ 2.0 & 1.0 \end{pmatrix}$$

and one can find its inverse as

$$J^{-1} = \begin{pmatrix} 0.1000 & -0.1000 \\ -0.2000 & 1.2000 \end{pmatrix}$$

Substituting all these values in the Newton's formula to get the first approximation as follows

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1.0 \\ 1.0 \end{pmatrix} - \begin{pmatrix} 0.1000 & -0.1000 \\ -0.2000 & 1.2000 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.1000 \\ 0.8000 \end{pmatrix}$$

Similarly, the Newton's method gives $x^{(5)} = 1.08828, \quad y^{(5)} = 0.84434$ accurate to within 10^{-7}.

40. Solve the following system using the Newton's method

$$\begin{array}{rcl} x + e^y &=& 68.1 \\ \sin x - y &=& -3.6 \end{array}$$

Start with initial approximation $x_0 = 2.5,\ y_0 = 4$, compute the first three approximations.

Solution: Solving the given nonlinear system using the Newton's method, we do the following:

$$\begin{array}{rcll} f_1(x,y) &=& x + e^y - 68.1, & f_{1x} = 1, \quad f_{1y} = e^y \\ f_2(x,y) &=& \sin x - y + 3.6, & f_{2x} = \cos x, \quad f_{2y} = -1 \end{array}$$

At the given initial approximation $x_0 = 2.5$ and $y_0 = 4.0$, we get

$$f_1(2.5, 4.0) = -11.0018, \quad \frac{\partial f_1}{\partial x} = f_{1x} = 1.0000, \quad \frac{\partial f_1}{\partial y} = f_{1y} = 54.5982$$

$$f_2(2.5, 4.0) = 0.1985, \quad \frac{\partial f_1}{\partial x} = f_{2x} = -0.8011, \quad \frac{\partial f_2}{\partial y} = f_{2y} = -1.0000$$

The Jacobian matrix J and its inverse J^{-1} at the given initial approximation can be calculated as

$$J = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 1.0000 & 54.5982 \\ \\ -0.8011 & -1.0000 \end{pmatrix}$$

and one can find its inverse as

$$J^{-1} = \begin{pmatrix} -0.0234 & -1.2775 \\ 0.0187 & 0.0234 \end{pmatrix}$$

Substituting all these values in the Newton's formula to get the first approximation as follows

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 4.0 \end{pmatrix} - \begin{pmatrix} -0.0234 & -1.2775 \\ 0.0187 & 0.0234 \end{pmatrix} \begin{pmatrix} -11.0018 \\ 0.1985 \end{pmatrix} = \begin{pmatrix} 2.4961 \\ 4.2016 \end{pmatrix}$$

Similarly, second and third approximations by Newton's method are as follows

$$x^{(2)} = 2.5188, \ y^{(2)} = 4.1835 \quad \text{and } x^{(3)} = 2.5188, \ y^{(3)} = 4.1833$$

Chapter 3 Systems of Linear Equations

1. Determine the matrix C given by the following expression

$$C = 2A - 3B,$$

if the matrices A and B are

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 2 & 1 & 4 \end{pmatrix}$$

Solution: Since

$$2A = \begin{pmatrix} 4 & -2 & 2 \\ -2 & 4 & 6 \\ 4 & 2 & 4 \end{pmatrix}$$

and

$$3B = \begin{pmatrix} 3 & 3 & 3 \\ 0 & 3 & 9 \\ 6 & 3 & 12 \end{pmatrix}$$

therefore,

$$C = 2A - 3B = \begin{pmatrix} 4-3 & -2-3 & 2-3 \\ -2+0 & 4-3 & 6-9 \\ 4-6 & 2-3 & 4-12 \end{pmatrix} = \begin{pmatrix} 1 & -5 & -1 \\ -2 & 1 & -3 \\ -2 & -1 & -8 \end{pmatrix}$$

2. Find the product AB and BA for the matrices of the Problem 1.

Solution: Multiplying the matrix A with the matrix B, we have

$$AB = \begin{pmatrix} 2-0+2 & 2-1+1 & 2-3+4 \\ -1+0+6 & -1+2+3 & -1+6+12 \\ 2+0+4 & 2+1+2 & 2+3+8 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 \\ 5 & 4 & 17 \\ 6 & 5 & 13 \end{pmatrix}$$

Similarly, multiplying the matrix B with the matrix A, we have

$$BA = \begin{pmatrix} 2-1+2 & -1+2+1 & 1+3+2 \\ 0-1+6 & 0+2+3 & 0++3+6 \\ 4-1+8 & -2+2+4 & 2+3+8 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 6 \\ 5 & 5 & 9 \\ 11 & 4 & 13 \end{pmatrix}$$

3. Show that the product AB of the following rectangular matrices is a singular matrix.

$$A = \begin{pmatrix} 6 & -3 \\ 1 & 4 \\ -2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & -1 & -2 \\ 3 & -4 & -1 \end{pmatrix}$$

Solution: First multiplying the matrix A with the matrix B, we have

$$AB = \begin{pmatrix} 12-9 & -6+12 & -12+3 \\ 2+12 & -1-16 & -2-4 \\ -4+3 & 2-4 & 4-1 \end{pmatrix} = \begin{pmatrix} 3 & 6 & -9 \\ 14 & -17 & -6 \\ -1 & -2 & 3 \end{pmatrix}$$

Then the determinant of the matrix AB can be calculated as

$$|AB| = \begin{vmatrix} 3 & 6 & -9 \\ 14 & -17 & -6 \\ -1 & -2 & 3 \end{vmatrix} = 3(-51-12) + 6(6-42) - 9(-28-17)$$

or

$$|AB| = -189 - 216 + 405 = 0$$

4. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 2 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix},$$

(a) Compute AB and BA and show that $AB \neq BA$.
(b) Find $(A+B)+C$ and $A+(B+C)$.
(c) Show that $(AB)^T = B^T A^T$

Solution: **(a)** Multiplying the matrix A with the matrix B, we have

$$AB = \begin{pmatrix} 1-2+3 & 1+2+0 & 2-2+6 \\ 0+1+2 & 0-1+0 & 0+1+4 \\ 2+0+2 & 2+0+0 & 4+0+4 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 6 \\ 3 & -1 & 5 \\ 4 & 2 & 8 \end{pmatrix}$$

Now multiplying the matrix B with the matrix A, we have

$$BA = \begin{pmatrix} 1+0+4 & 2-1+0 & 3+2+4 \\ -1+0-2 & -2-1+0 & -3+2-2 \\ 1+0+4 & 2+0+0 & 3+0+4 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 9 \\ -3 & -3 & -3 \\ 5 & 2 & 7 \end{pmatrix}$$

which shows that $AB =\neq BA$.

(b) Adding the matrices A and B, we have

$$A+B = \begin{pmatrix} 1+1 & 2+1 & 3+2 \\ 0-1 & -1+1 & 2-1 \\ 2+1 & 0+0 & 2+2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 \\ -1 & 0 & 1 \\ 3 & 0 & 4 \end{pmatrix}$$

Then

$$(A+B)+C = \begin{pmatrix} 2+1 & 3+0 & 5+1 \\ -1+0 & 0+1 & 1+2 \\ 3+2 & 0+0 & 4+1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 6 \\ -1 & 1 & 3 \\ 5 & 0 & 5 \end{pmatrix}$$

Similarly,

$$B+C = \begin{pmatrix} 1+1 & 1+0 & 2+1 \\ -1+0 & 1+1 & -1+2 \\ 1+2 & 0+0 & 2+1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 0 & 3 \end{pmatrix}$$

Then

$$A+(B+C) = \begin{pmatrix} 1+2 & 2+1 & 3+3 \\ 0-1 & -1+2 & 2+1 \\ 2+3 & 0+0 & 2+3 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 6 \\ -1 & 1 & 3 \\ 5 & 0 & 5 \end{pmatrix}$$

(c) Since we know from part (a) that product AB is equal to

$$AB = \begin{pmatrix} 2 & 3 & 6 \\ 3 & -1 & 5 \\ 4 & 2 & 8 \end{pmatrix}$$

Now interchanging rows and columns, gives

$$(AB)^T = \begin{pmatrix} 2 & 3 & 4 \\ 3 & -1 & 2 \\ 6 & 5 & 8 \end{pmatrix}$$

Since transpose of the matrix A is

$$A^T = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 0 \\ 3 & 2 & 2 \end{pmatrix}$$

and transpose of the matrix B is

$$B^T = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 2 \end{pmatrix}$$

Then the product $B^T A^T$ is equal to

$$B^T A^T = \begin{pmatrix} 1-2+3 & 0+1+2 & 2+0+2 \\ 1+2+0 & 0-1+0 & 2+0+0 \\ 2-2+6 & 0+1+4 & 4+0+4 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 \\ 3 & -1 & 2 \\ 6 & 5 & 8 \end{pmatrix}$$

Thus $(AB)^T = B^T A^T$.

5. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

then show that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution: Multiplying the matrix A with the matrix B, we have

$$AB = \begin{pmatrix} 1+1 & 0+1 \\ 0+1 & 0+1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Now finding the inverse of product AB as

$$(AB)^{-1} = \frac{1}{(2-1)} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Similarly, finding inverse of A and B, we get

$$A^{-1} = \frac{1}{(1-0)} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and

$$B^{-1} = \frac{1}{(1-0)}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Then

$$B^{-1}A^{-1} = \begin{pmatrix} 1+0 & -1+0 \\ -1+0 & 1+1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Thus $(AB)^{-1} = B^{-1}A^{-1}$.

6. Find a value of x and a value of y so that $AB^T = 0$, where $A = [1\ x\ 1]$ and $B = [-2\ 2\ y]$.

Solution: Multiplying the row matrix A and the column matrix B^T, we have

$$AB^T = \begin{pmatrix} 1 & x & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 2 \\ y \end{pmatrix} = -2 + 2x + y$$

Since $AB^T = 0$, then it gives, $x = (2-y)/2$. In particular, if $y = 1$, then $x = 1/2$, and so on.

7. Evaluate the determinant of each matrix

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ 1 & -5 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 1 & 6 \\ -3 & 6 & 4 \\ 5 & 0 & 9 \end{pmatrix}, \quad C = \begin{pmatrix} 17 & 46 & 7 \\ 20 & 49 & 8 \\ 23 & 52 & 9 \end{pmatrix}$$

Solution: First, we find the determinant of the matrix A as follows

$$det(A) = |A| = \begin{vmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ 1 & -5 & 1 \end{vmatrix} = 3(0+20) + 1(4-2) + (-1)(-10-0) = 72$$

Similarly, the determinant of the matrix B and the matrix C can be computed as follows

$$det(B) = |B| = \begin{vmatrix} 4 & 1 & 6 \\ -3 & 6 & 4 \\ 5 & 0 & 9 \end{vmatrix} = 4(54-0) + 1(20+27) + 6(0-30) = 83$$

and

$$det(C) = |C| = \begin{vmatrix} 17 & 46 & 7 \\ 20 & 49 & 8 \\ 23 & 52 & 9 \end{vmatrix} = 17(441-416) + 46(184-180) + 7(1040-1127) = 0$$

8. Find all zeros (values of x such that $f(x) = 0$) of the polynomial $f(x) = \det(A)$ where

$$A = \begin{pmatrix} x-1 & 3 & 2 \\ 3 & x & 1 \\ 2 & 1 & x-2 \end{pmatrix}$$

Solution: The determinant of the matrix A can be calculated as

$$\begin{array}{rcl} det(A) & = & \begin{vmatrix} x-1 & 3 & 2 \\ 3 & x & 1 \\ 2 & 1 & x-2 \end{vmatrix} \\ & = & (x-1)(x^2-2x-1)+3(2-3x+6)+2(3-2x) = x^3-3x^2-12x+31 \end{array}$$

Given

$$f(x) = det(A) = x^3 - 3x^2 - 12x + 31 = 0$$

Solving this cubic equation, we get, $x_1 = 4.0787, x_2 = 2.2698, x_3 = -3.3485$.

9. Compute the adjoint of each matrix A, and find the inverse of it if it exists

$$(a)\quad \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}, \quad (\mathbf{b})\quad \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 4 \\ 1 & 5 & -7 \end{pmatrix}, \quad (\mathbf{c})\quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution: (a) Since we know that for a 1×1 matrix $[a_{11}]$, the entry a_{11} itself is defined to be the minor and cofactor of a_{11}, therefore, the minors M_{ij} of all elements a_{ij} of the matrix A are calculated as follows

$$M_{11} = 4, \quad M_{12} = -3, \quad M_{21} = 2, \quad M_{22} = 1$$

Now using these minors we can have the cofactors of the matrix as follows

$$\begin{array}{rcrcrcr} A_{11} & = & (-1)^{1+1}M_{11} & = & M_{11} & = & 4 \\ A_{12} & = & (-1)^{1+2}M_{12} & = & -M_{12} & = & 3 \\ A_{21} & = & (-1)^{2+1}M_{21} & = & -M_{21} & = & -2 \\ A_{22} & = & (-1)^{2+2}M_{22} & = & M_{22} & = & 1 \end{array}$$

So that the matrix of the cofactor is

$$C = \begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix}$$

and the transpose of the cofactor matrix C is the adjoint of A, that is

$$Adj(A) = \begin{pmatrix} 4 & -2 \\ 3 & 1 \end{pmatrix}$$

Now by using the cofactor expansion along the first row, we can find the determinant of the matrix A as follows

$$det(A) = a_{11}A_{11} + a_{12}A_{12} = 4 + 6 = 10$$

The inverse of the matrix A can be calculated by using

$$A^{-1} = \frac{Adj(A)}{\det(A)} = \frac{1}{10}\begin{pmatrix} 4 & -2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 2/5 & -1/5 \\ 3/10 & 1/10 \end{pmatrix}$$

(b) Firstly, the minors M_{ij} of all elements a_{ij} of the matrix A are calculated as follows

$$\begin{aligned}
M_{11} &= \begin{vmatrix} 1 & 4 \\ 5 & -7 \end{vmatrix} = -7-20 = -27 \\
M_{12} &= \begin{vmatrix} 2 & 4 \\ 1 & -7 \end{vmatrix} = -14-4 = -18 \\
M_{13} &= \begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} = 10-1 = 9 \\
M_{21} &= \begin{vmatrix} 2 & -1 \\ 5 & -7 \end{vmatrix} = -14+5 = -9 \\
M_{22} &= \begin{vmatrix} 1 & -1 \\ 1 & -7 \end{vmatrix} = -7+1 = -6 \\
M_{23} &= \begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} = 5-2 = 3 \\
M_{31} &= \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} = 8+1 = 9 \\
M_{32} &= \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} = 4+2 = 6 \\
M_{33} &= \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1-4 = -3
\end{aligned}$$

Now using these minors we can have the cofactors of the matrix as follows

$$\begin{aligned}
A_{11} &= (-1)^{1+1}M_{11} = M_{11} = -27 \\
A_{12} &= (-1)^{1+2}M_{12} = -M_{12} = 18 \\
A_{13} &= (-1)^{1+3}M_{13} = M_{13} = 9 \\
A_{21} &= (-1)^{2+1}M_{21} = -M_{21} = 9 \\
A_{22} &= (-1)^{2+2}M_{22} = M_{22} = -6 \\
A_{23} &= (-1)^{2+3}M_{23} = -M_{23} = -3 \\
A_{31} &= (-1)^{3+1}M_{31} = M_{31} = 9 \\
A_{32} &= (-1)^{3+2}M_{32} = -M_{32} = -6 \\
A_{33} &= (-1)^{3+3}M_{33} = M_{33} = -3
\end{aligned}$$

So that the matrix of the cofactor is

$$C = \begin{pmatrix} -27 & 18 & 9 \\ 9 & -6 & -3 \\ 9 & -6 & -3 \end{pmatrix}$$

and the transpose of the cofactor matrix C is the adjoint of A, that is

$$Adj(A) = \begin{pmatrix} -27 & 9 & 9 \\ 18 & -6 & -6 \\ 9 & -3 & -3 \end{pmatrix}$$

Now by using the cofactor expansion along the first row, we can find the determinant of the matrix A as follows

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = -27 + 36 - 9 = 0$$

Since the determinant of the matrix is zero, therefore, the inverse of A does not exist.

(c) The minors M_{ij} of all elements a_{ij} of the matrix A are calculated as follows

$$\begin{aligned}
M_{11} &= \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} &= 0-1 &= -1 \\
M_{12} &= \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} &= 1-0 &= 1 \\
M_{13} &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} &= 1-0 &= 1 \\
M_{21} &= \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} &= 1-0 &= 1 \\
M_{22} &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} &= 1-0 &= 1 \\
M_{23} &= \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} &= 1-0 &= 1 \\
M_{31} &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} &= 1-0 &= 1 \\
M_{32} &= \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} &= 1-0 &= 1 \\
M_{33} &= \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} &= 0-1 &= -1
\end{aligned}$$

Now using these minors we can have the cofactors of the matrix as follows

$$\begin{aligned}
A_{11} &= (-1)^{1+1}M_{11} &= M_{11} &= -1 \\
A_{12} &= (-1)^{1+2}M_{12} &= -M_{12} &= -1 \\
A_{13} &= (-1)^{1+3}M_{13} &= M_{13} &= 1 \\
A_{21} &= (-1)^{2+1}M_{21} &= -M_{21} &= -1 \\
A_{22} &= (-1)^{2+2}M_{22} &= M_{22} &= 1 \\
A_{23} &= (-1)^{2+3}M_{23} &= -M_{23} &= -1 \\
A_{31} &= (-1)^{3+1}M_{31} &= M_{31} &= 1 \\
A_{32} &= (-1)^{3+2}M_{32} &= -M_{32} &= -1 \\
A_{33} &= (-1)^{3+3}M_{33} &= M_{33} &= -1
\end{aligned}$$

So that the matrix of the cofactor is

$$C = \begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$$

and the transpose of the cofactor matrix C is the adjoint of A, that is

$$Adj(A) = \begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$$

Now by using the cofactor expansion along the first row, we can find the determinant of the matrix A as follows

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = -1 - 1 - 0 = -2$$

To find the inverse of the matrix A, we use

$$A^{-1} = \frac{Adj(A)}{\det(A)} = -\frac{1}{2}\begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix}$$

10. Let

$$A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \\ 3 & -2 & 1 \end{pmatrix},$$

Show that $A(Adj\ A) = (Adj\ A)A = \det(A)\mathbf{I_3}$.

Solution: The minors M_{ij} of all elements a_{ij} of the matrix A are calculated as follows

$$M_{11} = \begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} = 2 - 0 = 2$$

$$M_{12} = \begin{vmatrix} -1 & 0 \\ 3 & 1 \end{vmatrix} = -1 - 0 = -1$$

$$M_{13} = \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} = 2 - 6 = -4$$

$$M_{21} = \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = 1 + 6 = 7$$

$$M_{22} = \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = 2 - 9 = -7$$

$$M_{23} = \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} = -4 - 3 = -7$$

$$M_{31} = \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = 0 - 6 = -6$$

$$M_{32} = \begin{vmatrix} 2 & 3 \\ -1 & 0 \end{vmatrix} = 0 + 3 = 3$$

$$M_{33} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 4 + 1 = 5$$

Now using these minors we can have the cofactors of the matrix as follows

$$\begin{array}{lllllll}
A_{11} & = & (-1)^{1+1}M_{11} & = & M_{11} & = & 2 \\
A_{12} & = & (-1)^{1+2}M_{12} & = & -M_{12} & = & 1 \\
A_{13} & = & (-1)^{1+3}M_{13} & = & M_{13} & = & -4 \\
A_{21} & = & (-1)^{2+1}M_{21} & = & -M_{21} & = & -7 \\
A_{22} & = & (-1)^{2+2}M_{22} & = & M_{22} & = & -7 \\
A_{23} & = & (-1)^{2+3}M_{23} & = & -M_{23} & = & 7 \\
A_{31} & = & (-1)^{3+1}M_{31} & = & M_{31} & = & -6 \\
A_{32} & = & (-1)^{3+2}M_{32} & = & -M_{32} & = & -3 \\
A_{33} & = & (-1)^{3+3}M_{33} & = & M_{33} & = & 5
\end{array}$$

So that the matrix of the cofactor is

$$C = \begin{pmatrix} 2 & 1 & -4 \\ -7 & -7 & 7 \\ -6 & -3 & 5 \end{pmatrix}$$

and the transpose of the cofactor matrix C is the adjoint of A, that is

$$Adj(A) = \begin{pmatrix} 2 & -7 & -6 \\ 1 & -7 & -3 \\ -4 & 7 & 5 \end{pmatrix}$$

Now by using the cofactor expansion along the first row, we can find the determinant of the matrix A as follows

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 4 + 1 - 12 = -7$$

Now finding the product

$$A(Adj(A)) = (Adj(A))A = \begin{pmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{pmatrix}$$

which can be also written as

$$A(Adj(A)) = (Adj(A))A = -7 \begin{pmatrix} 1 & 0 & \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \det(A)\mathbf{I_3}$$

11. Use the matrices of the Problem 9, solve the following systems using matrix inverse method

$$\textbf{(a)} \quad A\mathbf{x} = [1, 1]^T, \quad \textbf{(b)} \quad A\mathbf{x} = [2, 1, 3]^T, \quad \textbf{(c)} \quad A\mathbf{x} = [1, 0, 1]^T.$$

Solution: **(a)** Since the inverse of the matrix A exists, and is

$$A^{-1} = \begin{pmatrix} 2/5 & -1/5 \\ 3/10 & 1/10 \end{pmatrix}$$

Using the matrix inverse method, we get

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} 2/5 & -1/5 \\ 3/10 & 1/10 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/5 \\ 2/5 \end{pmatrix}$$

(b) Since the inverse of the matrix A does not exist, therefore, the matrix inverse method can not be applicable.
(c) Since the inverse of the matrix A exists, and is

$$A^{-1} = \begin{pmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix}$$

Using the matrix inverse method, we get

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

13. Solve the following systems using the Cramer's rule.
(a)

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & - & x_3 & = & 4 \\ 5x_1 & - & 2x_2 & - & x_3 & = & -2 \\ 2x_1 & + & 2x_2 & + & x_3 & = & 9 \end{array}$$

(b)

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & 3x_3 & = & 2 \\ 5x_1 & + & 3x_2 & + & x_3 & = & 3 \\ 2x_1 & + & 3x_2 & + & x_3 & = & -1 \end{array}$$

(c)

$$\begin{array}{rcrcrcr} 4x_1 & + & x_2 & - & 3x_3 & = & 9 \\ 3x_1 & + & 2x_2 & - & 6x_3 & = & -2 \\ x_1 & - & 5x_2 & + & 3x_3 & = & 1 \end{array}$$

Solution: **(a)** Writing the given system in matrix form

$$\begin{pmatrix} 1 & 3 & -1 \\ 5 & -2 & -1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 9 \end{pmatrix}$$

gives

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 5 & -2 & -1 \\ 2 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 4 \\ -2 \\ 9 \end{pmatrix}$$

The determinant of the matrix can be calculated as

$$|A| = \begin{vmatrix} 1 & 3 & -1 \\ 5 & -2 & -1 \\ 2 & 2 & 1 \end{vmatrix} = 1(-2+2) + 3(-2-5) - 1(10+4) = -35 \neq 0$$

which showed that the given matrix A is nonsingular. Then the matrices A_1, A_2, and A_3 can be computed as

$$A_1 = \begin{pmatrix} 4 & 3 & -1 \\ -2 & -2 & -1 \\ 9 & 2 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 4 & -1 \\ 5 & -2 & -1 \\ 2 & 9 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 3 & 4 \\ 5 & -2 & -2 \\ 2 & 2 & 9 \end{pmatrix}$$

Now finding the determinant of the matrices A_1, A_2, and A_3, we get

$$\begin{aligned} |A_1| &= 4(-2+2) + 3(-9+2) - 1(-4+18) = 0 - 21 - 14 = -35 \\ |A_2| &= 1(-2+9) + 4(-2-5) - 1(45+4) = 7 - 28 - 49 = -70 \\ |A_3| &= 1(-18+4) + 3(-4-45) + 4(10+4) = -14 - 147 + 56 = -105 \end{aligned}$$

Therefore, by using the Cramer's rule, we get

$$x_1 = \frac{|A_1|}{|A|} = \frac{-35}{-35} = 1$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{-70}{-35} = 2$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{-105}{-35} = 3$$

(b) The matrix form of the give system is

$$\begin{pmatrix} 1 & 1 & 3 \\ 5 & 3 & 1 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

gives

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 3 & 1 \\ 2 & 3 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

The determinant of the matrix can be calculated as

$$|A| = \begin{vmatrix} 1 & 1 & 3 \\ 5 & 3 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 1(3-3) + 1(2-5) + 3(15-6) = 0 - 3 + 27 = 24 \neq 0$$

which showed that the given matrix A is nonsingular. Then the matrices A_1, A_2, and A_3 can be computed as

$$A_1 = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 3 & 1 \\ -1 & 3 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 3 & 1 \\ 2 & -1 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 & 2 \\ 5 & 3 & 3 \\ 2 & 3 & -1 \end{pmatrix}$$

The determinant of the matrices A_1, A_2, and A_3 can be computed as

$$\begin{aligned}
|A_1| &= 2(3-3)+1(-1-3)+3(9+3) = 0-4+36 = 32 \\
|A_2| &= 1(3+1)+2(2-5)+3(-5-6) = 4-6-33 = -35 \\
|A_3| &= 1(-3-9)+1(6+5)+2(15-6) = -12+11+18 = 17
\end{aligned}$$

Therefore, by the Cramer's rule

$$x_1 = \frac{|A_1|}{|A|} = \frac{32}{24} = 1.3333$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{-35}{24} = -1.4583$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{17}{24} = 0.70833$$

(c) Writing the given system in matrix form

$$\begin{pmatrix} 4 & 1 & -3 \\ 3 & 2 & -6 \\ 1 & -5 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9 \\ -2 \\ 1 \end{pmatrix}$$

gives

$$A = \begin{pmatrix} 4 & 1 & -3 \\ 3 & 2 & -6 \\ 1 & -5 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 9 \\ -2 \\ 1 \end{pmatrix}$$

The determinant of the matrix can be calculated as

$$|A| = \begin{vmatrix} 4 & 1 & -3 \\ 3 & 2 & -6 \\ 1 & -5 & 3 \end{vmatrix} = 4(6-30)+1(-6-9)-3(-15-2) = -60 \neq 0$$

which showed that the given matrix A is nonsingular. Then the matrices A_1, A_2, and A_3 can be computed as

$$A_1 = \begin{pmatrix} 9 & 1 & -3 \\ -2 & 2 & -6 \\ 1 & -5 & 3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 4 & 9 & -3 \\ 3 & -2 & -6 \\ 1 & 1 & 3 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 4 & 1 & 9 \\ 3 & 2 & -2 \\ 1 & -5 & 1 \end{pmatrix}$$

Now finding the determinant of the matrices A_1, A_2, and A_3, we get

$$\begin{aligned}
|A_1| &= 9(6-30)+1(-6+6)-3(10-2) = -216+0-24 = -240 \\
|A_2| &= 4(-6+6)+9(-6-9)-3(3+2) = 0-135-15 = -150 \\
|A_3| &= 4(2-10)+1(-2-3)+9(-15-2) = -32-5-153 = -190
\end{aligned}$$

Therefore, by using the Cramer's rule, we get

$$x_1 = \frac{|A_1|}{|A|} = \frac{-240}{-60} = 4$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{-150}{-60} = 2.5$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{-190}{-60} = 3.1667$$

14. Use the simple Gaussian elimination method to show that the following system does not have a solution

$$\begin{array}{rcrcrcl} 3x_1 & + & x_2 & & & = & 1.5 \\ 2x_1 & - & x_2 & - & x_3 & = & 2 \\ 4x_1 & + & 3x_2 & + & x_3 & = & 0 \end{array}$$

Solution: The process begins with the augmented matrix form

$$\left(\begin{array}{rrr:r} 3 & 1 & 0 & 1.5 \\ 2 & -1 & -1 & 2 \\ 4 & 3 & 1 & 0 \end{array}\right)$$

Since $a_{11} = 3 \neq 0$, so we wish to eliminate the elements a_{21} and a_{31} by subtracting from the second and third rows the appropriate multiples of the first row. In this case the multiples are given

$$m_{21} = \frac{2}{3}, \quad \text{and} \quad m_{31} = \frac{4}{3}$$

Hence

$$\left(\begin{array}{rrr:r} 3 & 1 & 0 & 1.5 \\ 0 & -5/3 & -1 & 1 \\ 0 & 5/3 & 1 & -2 \end{array}\right)$$

As $a_{22}^{(1)} = -5/3 \neq 0$, therefore, we eliminate entry in $a_{32}^{(1)}$ position by subtracting the multiple $m_{32} = -1$ of the second row from the third row, to get

$$\left(\begin{array}{rrr:r} 3 & 1 & 0 & 1.5 \\ 0 & -5/3 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

Obviously, the original set of equations has been transformed to an upper-triangular form. Since the third diagonal element of the obtaining upper-triangular matrix is zero, which means that the coefficient matrix of the given system is singular and therefore, the given system has no unique solution. Now expressing the set in algebraic form yields

$$\begin{array}{rcrcrcr} 3x_1 & + & x_2 & & & = & 1.5 \\ & - & 5/3x_2 & - & x_3 & = & 1 \\ & & & & 0x_3 & = & -1 \end{array}$$

From the third equation, we find that $0 = -1$, which is not possible, therefore, this system has no solution.

15. Solve the following systems using the simple Gaussian elimination method

(a)

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & & & = & 0 \\ -x_1 & + & 2x_2 & - & x_3 & = & 1 \\ & - & x_2 & + & 4x_3 & = & 0 \end{array}$$

(b)

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 1 \\ 2x_1 & + & 3x_2 & + & 4x_3 & = & 3 \\ 4x_1 & + & 9x_2 & + & 16x_3 & = & 11 \end{array}$$

(c)

$$\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & - & x_3 & = & 1 \\ x_1 & - & 3x_2 & + & 2x_3 & = & 2 \\ 2x_1 & - & x_2 & + & x_3 & = & 3 \end{array}$$

(d)

$$\begin{array}{lcrcrcrcr} 2x_1 & + & x_2 & + & x_3 & - & x_4 & = & -3 \\ x_1 & + & 9x_2 & + & 8x_3 & + & 4x_4 & = & 15 \\ -x_1 & + & 3x_2 & + & 5x_3 & + & 2x_4 & = & 10 \\ & & x_2 & & & + & x_4 & = & 2 \end{array}$$

Solution: (a) The process begins with the augmented matrix form

$$\left(\begin{array}{rrrcr} 1 & -1 & 0 & \vdots & 0 \\ -1 & 2 & -1 & \vdots & 1 \\ 0 & -1 & 4 & \vdots & 0 \end{array}\right)$$

Since $a_{11} = 1 \neq 0$, so we wish to eliminate the elements a_{21} by subtracting from the second row the appropriate multiple of the first row. In this case the multiple is given as

$$m_{21} = \frac{-1}{1} = -1$$

Hence

$$\left(\begin{array}{rrrcr} 1 & -1 & 0 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 1 \\ 0 & -1 & 4 & \vdots & 0 \end{array}\right)$$

As $a_{22}^{(1)} = 1 \neq 0$, therefore, we eliminate entry in $a_{32}^{(1)}$ position by subtracting the multiple $m_{32} = \dfrac{-1}{1} = -1$ of the second row from the third row, to get

$$\left(\begin{array}{rrrcr} 1 & -1 & 0 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 1 \\ 0 & 0 & 3 & \vdots & 1 \end{array}\right)$$

Obviously, the original set of equations has been transformed to an upper-triangular form. Since all the diagonal elements of the obtaining upper-triangular matrix are nonzero, which means that the coefficient matrix of the given system is nonsingular and therefore, the given system has a unique solution. Now expressing the set in algebraic form yields

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & & & = & 0 \\ & & x_2 & - & x_3 & = & 1 \\ & & & & 3x_3 & = & 1 \end{array}$$

Now using backward substitution to give

$$\begin{array}{lcll} 3x_3 & = & 1, & \text{gives} \quad x_3 = 1/3 \\ x_2 & = & x_3 + 1 = 1/3 + 1, & \text{gives} \quad x_2 = 4/3 \\ x_1 & = & x_2, & \text{gives} \quad x_1 = 4/3 \end{array}$$

(b) The process begins with the augmented matrix form

$$\begin{pmatrix} 1 & 1 & 1 & \vdots & 1 \\ 2 & 3 & 4 & \vdots & 3 \\ 4 & 9 & 16 & \vdots & 11 \end{pmatrix}$$

Since $a_{11} = 1 \neq 0$, so we wish to eliminate the elements a_{21} and a_{31} by subtracting from the second and third rows the appropriate multiples of the first row. In this case the multiples are given

$$m_{21} = \frac{2}{1} = 2, \quad \text{and} \quad m_{31} = \frac{4}{1} = 4$$

Hence

$$\begin{pmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & 1 & 2 & \vdots & 1 \\ 0 & 5 & 12 & \vdots & 7 \end{pmatrix}$$

As $a_{22}^{(1)} = 1 \neq 0$, therefore, we eliminate entry in $a_{32}^{(1)}$ position by subtracting the multiple $m_{32} = 5$ of the second row from the third row, to get

$$\begin{pmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & 1 & 2 & \vdots & 1 \\ 0 & 0 & 2 & \vdots & 2 \end{pmatrix}$$

Obviously, the original set of equations has been transformed to an upper-triangular form. Since all the diagonal elements of the obtaining upper-triangular matrix are nonzero, which means that the coefficient matrix of the given system is nonsingular and therefore, the given system has a unique solution. Now expressing the set in algebraic form yields

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 1 \\ & & x_2 & + & 2x_3 & = & 1 \\ & & & & 2x_3 & = & 2 \end{array}$$

Now using backward substitution to give

$$\begin{array}{lcll} 2x_3 & = & 2, & \text{gives} \quad x_3 = 1 \\ x_2 & = & 1 - 2x_3 = 1 - 2, & \text{gives} \quad x_2 = -1 \\ x_1 & = & 1 - x_2 - x_3 = 1 + 1 - 1, & \text{gives} \quad x_1 = 1 \end{array}$$

(c) The augmented matrix form of the given system

$$\begin{pmatrix} 3 & 2 & -1 & \vdots & 1 \\ 1 & -3 & 2 & \vdots & 2 \\ 2 & -1 & 1 & \vdots & 3 \end{pmatrix}$$

Since $a_{11} = 3 \neq 0$, so we wish to eliminate the elements a_{21} and a_{31} by subtracting from the second and third rows the appropriate multiples of the first row. In this case the multiples are given

$$m_{21} = \frac{1}{3}, \quad \text{and} \quad m_{31} = \frac{2}{3}$$

Hence

$$\left(\begin{array}{rrrcr} 3 & 2 & -1 & \vdots & 1 \\ 0 & -11/3 & 7/3 & \vdots & 5/3 \\ 0 & -7/3 & 5/3 & \vdots & 7/3 \end{array}\right)$$

As $a_{22}^{(1)} = -11/3 \neq 0$, therefore, we eliminate entry in $a_{32}^{(1)}$ position by subtracting the multiple $m_{32} = 7/11$ of the second row from the third row, to get

$$\left(\begin{array}{rrrcr} 3 & 2 & -1 & \vdots & 1 \\ 0 & -11/3 & 7/3 & \vdots & 5/3 \\ 0 & 0 & 2/11 & \vdots & 14/11 \end{array}\right)$$

Obviously, the original set of equations has been transformed to an upper-triangular form. Since all the diagonal elements of the obtaining upper-triangular matrix are nonzero, which means that the coefficient matrix of the given system is nonsingular and therefore, the given system has a unique solution. Now expressing the set in algebraic form yields

$$\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & - & x_3 & = & 1 \\ & - & 11/3x_2 & + & 7/3x_3 & = & 5/3 \\ & & & & 2/11x_3 & = & 14/11 \end{array}$$

Now using backward substitution to give

$$\begin{array}{lcll} 2/11x_3 & = & 14/11, & \text{gives} \quad x_3 = 7 \\ -11/3x_2 & = & 5/3 - 7/3x_3 = -44/3, & \text{gives} \quad x_2 = 4 \\ 3x_1 & = & 1 - 2x_2 + x_3 = 0, & \text{gives} \quad x_1 = 0 \end{array}$$

(d) The augmented matrix form of the given system

$$\left(\begin{array}{rrrrcr} 2 & 1 & 1 & -1 & \vdots & -3 \\ 1 & 9 & 8 & 4 & \vdots & 15 \\ -1 & 3 & 5 & 2 & \vdots & 10 \\ 0 & 1 & 0 & 1 & \vdots & 2 \end{array}\right)$$

Since $a_{11} = 2 \neq 0$, so we wish to eliminate the elements a_{21} and a_{31} by subtracting from the second, third, and fourth rows the appropriate multiples of the first row. In this case the multiples are given

$$m_{21} = \frac{1}{2}, \quad \text{and} \quad m_{31} = \frac{-1}{2}$$

Hence

$$\left(\begin{array}{cccc:c} 2 & 1 & 1 & -1 & -3 \\ 0 & 17/2 & 15/2 & 9/2 & 33/2 \\ 0 & 7/2 & 11/2 & 3/2 & 17/2 \\ 0 & 1 & 0 & 1 & 2 \end{array}\right)$$

As $a_{22}^{(1)} = 17/2 \neq 0$, therefore, we eliminate entries in $a_{32}^{(1)}$ and $a_{42}^{(1)}$ positions by subtracting the multiples $m_{32} = 7/17$ and $m_{42} = 2/17$ of the second row from the third and fourth rows, to get

$$\left(\begin{array}{cccc:c} 2 & 1 & 1 & -1 & -3 \\ 0 & 17/2 & 15/2 & 9/2 & 33/2 \\ 0 & 0 & 41/17 & -6/17 & 29/17 \\ 0 & 0 & -15/17 & 8/17 & 1/17 \end{array}\right)$$

As $a_{33}^{(1)} = 41/17 \neq 0$, therefore, we eliminate entry in $a_{43}^{(1)}$ position by subtracting the multiple $m_{42} = -15/41$ of the third row from the fourth row, to get

$$\left(\begin{array}{cccc:c} 2 & 1 & 1 & -1 & -3 \\ 0 & 17/2 & 15/2 & 9/2 & 33/2 \\ 0 & 0 & 41/17 & -6/17 & 29/17 \\ 0 & 0 & 0 & 14/41 & 28/41 \end{array}\right)$$

Obviously, the original set of equations has been transformed to an upper-triangular form. Since all the diagonal elements of the obtaining upper-triangular matrix are nonzero, which means that the coefficient matrix of the given system is nonsingular and therefore, the given system has a unique solution. Now expressing the set in algebraic form yields

$$\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & + & x_3 & - & x_4 & = & -3 \\ & & 17/2x_2 & + & 15/2x_3 & + & 9/2x_4 & = & 33/2 \\ & & & & 41/17x_3 & - & 6/17x_4 & = & 29/17 \\ & & & & & & 14/41x_4 & = & 28/41 \end{array}$$

Now using backward substitution, we get

$$x_1 = -1, \quad x_2 = 0, \quad x_3 = 1, \quad x_4 = 2$$

16. Find a value of k so that the following linear system has a non-trivial solution, and solve it in this case.

$$\begin{array}{rcrcrcr} 2x_1 & + & 2x_2 & + & 3x_3 & = & 0 \\ 3x_1 & + & kx_2 & + & 5x_3 & = & 0 \\ x_1 & + & 7x_2 & + & 3x_3 & = & 0 \end{array}$$

Solution: The augmented matrix form of the given system

$$\begin{pmatrix} 2 & 2 & 3 & \vdots & 0 \\ 3 & k & 5 & \vdots & 0 \\ 1 & 7 & 3 & \vdots & 0 \end{pmatrix}$$

Since $a_{11} = 2 \neq 0$, so we wish to eliminate the elements a_{21} and a_{31} by subtracting from the second and third rows the appropriate multiples of the first row. In this case the multiples are given

$$m_{21} = \frac{3}{2}, \quad \text{and} \quad m_{31} = \frac{1}{2}$$

Hence

$$\begin{pmatrix} 2 & 2 & 3 & \vdots & 0 \\ 0 & k-3 & 1/2 & \vdots & 0 \\ 0 & 6 & 3/2 & \vdots & 0 \end{pmatrix}$$

If $a_{22}^{(1)} = k - 3 \neq 0$, then, we eliminate entry in $a_{32}^{(1)}$ position by subtracting the multiple $m_{32} = 6/k - 3$ of the second row from the third row, to get

$$\begin{pmatrix} 2 & 2 & 3 & \vdots & 0 \\ 0 & k-3 & 1/2 & \vdots & 0 \\ 0 & 0 & (3k-15)/2(k-3) & \vdots & 0 \end{pmatrix}$$

Since we know that a homogeneous system of n equations in n unknowns has a solution other than trivial solution if and only if the determinant of the coefficient matrix equal to zero, therefore, we put

$$(3k - 15)/2(k - 3) = 0, \quad \text{gives} \quad k = 5$$

Now expressing the set in algebraic form yields

$$\begin{array}{rcrcrcl} 2x_1 & + & 2x_2 & + & 3x_3 & = & 0 \\ & & 2x_2 & + & 1/2x_3 & = & 0 \\ & & & & 0x_3 & = & 0 \end{array}$$

and this underdetermined linear system has infinitely many solutions. If we take $x_3 = 4$, then we get

$$x_1 = -5, \quad x_2 = -1, \quad x_3 = 4$$

17. Determine the rank of each matrix

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ 1 & -5 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 1 & 6 \\ -3 & 6 & 4 \\ 5 & 0 & 9 \end{pmatrix}, \quad C = \begin{pmatrix} 17 & 46 & 7 \\ 20 & 49 & 8 \\ 23 & 52 & 9 \end{pmatrix}$$

Solution: Applying the forward elimination step of the simple Gaussian elimination on the given matrix A and eliminating the elements below the first pivot (first diagonal element) to

$$\begin{pmatrix} 3 & 1 & -1 \\ 0 & -2/3 & 14/3 \\ 0 & -16/3 & 4/3 \end{pmatrix}$$

We finished with the first elimination step. The second pivot is in $(2,2)$ position but after eliminating the element below it we find the triangular form to be

$$\begin{pmatrix} 3 & 1 & -1 \\ 0 & -2/3 & 14/3 \\ 0 & 0 & -36 \end{pmatrix}$$

Since the number of pivots are three, therefore, the rank of the given matrix is 3. Note that the original matrix is nonsingular as the rank of 3×3 matrix is 3.

Now applying the forward elimination step of the simple Gaussian elimination on the given matrix B and eliminating the elements below the first pivot (first diagonal element) to

$$\begin{pmatrix} 4 & 1 & 6 \\ 0 & 27/4 & 17/2 \\ 0 & -5/4 & 3/2 \end{pmatrix}$$

We finished with the first elimination step. The second pivot is in $(2,2)$ position but after eliminating the element below it we find the triangular form to be

$$\begin{pmatrix} 4 & 1 & 6 \\ 0 & 27/4 & 17/2 \\ 0 & 0 & 83/27 \end{pmatrix}$$

Since the number of pivots are three, therefore, the rank of the given matrix is 3. Note that this matrix is also nonsingular.

Repeating the same above procedure in finding the rank of the matrix C, we get

$$\begin{pmatrix} 17 & 46 & 7 \\ 0 & -87/17 & -4/17 \\ 0 & -174/17 & -8/17 \end{pmatrix}$$

and

$$\begin{pmatrix} 17 & 46 & 7 \\ 0 & -87/17 & -4/17 \\ 0 & 0 & 0 \end{pmatrix}$$

Since the number of pivots are two, therefore, the rank of the given matrix is 2. Note that the original matrix C is singular as the rank of 3×3 matrix is 2.

19. Determine the rank of each matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0.1 & 0.2 & 0.3 \\ 0.4 & 0.5 & 0.6 \\ 0.7 & 0.8 & 0.901 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 5 & 7 & 9 \\ 4 & 6 & 8 & 10 \end{pmatrix}$$

Solution: Applying the forward elimination step of the simple Gaussian elimination on the given matrix A and eliminating the elements below the first pivot (first diagonal element) to

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 3/2 & -1 \end{pmatrix}$$

Replacing second row by third row, we get

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Since the number of pivots are three, therefore, the rank of the given matrix is 3. Note that the original matrix is nonsingular as the rank of 3×3 matrix is 3.

Now applying the forward elimination step of the simple Gaussian elimination on the given matrix B and eliminating the elements below the first pivot (first diagonal element) to

$$\begin{pmatrix} 0.1 & 0.2 & 0.3 \\ 0.0 & -0.3 & -0.6 \\ 0.0 & -0.6 & -1.199 \end{pmatrix}$$

We finished with the first elimination step. The second pivot is in $(2, 2)$ position but after eliminating the element below it we find the triangular form to be

$$\begin{pmatrix} 0.1 & 0.2 & 0.3 \\ 0.0 & -0.3 & -0.6 \\ 0.0 & 0.0 & 0.001 \end{pmatrix}$$

Since the number of pivots are three, therefore, the rank of the given matrix is 3. Note that this matrix is nonsingular.

Repeating the same above procedure in finding the rank of the matrix C, we get

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \end{pmatrix}$$

Replacing second row by third row, we get

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & -4 & -6 \end{pmatrix}$$

We finished with the first elimination step. The second pivot is in $(2, 2)$ position but after eliminating the element below it we find the following form

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since the number of pivots are two, therefore, the rank of the given matrix is 2. Note that the original matrix C is singular as the rank of 4×4 matrix is 2.

20. Solve the Problem 15 using the Gaussian elimination with partial pivoting and complete pivoting.

Solution: Partial Pivoting

(a)

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & & & = & 0 \\ -x_1 & + & 2x_2 & - & x_3 & = & 1 \\ & - & x_2 & + & 4x_3 & = & 0 \end{array}$$

For the first elimination step, since $a_{11} = 1$ and $a_{21} = -1$ are both largest absolute coefficients of first variable x_1, therefore, we can choose $a_{11} = 1$ as the first pivotal element. Eliminate first variable x_1 from the second row by subtracting the multiple $m_{21} = \dfrac{-1}{1} = -1$ of row 1 from row 2, gives

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & & & = & 0 \\ & & x_2 & - & x_3 & = & 1 \\ & - & x_2 & + & 4x_3 & = & 0 \end{array}$$

For the second elimination step, eliminate second variable x_2 from the third row by subtracting the multiple $m_{32} = \dfrac{-1}{1} = -1$ of row 2 from row 3, gives

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & & & = & 0 \\ & & x_2 & - & x_3 & = & 1 \\ & & & & 3x_3 & = & 1 \end{array}$$

Obviously, the original set of equations has been transformed to an equivalent upper-triangular form. Now using backward substitution to get the solution

$$x_1 = 4/3, \quad x_2 = 4/3, \quad x_3 = 1/3$$

(b)

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 1 \\ 2x_1 & + & 3x_2 & + & 4x_3 & = & 3 \\ 4x_1 & + & 9x_2 & + & 16x_3 & = & 11 \end{array}$$

For the first elimination step, since 4 is the largest absolute coefficient of first variable x_1, therefore, the first row and the third row are interchange, giving us

$$\begin{array}{rcrcrcr} 4x_1 & + & 9x_2 & + & 16x_3 & = & 11 \\ 2x_1 & + & 3x_2 & + & 4x_3 & = & 3 \\ x_1 & + & x_2 & + & x_3 & = & 1 \end{array}$$

Eliminate first variable x_1 from the second and third rows by subtracting the multiples $m_{21} = \dfrac{2}{4}$ and $m_{31} = \dfrac{1}{4}$ of row 1 from row 2 and row 3 respectively, gives

$$\begin{array}{rcrcrcr} 4x_1 & + & 9x_2 & + & 16x_3 & = & 11 \\ & - & 3/2x_2 & - & 4x_3 & = & -5/2 \\ & - & 5/4x_2 & - & x_3 & = & -7/5 \end{array}$$

For the second elimination step, $-3/2$ is the largest absolute coefficient of second variable x_2, so eliminate second variable x_2 from the third row by subtracting the multiple $m_{32} = \frac{5}{6}$ of row 2 from row 3, gives

$$\begin{array}{rcrcrcr} 4x_1 & + & 9x_2 & + & 16x_3 & = & 11 \\ & - & 3/2x_2 & - & 4x_3 & = & -5/2 \\ & & & & 1/3x_3 & = & 1/3 \end{array}$$

Obviously, the original set of equations has been transformed to an equivalent upper-triangular form. Now using backward substitution to get the solution of the linear system which is

$$x_1 = 1, \quad x_2 = -1, \quad x_3 = 1$$

(c)

$$\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & - & x_3 & = & 1 \\ x_1 & - & 3x_2 & + & 2x_3 & = & 2 \\ 2x_1 & - & x_2 & + & x_3 & = & 3 \end{array}$$

Since 3 is the largest absolute coefficient of first variable x_1 in the first row, therefore, eliminate first variable x_1 from the second and third rows by subtracting the multiples $m_{21} = \frac{1}{3}$ and $m_{31} = \frac{2}{3}$ of row 1 from row 2 and row 3 respectively, gives

$$\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & - & x_3 & = & 1 \\ & - & 11/3x_2 & + & 7/3x_3 & = & 5/3 \\ & - & 7/3x_2 & + & 5/3x_3 & = & 7/3 \end{array}$$

For the second elimination step, $-11/3$ is the largest absolute coefficient of second variable x_2 in the second row, so eliminate second variable x_2 from the third row by subtracting the multiple $m_{32} = \frac{7}{11}$ of row 2 from row 3, gives

$$\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & - & x_3 & = & 1 \\ & - & 11/3x_2 & + & 7/3x_3 & = & 5/3 \\ & & & & 2/11x_3 & = & 14/11 \end{array}$$

Since the original set of equations has been transformed to an equivalent upper-triangular form, therefore, using backward substitution to get the solution of the linear system which is

$$x_1 = 0, \quad x_2 = 4, \quad x_3 = 7$$

(d)

$$\begin{array}{lclclclcr} 2x_1 & + & x_2 & + & x_3 & - & x_4 & = & -3 \\ x_1 & + & 9x_2 & + & 8x_3 & + & 4x_4 & = & 15 \\ -x_1 & + & 3x_2 & + & 5x_3 & + & 2x_4 & = & 10 \\ & & x_2 & & & + & x_4 & = & 2 \end{array}$$

Since 2 is the largest absolute coefficient of first variable x_1 in the first row, therefore, eliminate first variable x_1 from the second and third rows by subtracting the multiples $m_{21} = \frac{1}{2}$ and $m_{31} = \frac{-1}{2}$ of row 1 from row 2 and row 3 respectively, gives

$$\begin{array}{rcrcrcrcl} 2x_1 & + & x_2 & + & x_3 & - & x_4 & = & -3 \\ & & 17/2x_2 & + & 15/2x_3 & + & 9/2x_4 & = & 33/2 \\ & & 7/2x_2 & + & 11/2x_3 & + & 3/2x_4 & = & 17/2 \\ & & x_2 & & & + & x_4 & = & 2 \end{array}$$

For the second elimination step, 17/2 is the largest absolute coefficient of second variable x_2 in the second row, so eliminate second variable x_2 from the third and fourth rows by subtracting the multiples $m_{32} = \frac{7}{17}$ and $m_{42} = \frac{2}{17}$ of row 2 from row 3 and row 4 respectively, gives

$$\begin{array}{rcrcrcrcl} 2x_1 & + & x_2 & + & x_3 & - & x_4 & = & -3 \\ & & 17/2x_2 & + & 15/2x_3 & + & 9/2x_4 & = & 33/2 \\ & & & & 41/17x_3 & - & 6/17x_4 & = & 29/17 \\ & & & - & 15/17x_3 & + & 8/17x_4 & = & 1/17 \end{array}$$

For the third elimination step, 41/17 is the largest absolute coefficient of third variable x_3 in the third row, so eliminate third variable x_3 from the fourth row by subtracting the multiple $m_{43} = \frac{15}{41}$ of row 3 from row 4, gives

$$\begin{array}{rcrcrcrcl} 2x_1 & + & x_2 & + & x_3 & - & x_4 & = & -3 \\ & & 17/2x_2 & + & 15/2x_3 & + & 9/2x_4 & = & 33/2 \\ & & & & 41/17x_3 & - & 6/17x_4 & = & 29/17 \\ & & & & & & 14/41x_4 & = & 28/41 \end{array}$$

Since the original set of equations has been transformed to an equivalent upper-triangular form, therefore, using backward substitution to get the solution of the linear system which is

$$x_1 = -1, \quad x_2 = 0, \quad x_3 = 1, \quad x_4 = 2$$

Complete Pivoting

(a)

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & & & = & 0 \\ -x_1 & + & 2x_2 & - & x_3 & = & 1 \\ & - & x_2 & + & 4x_3 & = & 0 \end{array}$$

Solution: For the first elimination step, since 4 is the largest absolute coefficient of variable x_3 in the given system, therefore, the first row and the third row are interchange as well as the first column and third column. Then eliminate third variable x_3 from the second row by subtracting the multiple $m_{21} = \frac{-1}{4}$ of row 1 from row 2, gives

$$\begin{array}{rcrcrcl} 4x_3 & - & x_2 & & & = & 0 \\ & & 7/4x_2 & - & x_1 & = & 1 \\ & - & x_2 & + & x_1 & = & 0 \end{array}$$

For the second elimination step, $\dfrac{7}{4}$ is the largest absolute coefficient of second variable x_2 in the remaining system of equations, so eliminate second variable x_2 from the third row by subtracting the multiple $m_{32} = \dfrac{-4}{7}$ of row 2 from row 3, gives

$$\begin{array}{rcrcrcl} 4x_3 & - & x_2 & & & = & 0 \\ & & 7/4x_2 & - & x_1 & = & 1 \\ & & & & 3/7x_1 & = & 4/7 \end{array}$$

Obviously, the original set of equations has been transformed to an equivalent upper-triangular form. Now using backward substitution to get the solution of the given system as

$$x_1 = 4/3, \quad x_2 = 4/3, \quad x_3 = 1/3$$

(b)

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 1 \\ 2x_1 & + & 3x_2 & + & 4x_3 & = & 3 \\ 4x_1 & + & 9x_2 & + & 16x_3 & = & 11 \end{array}$$

For the first elimination step, since 16 is the largest absolute coefficient of variable x_3 in the given system, therefore, the first row and the third row are interchange as well as the first column and third column, we get

$$\begin{array}{rcrcrcl} 16x_3 & + & 9x_2 & + & 4x_1 & = & 11 \\ 4x_3 & + & 3x_2 & + & 2x_1 & = & 3 \\ x_3 & + & 9x_2 & + & x_1 & = & 1 \end{array}$$

Then eliminate third variable x_3 from the second and the third rows by subtracting the multiples $m_{21} = \dfrac{4}{16}$ and $m_{31} = \dfrac{1}{16}$ of row 1 from rows 2 and 3 respectively, gives

$$\begin{array}{rcrcrcl} 16x_3 & + & 9x_2 & + & 4x_1 & = & 11 \\ & & 3/4x_2 & + & x_1 & = & 1/4 \\ & & 7/16x_2 & + & 3/4x_1 & = & 5/16 \end{array}$$

For the second elimination step, 1 is the largest absolute coefficient of first variable x_1 in the second row and the third column, so the second and third columns are interchange, giving us

$$\begin{array}{rcrcrcl} 16x_3 & + & 4x_1 & + & 9x_2 & = & 11 \\ & & x_1 & + & 3/4x_2 & = & 1/4 \\ & & 3/4x_1 & + & 7/16x_2 & = & 5/16 \end{array}$$

Eliminate first variable x_1 from the third row by subtracting the multiple $m_{32} = \dfrac{3}{4}$ of row 2 from row 3, gives

$$\begin{array}{rcrcrcl} 16x_3 & + & 4x_1 & + & 9x_2 & = & 11 \\ & & x_1 & + & 3/4x_2 & = & 1/4 \\ & & & - & 1/8x_2 & = & 1/8 \end{array}$$

The original set of equations has been transformed to an equivalent upper-triangular form. Now using backward substitution to get the solution of the linear system which is

$$x_1 = 1, \quad x_2 = -1, \quad x_3 = 1$$

(c)

$$\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & - & x_3 & = & 1 \\ x_1 & - & 3x_2 & + & 2x_3 & = & 2 \\ 2x_1 & - & x_2 & + & x_3 & = & 3 \end{array}$$

For the first elimination step, since 3 is the largest absolute coefficient of variable x_1 in the first row and first column, therefore, no need to do any interchanging. Eliminate first variable x_1 from the second and the third rows by subtracting the multiples $m_{21} = \dfrac{1}{3}$ and $m_{31} = \dfrac{2}{3}$ of row 1 from rows 2 and 3 respectively, gives

$$\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & - & x_3 & = & 1 \\ & - & 11/3x_2 & + & 7/3x_3 & = & 5/3 \\ & - & 7/3x_2 & + & 5/3x_3 & = & 7/3 \end{array}$$

For the second elimination step, $-11/3$ is the largest absolute coefficient of second variable x_2 in the second row and the second column, so again no need to do any interchanging. Then eliminate second variable x_2 from the third row by subtracting the multiple $m_{32} = \dfrac{7}{11}$ of row 2 from row 3, gives

$$\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & - & x_3 & = & 1 \\ & - & 11/3x_2 & + & 7/3x_3 & = & 5/3 \\ & & & & 2/11x_3 & = & 14/11 \end{array}$$

Since the original set of equations has been transformed to an equivalent upper-triangular form, therefore, using backward substitution to get the solution of the linear system which is

$$x_1 = 0, \quad x_2 = 4, \quad x_3 = 7$$

(d)

$$\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & + & x_3 & - & x_4 & = & -3 \\ x_1 & + & 9x_2 & + & 8x_3 & + & 4x_4 & = & 15 \\ -x_1 & + & 3x_2 & + & 5x_3 & + & 2x_4 & = & 10 \\ & & x_2 & & & + & x_4 & = & 2 \end{array}$$

For the first elimination step, since 9 is the largest absolute coefficient of variable x_2 in the given system, therefore, the first row and the second row are interchange as well as the first column and second column, we get

$$\begin{array}{rcrcrcrcr} 9x_2 & + & x_1 & + & 8x_3 & + & 4x_4 & = & 15 \\ x_2 & + & 2x_1 & + & x_3 & - & x_4 & = & -3 \\ 3x_2 & - & x_1 & + & 5x_3 & + & 2x_4 & = & 10 \\ x_2 & & & & & + & x_4 & = & 2 \end{array}$$

Now eliminate second variable x_2 from second, third, and fourth rows by subtracting the multiples $m_{21} = \dfrac{1}{9}$, $m_{31} = \dfrac{3}{9}$, and $m_{41} = \dfrac{1}{9}$ of row 1 from rows 2, 3, and 4 respectively, gives

$$\begin{array}{rcrcrcrcr} 9x_2 & + & x_1 & + & 8x_3 & + & 4x_4 & = & 15 \\ & & 17/9x_1 & + & 1/9x_3 & - & 13/9x_4 & = & -14/3 \\ & - & 4/3x_1 & + & 7/3x_3 & + & 2/3x_4 & = & 5 \\ & - & 1/9x_1 & - & 8/9x_3 & + & 5/9x_4 & = & 1/3 \end{array}$$

Since 7/3 is the largest absolute coefficient of third variable x_3 in the third row and third column, therefore, the second row and the third row are interchange as well as second column and third column, we get

$$\begin{array}{rcrcrcrcl} 9x_2 & + & 8x_3 & + & x_1 & + & 4x_4 & = & 15 \\ & & 7/3x_3 & - & 4/3x_1 & + & 2/3x_4 & = & 5 \\ & & 1/9x_3 & + & 17/9x_1 & - & 13/9x_4 & = & -14/3 \\ & - & 8/9x_3 & - & 1/9x_1 & + & 5/9x_4 & = & 1/3 \end{array}$$

Now eliminate third variable x_3 from the third and the fourth rows by subtracting the multiples $m_{32} = \dfrac{1}{21}$ and $m_{42} = \dfrac{-8}{21}$ of row 2 from row 3 and row 4 respectively, gives

$$\begin{array}{rcrcrcrcl} 9x_2 & + & 8x_3 & + & x_1 & + & 4x_4 & = & 15 \\ & & 7/3x_3 & - & 4/3x_1 & + & 2/3x_4 & = & 5 \\ & & & & 41/21x_1 & - & 31/21x_4 & = & -103/21 \\ & & & - & 13/21x_1 & + & 17/21x_4 & = & 47/21 \end{array}$$

For the third elimination step, 41/21 is the largest absolute coefficient of first variable x_1 in the third row and the third column, so no need to do any interchanging. Now eliminate first variable x_1 from the fourth row by subtracting the multiple $m_{43} = \dfrac{-13}{41}$ of row 3 from row 4, gives

$$\begin{array}{rcrcrcrcl} 9x_2 & + & 8x_3 & + & x_1 & + & 4x_4 & = & 15 \\ & & 7/3x_3 & - & 4/3x_1 & + & 2/3x_4 & = & 5 \\ & & & & 41/21x_1 & - & 31/21x_4 & = & -103/21 \\ & & & & & & 14/41x_4 & = & 28/41 \end{array}$$

Since the original set of equations has been transformed to an equivalent upper-triangular form, therefore, using backward substitution to get the solution of the linear system which is

$$x_1 = -1, \quad x_2 = 0, \quad x_3 = 1, \quad x_4 = 2$$

21. Solve the following linear systems using the Gaussian elimination with partial pivoting and without pivoting

(a)

$$\begin{array}{rcrcl} 1.001x_1 & + & 1.5x_2 & = & 0 \\ 2x_1 & + & 3x_2 & = & 1 \end{array}$$

(b)

$$\begin{array}{rcrcl} x_1 & + & 1.001x_2 & = & 2.001 \\ x_1 & + & x_2 & = & 2 \end{array}$$

(c)

$$\begin{array}{rcrcl} 6.122x_1 & + & 1500.5x_2 & = & 1506.622 \\ 2000x_1 & + & 3x_2 & = & 2003 \end{array}$$

Solution: Without Pivoting

(a) The process begins with the augmented matrix form

$$\begin{pmatrix} 1.001 & 1.5 & \vdots & 0 \\ 2 & 3 & \vdots & 1 \end{pmatrix}$$

Since $a_{11} = 1.001 \neq 0$, so we wish to eliminate the elements a_{21} by subtracting from the second row the appropriate multiple of the first row. In this case the multiple is given as

$$m_{21} = \frac{2}{1.001} = 1.9980$$

Hence

$$\begin{pmatrix} 1.001 & 1.5 & \vdots & 0 \\ 0 & 0.003 & \vdots & 1 \end{pmatrix}$$

Obviously, the original set of equations has been transformed to an upper-triangular form. Since all the diagonal elements of the obtaining upper-triangular matrix are nonzero, which means that the coefficient matrix of the given system is nonsingular and therefore, the given system has a unique solution. Now expressing the set in algebraic form yields

$$\begin{pmatrix} 1.001x_1 & + & 1.5x_2 & = & 0 \\ & & 0.003x_2 & = & 1 \end{pmatrix}$$

Now using backward substitution to give

$$\begin{array}{rcllr} 0.003x_2 & = & 1, & \text{gives} & x_2 = 333.6667 \\ 1.001x_1 & = & 0 - 1.5x_2, & \text{gives} & x_1 = -500.0000 \end{array}$$

(c) The process begins with the augmented matrix form

$$\begin{pmatrix} 6.122 & 1500.5 & \vdots & 1506.622 \\ 2000 & 3 & \vdots & 2003 \end{pmatrix}$$

Since $a_{11} = 6.122 \neq 0$, so we wish to eliminate the elements a_{21} by subtracting from the second row the appropriate multiple of the first row. In this case the multiple is given as

$$m_{21} = \frac{2000}{6.122} = 326.69$$

Hence

$$\begin{pmatrix} 6.122 & 1500.5 & \vdots & 1506.622 \\ 0 & -490196.28 & \vdots & -490196.28 \end{pmatrix}$$

The original set of equations has been transformed to an upper-triangular form. Since all the diagonal elements of the obtaining upper-triangular matrix are nonzero, which means that the coefficient matrix of the given system is nonsingular. Expressing the set in algebraic form yields

$$\begin{pmatrix} 6.122x_1 & + & 1500.5x_2 & = & 1506.622 \\ & - & 490196.28x_2 & = & -490196.28 \end{pmatrix}$$

Now using backward substitution to give

$$x_1 = 1, quad x_2 = 1$$

Partial Pivoting

(a) For the first elimination step, since 2 is the largest absolute coefficient of first variable x_1, therefore, the first row and the second row are interchange, giving us

$$\begin{array}{lcccc} 2x_1 & + & 3x_2 & = & 1 \\ 1.001x_1 & + & 1.5x_2 & = & 0 \end{array}$$

Eliminate first variable x_1 from the second row by subtracting the multiple $m_{21} = \frac{1.001}{2}$ of row 1 from row 2, gives

$$\begin{array}{ccccc} 2x_1 & + & 3x_2 & = & 1 \\ & - & 0.0015x_2 & = & -0.5005 \end{array}$$

Since all the diagonal elements of the obtaining upper-triangular matrix are nonzero, which means that the coefficient matrix of the given system is nonsingular. Now using backward substitution to give

$$x_1 = -500.0000 quad \text{and} \quad x_2 = 333.6667$$

(c) For the first elimination step, since 2000 is the largest absolute coefficient of first variable x_1, therefore, the first row and the second row are interchange, giving us

$$\begin{array}{lcccc} 2000x_1 & + & 3x_2 & = & 2003 \\ 6.122x_1 & + & 1500.5x_2 & = & 1506.622 \end{array}$$

Eliminate first variable x_1 from the second row by subtracting the multiple $m_{21} = \frac{6.122}{2000}$ of row 1 from row 2, gives

$$\begin{array}{ccccc} 2000x_1 & + & 3x_2 & = & 2003 \\ & + & 1500.49x_2 & = & 1500.49 \end{array}$$

Since all the diagonal elements of the obtaining upper-triangular matrix are nonzero, which means that the coefficient matrix of the given system is nonsingular. Now using backward substitution to give

$$x_1 = 1.0000 quad \text{and} \quad x_2 = 1.0000$$

22. The elements of the matrix A, the Hilbert matrix, are defined by

$$a_{ij} = 1/(i + j - 1), \quad \text{for} \quad i, j = 1, 2, \ldots, n$$

Find the solution of the system $A\mathbf{x} = \mathbf{b}$ for $n = 3$ and $\mathbf{b} = [\mathbf{1}, \mathbf{2}, \mathbf{3}]^{\mathbf{T}}$, using the Gaussian elimination by without pivoting, partial pivoting, and complete pivoting.

Solution: The Hilbert matrix for $n = 3$ is

$$\begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}$$

Without Pivoting Writing the augmented matrix form

$$\begin{pmatrix} 1 & 1/2 & 1/3 & 1 \\ 1/2 & 1/3 & 1/4 & 2 \\ 1/3 & 1/4 & 1/5 & 3 \end{pmatrix}$$

Since $a_{11} = 1 \neq 0$, so we wish to eliminate the elements a_{21} and a_{31} by subtracting from the second row and the third row the appropriate multiples of the first row. In this case the multiples are given as

$$m_{21} = \frac{1/2}{1} = \frac{1}{2}$$

and

$$m_{31} = \frac{1/3}{1} = \frac{1}{3}$$

Hence

$$\begin{pmatrix} 1 & 1/2 & 1/3 & 1 \\ 0 & 1/12 & 1/12 & 3/2 \\ 0 & 1/12 & 4/45 & 8/3 \end{pmatrix}$$

For the second elimination step, we eliminate second variable x_2 from the third row by subtracting the multiple $m_{32} = \dfrac{1/12}{1/12} = 1$ of row 2 from row 3, gives

$$\begin{pmatrix} 1 & 1/2 & 1/3 & 1 \\ 0 & 1/12 & 1/12 & 3/2 \\ 0 & 0 & 1/180 & 7/6 \end{pmatrix}$$

Since the original set of equations has been transformed to an upper-triangular form, therefore, using the backward substitution give

$$x_1 = 27, \quad x_2 = -192, \quad x_3 = 210.$$

23. Solve the following systems using the Gauss-Jordan method

(a)

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & + & x_3 & = & 1 \\ 2x_1 & + & 4x_2 & + & x_3 & = & 9 \\ 3x_1 & + & 5x_2 & - & 2x_3 & = & 11 \end{array}$$

(b)

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 1 \\ 2x_1 & - & x_2 & + & 3x_3 & = & 4 \\ 3x_1 & + & 2x_2 & - & 2x_3 & = & -2 \end{array}$$

(c)

$$\begin{array}{rcrcrcrcr} 2x_1 & + & 3x_2 & + & 6x_3 & + & x_4 & = & 2 \\ x_1 & + & x_2 & - & 2x_3 & + & 4x_4 & = & 1 \\ 3x_1 & + & 5x_2 & - & 2x_3 & + & 2x_4 & = & 11 \\ 2x_1 & + & 2x_2 & + & 2x_3 & - & 3x_4 & = & 2 \end{array}$$

Solution: (a) Writing the given system in the augmented matrix form

$$\begin{pmatrix} 1 & 4 & 1 & \vdots & 1 \\ 2 & 4 & 1 & \vdots & 9 \\ 3 & 5 & -2 & \vdots & 11 \end{pmatrix}$$

The first elimination step is to eliminate elements $a_{21} = 2$ and $a_{31} = 3$ by subtracting the multiples $m_{21} = 2$ and $m_{31} = 3$ of row 1 from rows 2 and 3 respectively, gives

$$\begin{pmatrix} 1 & 4 & 1 & \vdots & 1 \\ 0 & -4 & -1 & \vdots & 7 \\ 0 & -7 & -5 & \vdots & 8 \end{pmatrix}$$

The second row is now divided by -4 to give

$$\begin{pmatrix} 1 & 4 & 1 & \vdots & 1 \\ 0 & 1 & 1/4 & \vdots & -7/4 \\ 0 & -7 & -5 & \vdots & 8 \end{pmatrix}$$

The second elimination step is to eliminate elements in positions $a_{12}^{(1)} = 4$ and $a_{32} = -7$ by subtracting the multiples $m_{12} = 4$ and $m_{32} = -7$ of row 2 from rows 1 and 3 respectively, gives

$$\begin{pmatrix} 1 & 0 & 0 & \vdots & 8 \\ 0 & 1 & 1/4 & \vdots & -7/4 \\ 0 & 0 & -13/4 & \vdots & -17/4 \end{pmatrix}$$

The third row is now divided by $-13/4$ to give

$$\begin{pmatrix} 1 & 0 & 0 & \vdots & 8 \\ 0 & 1 & 1/4 & \vdots & -7/4 \\ 0 & 0 & 1 & \vdots & 17/13 \end{pmatrix}$$

The third elimination step is to eliminate elements in positions $a_{23}^{(1)} = -1$ by subtracting the multiples $m_{23} = -1/4$ and $m_{13} = 2$ of row 3 from row 2, gives

$$\begin{pmatrix} 1 & 0 & 0 & \vdots & 8 \\ 0 & 1 & 0 & \vdots & -27/13 \\ 0 & 0 & 1 & \vdots & 17/13 \end{pmatrix}$$

Obviously, the original set of equations has been transformed to a diagonal form. Now expressing the set in algebraic form yields

$$\begin{aligned} x_1 &= 8 \\ x_2 &= -27/13 \\ x_3 &= 17/13 \end{aligned}$$

which is the required solution of the given system.
(c) Writing the given system in the augmented matrix form

$$\left(\begin{array}{cccccc} 2 & 3 & 6 & 1 & \vdots & 2 \\ 1 & 1 & -2 & 4 & \vdots & 1 \\ 3 & 5 & -2 & 2 & \vdots & 11 \\ 2 & 2 & 2 & -3 & \vdots & 2 \end{array}\right)$$

Divide the first row by 2 to give

$$\left(\begin{array}{cccccc} 1 & 3/2 & 3 & 1/2 & \vdots & 1 \\ 1 & 1 & -2 & 4 & \vdots & 1 \\ 3 & 5 & -2 & 2 & \vdots & 11 \\ 2 & 2 & 2 & -3 & \vdots & 2 \end{array}\right)$$

The first elimination step is to eliminate elements $a_{21} = 1, a_{31} = 3$, and $a_{41} = 2$ by subtracting the multiples $m_{21} = 1, m_{31} = 3$, and $m_{41} = 2$ of row 1 from rows 2, 3, and 4 respectively, gives

$$\left(\begin{array}{cccccc} 1 & 3/2 & 3 & 1/2 & \vdots & 1 \\ 0 & -1/2 & -5 & 7/2 & \vdots & 0 \\ 0 & 1/2 & -11 & 1/2 & \vdots & 8 \\ 0 & -1 & -4 & -4 & \vdots & 0 \end{array}\right)$$

Multiply the second row by -2 to give

$$\left(\begin{array}{cccccc} 1 & 3/2 & 3 & 1/2 & \vdots & 1 \\ 0 & 1 & 10 & -7 & \vdots & 0 \\ 0 & 1/2 & -11 & 1/2 & \vdots & 8 \\ 0 & -1 & -4 & -4 & \vdots & 0 \end{array}\right)$$

The second elimination step is to eliminate elements in positions $a_{12}^{(1)} = 3/2, a_{32}^{(1)} = 1/2$, and $a_{42}^{(1)} = -1$ by subtracting the multiples $m_{12} = 3/2, m_{32} = 1/2$, and $m_{42} = -1$ of row 2 from rows 1, 3, and 4 respectively, gives

$$\left(\begin{array}{cccccc} 1 & 0 & -12 & 11 & \vdots & 1 \\ 0 & 1 & 10 & -7 & \vdots & 0 \\ 0 & 0 & -16 & 4 & \vdots & 8 \\ 0 & 0 & 6 & -11 & \vdots & 0 \end{array}\right)$$

The third row is now divided by -16 to give

$$\begin{pmatrix} 1 & 0 & -12 & 11 & \vdots & 1 \\ 0 & 1 & 10 & -7 & \vdots & 0 \\ 0 & 0 & 1 & -1/4 & \vdots & -1/2 \\ 0 & 0 & 6 & -11 & \vdots & 0 \end{pmatrix}$$

The third elimination step is to eliminate elements $6, 10$, and 3 in the column 3 by subtracting the multiples $m_{43} = 6, m_{23} = 10$, and $m_{13} = 3$ of row 3 from rows $4, 2$, and 1, gives

$$\begin{pmatrix} 1 & 0 & 0 & 8 & \vdots & -5 \\ 0 & 1 & 0 & -9/2 & \vdots & 5 \\ 0 & 0 & 1 & -1/4 & \vdots & -1/2 \\ 0 & 0 & 0 & -19/2 & \vdots & 3 \end{pmatrix}$$

Multiply the fourth row by $-2/19$ to give

$$\begin{pmatrix} 1 & 0 & 0 & 8 & \vdots & -5 \\ 0 & 1 & 0 & -9/2 & \vdots & 5 \\ 0 & 0 & 1 & -1/4 & \vdots & -1/2 \\ 0 & 0 & 0 & 1 & \vdots & -6/19 \end{pmatrix}$$

The fourth elimination step is to eliminate elements $-1/4, -9/2$, and 8 in the column 4 by subtracting the multiples $m_{34} = -1/4, m_{24} = -9/2$, and $m_{14} = 8$ of row 4 from rows $3, 2$, and 1, gives

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \vdots & -47/19 \\ 0 & 1 & 0 & 0 & \vdots & 68/19 \\ 0 & 0 & 1 & 0 & \vdots & -11/19 \\ 0 & 0 & 0 & 1 & \vdots & -6/19 \end{pmatrix}$$

Obviously, the original set of equations has been transformed to a diagonal form. Now expressing the set in algebraic form yields

$$\begin{aligned} x_1 &= -47/19 \\ x_2 &= 68/19 \\ x_3 &= -11/19 \\ x_4 &= -6/19 \end{aligned}$$

which is the required solution of the given system.

24. The following set of linear equations have common coefficients matrix but different right side terms.

(a)

$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 &= 0 \\ 3x_1 + x_2 - 2x_3 &= -2 \\ x_1 + 3x_2 + 4x_3 &= -3 \end{aligned}$$

(b)

$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & + & 5x_3 & = & 1 \\ 3x_1 & + & x_2 & - & 2x_3 & = & 2 \\ x_1 & + & 3x_2 & + & 4x_3 & = & 4 \end{array}$$

(c)

$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & + & 5x_3 & = & -5 \\ 3x_1 & + & x_2 & - & 2x_3 & = & 6 \\ x_1 & + & 3x_2 & + & 4x_3 & = & -1 \end{array}$$

the coefficients and three sets of right side terms may be combined into an augmented matrix form

$$\begin{pmatrix} 2 & 3 & 5 & \vdots & 0 & 1 & -5 \\ 3 & 1 & -2 & \vdots & -2 & 2 & 6 \\ 1 & 3 & 4 & \vdots & -3 & 4 & -1 \end{pmatrix}.$$

If we apply the Gauss-Jordan method to this augmented matrix form and reduce the first three columns to the unity matrix form, the solution for the three problems are automatically obtained in the fourth, fifth and the sixth columns when elimination is completed. Calculate the solution in this way.

Solution: Consider the augmented matrix form

$$\begin{pmatrix} 2 & 3 & 5 & \vdots & 0 & 1 & -5 \\ 3 & 1 & -2 & \vdots & -2 & 2 & 6 \\ 1 & 3 & 4 & \vdots & -3 & 4 & -1 \end{pmatrix}$$

The third row is now divided by -16 to give

$$\begin{pmatrix} 1 & 3/2 & 5/2 & \vdots & 0 & 1/2 & -5/2 \\ 3 & 1 & -2 & \vdots & -2 & 2 & 6 \\ 1 & 3 & 4 & \vdots & -3 & 4 & -1 \end{pmatrix}$$

The first elimination step is to eliminate elements $a_{21} = 3$ and $a_{31} = 1$ by subtracting the multiples $m_{21} = 3$ and $m_{31} = 1$ of row 1 from rows 2 and 3 respectively, gives

$$\begin{pmatrix} 1 & 3/2 & 5/2 & \vdots & 0 & 1/2 & -5/2 \\ 0 & -7/2 & -19/2 & \vdots & -2 & 1/2 & 27/2 \\ 0 & 3/2 & 3/2 & \vdots & -3 & 7/2 & 3/2 \end{pmatrix}$$

Multiply the second row by $-2/7$ to give

$$\begin{pmatrix} 1 & 3/2 & 5/2 & \vdots & 0 & 1/2 & -5/2 \\ 0 & 1 & 19/7 & \vdots & 4/7 & -1/7 & -27/7 \\ 0 & 3/2 & 3/2 & \vdots & -3 & 7/2 & 3/2 \end{pmatrix}$$

The second elimination step is to eliminate elements in positions $a_{12}^{(1)} = 3/2$ and $a_{32}^{(1)} = 3/2$ by subtracting the multiples $m_{12} = 3/2$ and $m_{32} = 3/2$ of row 2 from rows 1 and 3 respectively, gives

$$\left(\begin{array}{ccccccc} 1 & 0 & -11/7 & \vdots & -6/7 & 5/7 & 23/7 \\ 0 & 1 & 19/7 & \vdots & 4/7 & -1/7 & -27/7 \\ 0 & 0 & -18/7 & \vdots & -27/7 & 26/7 & 51/7 \end{array}\right)$$

The third row is now divided by $-18/7$ to give

$$\left(\begin{array}{ccccccc} 1 & 0 & -11/7 & \vdots & -6/7 & 5/7 & 23/7 \\ 0 & 1 & 19/7 & \vdots & 4/7 & -1/7 & -27/7 \\ 0 & 0 & 1 & \vdots & 3/2 & -13/9 & -17/6 \end{array}\right)$$

The third elimination step is to eliminate elements $19/7$ and $-11/7$ in column 3 by subtracting the multiples $m_{23} = 19/7$ and $m_{13} = -11/7$ of row 3 from rows 2 and 1 respectively, gives

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & \vdots & 3/2 & -14/9 & -7/6 \\ 0 & 1 & 0 & \vdots & -7/2 & 34/9 & 23/6 \\ 0 & 0 & 1 & \vdots & 3/2 & -13/9 & -17/6 \end{array}\right)$$

Obviously, the original set of equations has been transformed to a diagonal form. Thus Now expressing the set in algebraic form for the first linear system yields

$$\begin{aligned} x_1 &= 3/2 \\ x_2 &= -7/2 \\ x_3 &= 3/2 \end{aligned}$$

for second linear system

$$\begin{aligned} x_1 &= -14/9 \\ x_2 &= 23/6 \\ x_3 &= -13/9 \end{aligned}$$

and for third linear system

$$\begin{aligned} x_1 &= -7/6 \\ x_2 &= 34/9 \\ x_3 &= -17/6 \end{aligned}$$

25. Calculate the inverse of each matrix using the Gauss-Jordan method

$$\textbf{(a)} \quad \begin{pmatrix} 3 & -9 & 5 \\ 0 & 5 & 1 \\ -1 & 6 & 3 \end{pmatrix}, \quad \textbf{(b)} \quad \begin{pmatrix} 1 & 4 & 5 \\ 2 & 1 & 2 \\ 8 & 1 & 1 \end{pmatrix}, \quad \textbf{(c)} \quad \begin{pmatrix} 5 & -2 & 0 & 0 \\ -2 & 5 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 5 \end{pmatrix}.$$

Solution: **(a)** Consider the augmented matrix form

$$\left(\begin{array}{ccccccc} 3 & -9 & 5 & \vdots & 1 & 0 & 0 \\ 0 & 5 & 1 & \vdots & 0 & 1 & 0 \\ -1 & 6 & 3 & \vdots & 0 & 0 & 1 \end{array}\right)$$

Using Gauss-Jordan procedure, we get

$$\left(\begin{array}{rrr|rrr} 1 & -3 & 1.6667 & 0.3333 & 0 & 0 \\ 0 & 5 & 1 & 0 & 1 & 0 \\ -1 & 6 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{rrr|rrr} 1 & -3 & 1.6667 & 0.3333 & 0 & 0 \\ 0 & 5 & 1 & 0 & 1 & 0 \\ 0 & 3 & 4.6667 & 0.3333 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{rrr|rrr} 1 & -3 & 1.6667 & 0.3333 & 0 & 0 \\ 0 & 1 & 0.2000 & 0 & 0.2000 & 0 \\ 0 & 3 & 4.6667 & 0.3333 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{rrr|rrr} 1 & 0 & 2.2667 & 0.3333 & 0.6000 & 0 \\ 0 & 1 & 0.2000 & 0 & 0.2000 & 0 \\ 0 & 0 & 4.0667 & 0.3333 & -0.6000 & 1 \end{array}\right)$$

$$\left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 0.1475 & 0.9344 & -0.5574 \\ 0 & 1 & 0.2000 & 0 & 0.2000 & 0 \\ 0 & 0 & 1 & 0.0820 & -0.1475 & 0.2459 \end{array}\right)$$

$$\left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 0.1475 & 0.9344 & -0.5574 \\ 0 & 1 & 0 & -0.0164 & 0.2295 & -0.0492 \\ 0 & 0 & 1 & 0.0820 & -0.1475 & 0.2459 \end{array}\right)$$

Thus

$$A^{-1} = \begin{pmatrix} 0.1475 & 0.9344 & -0.5574 \\ -0.0164 & 0.2295 & -0.0492 \\ 0.0820 & -0.1475 & 0.2459 \end{pmatrix}$$

(b) Consider the augmented matrix form

$$\left(\begin{array}{rrr|rrr} 1 & 4 & 5 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 8 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

Using Gauss-Jordan procedure, we get

$$\left(\begin{array}{rrr|rrr} 1 & 4 & 5 & 1 & 0 & 0 \\ 0 & -7 & -8 & -2 & 1 & 0 \\ 8 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{pmatrix} 1 & 4 & 5 & \vdots & 1 & 0 & 0 \\ 0 & -7 & -8 & \vdots & -2 & 1 & 0 \\ 0 & -31 & -39 & \vdots & -8 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0.4286 & \vdots & -0.1429 & 0.5714 & 0 \\ 0 & 1 & 1.1429 & \vdots & 0.2857 & -0.1429 & 0 \\ 0 & -31 & -39 & \vdots & -8 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0.4286 & \vdots & -0.1429 & 0.5714 & 0 \\ 0 & 1 & 1.1429 & \vdots & 0.2857 & -0.1429 & 0 \\ 0 & 0 & -3.5714 & \vdots & 0.8571 & -4.4286 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & \vdots & -0.0400 & 0.0400 & 0.1200 \\ 0 & 1 & 1.1429 & \vdots & 0.2857 & -0.1429 & 0 \\ 0 & 0 & 1 & \vdots & -0.2400 & 1.2400 & -0.2800 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & \vdots & -0.0400 & 0.0400 & 0.1200 \\ 0 & 1 & 0 & \vdots & 0.5600 & -1.5600 & 0.3200 \\ 0 & 0 & 1 & \vdots & -0.2400 & 1.2400 & -0.2800 \end{pmatrix}$$

Thus

$$A^{-1} = \begin{pmatrix} -0.0400 & 0.0400 & 0.1200 \\ 0.5600 & -1.5600 & 0.3200 \\ -0.2400 & 1.2400 & -0.2800 \end{pmatrix}$$

(c) Consider the augmented matrix form

$$\begin{pmatrix} 5 & -2 & 5 & 0 & \vdots & 1 & 0 & 0 & 0 \\ -2 & 5 & -2 & 0 & \vdots & 0 & 1 & 0 & 0 \\ 0 & -2 & 5 & -2 & \vdots & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 5 & \vdots & 0 & 0 & 0 & 1 \end{pmatrix}$$

Using Gauss-Jordan procedure, we get

$$\begin{pmatrix} 1 & -0.4 & 0 & 0 & \vdots & 0.2000 & 0 & 0 & 0 \\ 0 & 4.2 & -2 & 0 & \vdots & 0.4000 & 1 & 0 & 0 \\ 0 & -2 & 5 & -2 & \vdots & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 5 & \vdots & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -0.4 & 0 & 0 & \vdots & 0.2000 & 0 & 0 & 0 \\ 0 & 4.2 & -2 & 0 & \vdots & 0.4000 & 1 & 0 & 0 \\ 0 & -2 & 5 & -2 & \vdots & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 5 & \vdots & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -0.1905 & 0 & \vdots & 0.2381 & 0.0952 & 0 & 0 \\ 0 & 1 & -0.4762 & 0 & \vdots & 0.0952 & 0.2381 & 0 & 0 \\ 0 & -2 & 5 & -2 & \vdots & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 5 & \vdots & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -0.1905 & 0 & \vdots & 0.2381 & 0.0952 & 0 & 0 \\ 0 & 1 & -0.4762 & 0 & \vdots & 0.0952 & 0.2381 & 0 & 0 \\ 0 & 0 & 4.0476 & -2 & \vdots & 0.1905 & 0.4762 & 1 & 0 \\ 0 & 0 & -2 & 5 & \vdots & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & -0.0941 & \vdots & 0.2471 & 0.1176 & 0.0471 & 0 \\ 0 & 1 & -0.4762 & 0 & \vdots & 0.0952 & 0.2381 & 0 & 0 \\ 0 & 0 & 1 & -0.4941 & \vdots & 0.0471 & 0.1176 & 0.2471 & 0 \\ 0 & 0 & -2 & 5 & \vdots & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & -0.0941 & \vdots & 0.2471 & 0.1176 & 0.0471 & 0 \\ 0 & 1 & 0 & -0.2353 & \vdots & 0.1176 & 0.2941 & 0.1176 & 0 \\ 0 & 0 & 1 & -0.4941 & \vdots & 0.0471 & 0.1176 & 0.2471 & 0 \\ 0 & 0 & -2 & 5 & \vdots & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & -0.0941 & \vdots & 0.2471 & 0.1176 & 0.0471 & 0 \\ 0 & 1 & 0 & -0.2353 & \vdots & 0.1176 & 0.2941 & 0.1176 & 0 \\ 0 & 0 & 1 & -0.4941 & \vdots & 0.0471 & 0.1176 & 0.2471 & 0 \\ 0 & 0 & 0 & 4.0118 & \vdots & 0.0941 & 0.2353 & 0.4941 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \vdots & 0.2493 & 0.1232 & 0.0587 & 0.0235 \\ 0 & 1 & 0 & -0.2353 & \vdots & 0.1176 & 0.2941 & 0.1176 & 0 \\ 0 & 0 & 1 & -0.4941 & \vdots & 0.0471 & 0.1176 & 0.2471 & 0 \\ 0 & 0 & 0 & 1 & \vdots & 0.0235 & 0.0587 & 0.1232 & 0.2493 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \vdots & 0.2493 & 0.1232 & 0.0587 & 0.0235 \\ 0 & 1 & 0 & 0 & \vdots & 0.1232 & 0.3079 & 0.1466 & 0.0587 \\ 0 & 0 & 1 & -0.4941 & \vdots & 0.0471 & 0.1176 & 0.2471 & 0 \\ 0 & 0 & 0 & 1 & \vdots & 0.0235 & 0.0587 & 0.1232 & 0.2493 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \vdots & 0.2493 & 0.1232 & 0.0587 & 0.0235 \\ 0 & 1 & 0 & 0 & \vdots & 0.1232 & 0.3079 & 0.1466 & 0.0587 \\ 0 & 0 & 1 & 0 & \vdots & 0.0587 & 0.1466 & 0.3079 & 0.1232 \\ 0 & 0 & 0 & 1 & \vdots & 0.0235 & 0.0587 & 0.1232 & 0.2493 \end{pmatrix}$$

Thus

$$A^{-1} = \begin{pmatrix} 0.2493 & 0.1232 & 0.0587 & 0.0235 \\ 0.1232 & 0.3079 & 0.1466 & 0.0587 \\ 0.0587 & 0.1466 & 0.3079 & 0.1232 \\ 0.0235 & 0.0587 & 0.1232 & 0.2493- \end{pmatrix}$$

26. Find the inverse of the Hilbert matrix of size 4×4 using the Gauss-Jordan method. Then solve the linear system $A\mathbf{x} = [1, 2, 3, 4]^T$.

Solution: Since the Hilbert matrix of size 4×4 is

$$A = \begin{pmatrix} 1 & 0.5 & 0.3333 & 0.25 \\ 0.5 & 0.3333 & 0.25 & 0.2 \\ 0.3333 & 0.25 & 0.2 & 0.1667 \\ 0.25 & 0.2 & 0.1667 & 0.1429 \end{pmatrix}$$

Consider the augmented matrix form

$$\begin{pmatrix} 1 & 0.5 & 0.33 & 0.25 & \vdots & 1 & 0 & 0 & 0 \\ 0.5 & 0.33 & 0.25 & 0.2 & & & & & \\ & & \vdots & 0 & 1 & 0 & 0 & & \\ 0.33 & 0.25 & 0.2 & 0.17 & & & & & \\ & & \vdots & 0 & 0 & 1 & 0 & & \\ 0.25 & 0.2 & 0.17 & 0.14 & \vdots & 0 & 0 & 0 & 1 \end{pmatrix}$$

Using Gauss-Jordan procedure, we get

$$\begin{pmatrix} 1 & 0.50 & 0.33 & 0.25 & \vdots & 1 & 0 & 0 & 0 \\ 0 & 0.08 & 0.08 & 0.08 & \vdots & -0.50 & 1 & 0 & 0 \\ 0.33 & 0.25 & 0.20 & 0.17 & \vdots & 0 & 0 & 1 & 0 \\ 0.25 & 0.20 & 0.17 & 0.14 & \vdots & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0.50 & 0.33 & 0.25 & \vdots & 1 & 0 & 0 & 0 \\ 0 & 0.08 & 0.08 & 0.08 & \vdots & -0.50 & 1 & 0 & 0 \\ 0 & 0.08 & 0.09 & 0.08 & \vdots & -0.33 & 0 & 1 & 0 \\ 0.25 & 0.20 & 0.17 & 0.14 & \vdots & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0.50 & 0.33 & 0.25 & \vdots & 1 & 0 & 0 & 0 \\ 0 & 0.08 & 0.08 & 0.08 & \vdots & -0.50 & 1 & 0 & 0 \\ 0 & 0.08 & 0.09 & 0.08 & \vdots & -0.33 & 0 & 1 & 0 \\ 0 & 0.08 & 0.08 & 0.08 & \vdots & -0.25 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -0.17 & -0.20 & \vdots & 4 & -6 & 0 & 0 \\ 0 & 1 & 1 & 0.90 & \vdots & -6 & 12 & 0 & 0 \\ 0 & 0.08 & 0.09 & 0.08 & \vdots & -0.33 & 0 & 1 & 0 \\ 0 & 0.08 & 0.08 & 0.08 & \vdots & -0.25 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -0.17 & -0.20 & \vdots & 4 & -6 & 0 & 0 \\ 0 & 1 & 1 & 0.90 & \vdots & -6 & 12 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.01 & \vdots & 0.17 & -1 & 1 & 0 \\ 0 & 0.08 & 0.08 & 0.08 & \vdots & -0.25 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -0.17 & -0.20 & \vdots & 4.00 & -6 & 0 & 0 \\ 0 & 1 & 1 & 0.90 & \vdots & -6.00 & 12 & 0 & 0 \\ 0 & 0 & 0.01 & 0.01 & \vdots & 0.17 & -1 & 1 & 0 \\ 0 & 0 & 0.01 & 0.01 & \vdots & 0.20 & -0.90 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0.05 & \vdots & 9 & -36 & 30 & 0 \\ 0 & 1 & 1 & 0.90 & \vdots & -6 & 12 & 0 & 0 \\ 0 & 0 & 1 & 1.50 & \vdots & 30 & -180 & 180 & 0 \\ 0 & 0 & 0.01 & 0.01 & \vdots & 0.20 & -0.90 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0.05 & \vdots & 9 & -36 & 30 & 0 \\ 0 & 1 & 0 & -0.60 & \vdots & -36 & 192 & -180 & 0 \\ 0 & 0 & 1 & 1.50 & \vdots & 30 & -180 & 180 & 0 \\ 0 & 0 & 0.01 & 0.01 & \vdots & 0.20 & -0.90 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0.05 & \vdots & 9 & -36 & 30 & 0 \\ 0 & 1 & 0 & -0.60 & \vdots & -36 & 192 & -180 & 0 \\ 0 & 0 & 1 & 1.50 & \vdots & 30 & -180 & 180 & 0 \\ 0 & 0 & 0 & 0.00 & \vdots & -0.05 & 0.60 & -1.50 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \vdots & 16 & -120 & 240 & -140 \\ 0 & 1 & 0 & -0.60 & \vdots & -36 & 192 & -180 & 0 \\ 0 & 0 & 1 & 1.50 & \vdots & 30 & -180 & 180 & 0 \\ 0 & 0 & 0 & 1.00 & \vdots & -140 & 1680 & -4200 & 2800 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \vdots & 16 & -120 & 240 & -140 \\ 0 & 1 & 0 & 0 & \vdots & -120 & 1200 & -2700 & 1680 \\ 0 & 0 & 1 & 1.50 & \vdots & 30 & -180 & 180 & 0 \\ 0 & 0 & 0 & 1 & \vdots & -140 & 1680 & -4200 & 2800 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \vdots & 16 & -120 & 240 & -140 \\ 0 & 1 & 0 & 0 & \vdots & -120 & 1200 & -2700 & 1680 \\ 0 & 0 & 1 & 0 & \vdots & 240 & -2700 & 6480 & -4200 \\ 0 & 0 & 0 & 1 & \vdots & -140 & 1680 & -4200 & 2800 \end{pmatrix}$$

Thus

$$A^{-1} = \begin{pmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{pmatrix}$$

Then solving $\mathbf{x} = A^{-1}\mathbf{b}$, we get

$$\mathbf{x} = [-64, 900, -2520, 1820]^T$$

27. Find the LU decomposition of each matrix A using the Doolittle's method, and then solve the systems.

(a)

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -3 & 4 & -1 \\ 1 & -1 & 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$$

(b)

$$A = \begin{pmatrix} 7 & 6 & 5 \\ 5 & 4 & 3 \\ 3 & 7 & 6 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

(c)

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 2 & 1 \\ 3 & 3 & 4 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix}.$$

(d)

$$A = \begin{pmatrix} 2 & 4 & -6 \\ 1 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -4 \\ 10 \\ 5 \end{pmatrix}.$$

(e)

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -1 & 1 \\ 2 & -2 & -1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}.$$

(f)

$$A = \begin{pmatrix} 1 & 5 & 3 \\ 2 & 4 & 6 \\ 1 & 3 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 11 \\ 5 \end{pmatrix}$$

Solution: (a) The factorization of A is

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 0.5 & -0.2 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 0 & 2.5 & 0.5 \\ 0 & 0 & 0.6 \end{pmatrix}$$

Now solving the system $L\mathbf{y} = \mathbf{b}$ using forward substitution, we get

$$\mathbf{y} = [4.000, 11.000, 6.2000]^T$$

Again solving the system $U\mathbf{x} = \mathbf{y}$ using backward substitution, we get

$$\mathbf{x} = [-2.00, 2.333, 10.333]^T$$

(b) The factorization of A is

$$LU = \begin{pmatrix} 1.0000 & 0 & 0 \\ 0.7143 & 1.0000 & 0 \\ 0.4286 & -15.5000 & 1.0000 \end{pmatrix} \begin{pmatrix} 7.0000 & 6.0000 & 5.0000 \\ 0 & -0.2857 & -0.5714 \\ 0 & 0 & -5.0000 \end{pmatrix}$$

Now solving the system $L\mathbf{y} = \mathbf{b}$ using forward substitution, we get

$$\mathbf{y} = [2.000, -0.4286, -5.5000]^T$$

Again solving the system $U\mathbf{x} = \mathbf{y}$ using backward substitution, we get

$$\mathbf{x} = [0.100, -0.700, 1.1000]^T$$

(c) The factorization of A is

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1.5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now solving the system $L\mathbf{y} = \mathbf{b}$ using forward substitution, we get

$$\mathbf{y} = [0, -4, 1]^T$$

Again solving the system $U\mathbf{x} = \mathbf{y}$ using backward substitution, we get

$$\mathbf{x} = [3, -4, 1]^T$$

(d) The factorization of A is

$$LU = \begin{pmatrix} 1.0 & 0 & 0 \\ 0.5000 & 1.0000 & 0 \\ 0.5000 & 0.3333 & 1.0 \end{pmatrix} \begin{pmatrix} 2 & 4 & -6 \\ 0 & 3 & 6 \\ 0 & 0 & 3 \end{pmatrix}$$

Now solving the system $L\mathbf{y} = \mathbf{b}$ using forward substitution, we get

$$\mathbf{y} = [-4, 12, 3]^T$$

Again solving the system $U\mathbf{x} = \mathbf{y}$ using backward substitution, we get

$$\mathbf{x} = [-3, 2, 1]^T$$

(e) The factorization of A is

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

Now solving the system $L\mathbf{y} = \mathbf{b}$ using forward substitution, we get

$$\mathbf{y} = [2, 0, -1]^T$$

Again solving the system $U\mathbf{x} = \mathbf{y}$ using backward substitution, we get

$$\mathbf{x} = [1, -1, 1]^T$$

(f) The factorization of A is

$$LU = \begin{pmatrix} 1.0 & 0 & 0 \\ 2.0 & 1.0 & 0 \\ 1.0 & 0.3333 & 1.0 \end{pmatrix} \begin{pmatrix} 1 & 5 & 3 \\ 0 & -6 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Now solving the system $L\mathbf{y} = \mathbf{b}$ using forward substitution, we get

$$\mathbf{y} = [4, 3, 0]^T$$

Again solving the system $U\mathbf{x} = \mathbf{y}$ using backward substitution, we get

$$\mathbf{x} = [6.50, -0.50, 0]^T$$

28. Solve the Problem 27 by the LU decomposition using the Crout's method.

Solution: (a) The factorization of A is

$$LU = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 2.5 & 0 \\ 1 & -0.5 & 0.6 \end{pmatrix} \begin{pmatrix} 1 & -0.5 & 0.5 \\ 0 & 1 & 0.2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{y} = [2, 4.4, 10.3333]^T, \quad \mathbf{x} = [-2, 2.3333, 10.3333]^T$$

(c) The factorization of A is

$$LU = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{y} = [0, -4, 1]^T, \quad \mathbf{x} = [3, -4, 1]^T$$

(e) The factorization of A is

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{y} = [2, 0, 1]^T, \quad \mathbf{x} = [1, -1, 1]^T$$

29. Solve the following system by the LU decomposition using the Cholesky method.

(a)

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

(b)

$$A = \begin{pmatrix} 10 & 2 & 1 \\ 2 & 10 & 3 \\ 1 & 3 & 10 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 7 \\ -4 \\ 3 \end{pmatrix}.$$

(c)

$$A = \begin{pmatrix} 5 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 1 & 6 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(d)

$$A = \begin{pmatrix} 3 & 4 & -6 & 0 \\ 4 & 5 & 3 & 1 \\ -6 & 3 & 3 & 1 \\ 0 & 1 & 1 & 3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 4 \\ 5 \\ 2 \\ 3 \end{pmatrix}.$$

Solution: (a) The lower-triangular matrix L is

$$L = \begin{pmatrix} 1.414 & 0 & 0 \\ -0.707 & 1.223 & 0 \\ 0.707 & -0.408 & 1.155 \end{pmatrix}$$

Now solving the system $L\mathbf{y} = \mathbf{b}$ using forward substitution, we get

$$\mathbf{y} = [0.7071, 2.0412, 2.8868]^T$$

Again solving the system $L^T\mathbf{x} = \mathbf{y}$ using backward substitution, we get

$$\mathbf{x} = [0.5, 2.5, 2.5]^T$$

(b) The lower-triangular matrix L is

$$L = \begin{pmatrix} 3.1623 & 0 & 0 \\ 0.6325 & 3.0984 & 0 \\ 0.3162 & 0.9037 & 3.0139 \end{pmatrix}$$

Now solving the system $L\mathbf{y} = \mathbf{b}$ using forward substitution, we get

$$\mathbf{y} = [2.2136, -1.7428, 1.2857]^T$$

Again solving the system $L^T\mathbf{x} = \mathbf{y}$ using backward substitution, we get

$$\mathbf{x} = [0.7947, -0.6869, 0.4266]^T$$

(c) The lower-triangular matrix L is

$$L = \begin{pmatrix} 2.236 & 0 & 0 \\ 0.894 & 1.789 & 0 \\ 1.342 & -0.112 & 2.046 \end{pmatrix}$$

Now solving the system $L\mathbf{y} = \mathbf{b}$ using forward substitution, we get

$$\mathbf{y} = [0.4472, 0.3354, 0.2138]^T$$

Again solving the system $L^T\mathbf{x} = \mathbf{y}$ using backward substitution, we get

$$\mathbf{x} = [0.0597, 0.1940, 0.1045]^T$$

(d) The lower-triangular matrix L is

$$L = \begin{pmatrix} 5.5678 & 0 & 0 & 0 \\ 0.7184 & 2.1175 & 0 & 0 \\ -1.0776 & 1.7824 & 2.9431 & 0 \\ 0 & 0.4723 & 0.0538 & 1.6656 \end{pmatrix}$$

Now solving the system $L\mathbf{y} = \mathbf{b}$ using forward substitution, we get

$$\mathbf{y} = [0.7184, 2.1175, -0.3398, 1.2118]^T$$

Again solving the system $L^T\mathbf{x} = \mathbf{y}$ using backward substitution, we get

$$\mathbf{x} = [-0.0180, 0.9461, -0.1287, 0.7275]^T$$

30. Solve the following tridiagonal systems using the LU decomposition.

(a)

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

(b)

$$A = \begin{pmatrix} 2 & 3 & 0 & 0 \\ 3 & 2 & 3 & 0 \\ 0 & 3 & 2 & 3 \\ 0 & 0 & 3 & 2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 6 \\ 7 \\ 5 \\ 3 \end{pmatrix}.$$

(c)

$$A = \begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 3 \\ -1 \\ 2 \\ 5 \end{pmatrix}.$$

Solution: (a) The factorization of A is

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ -0.3333 & 1 & 0 \\ 0 & -0.375 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 0 \\ 0 & 2.6667 & -1 \\ 0 & 0 & 2.6250 \end{pmatrix}$$

$$\mathbf{y} = [1, 2.3333, 3.8750]^T, \quad \mathbf{x} = [0.8095, 1.4286, 1.4762]^T$$

(b) The factorization of A is

$$LU = \begin{pmatrix} 2 & 3 & 0 & 0 \\ 0 & -2.5 & 3 & 0 \\ 0 & 0 & 5.6 & 3 \\ 0 & 0 & 0 & 0.3929 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1.5 & 1 & 0 & 0 \\ 0 & -1.2 & 1 & 0 \\ 0 & 0 & 0.5357 & 1 \end{pmatrix}$$

$$\mathbf{y} = [6.0, -2.0, 2.60, 1.6071]^T, \quad \mathbf{x} = [4.9091, -1.2727, -1.7273, 4.0909]^T$$

(c) The factorization of A is

$$LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -0.25 & 1 & 0 & 0 \\ 0 & -0.2667 & 1 & 0 \\ 0 & 0 & -0.2679 & 1 \end{pmatrix} \begin{pmatrix} 4 & -1 & 0 & 0 \\ 0 & 3.75 & -1 & 0 \\ 0 & 0 & 3.7333 & -1 \\ 0 & 0 & 0 & 3.7321 \end{pmatrix}$$

$$\mathbf{y} = [3, -0.25, 1.9333, 5.5179]^T, \quad \mathbf{x} = [0.7943, 0.1770, 0.9139, 1.4785]^T$$

31. Find $\|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$, and $\|\mathbf{x}\|_\infty$ for the following vectors:

(**a**) $[2, -1, -6, 3]^T$, (**b**) $[\sin k, \cos k, 3^k]^T$ for a fixed integer k, (**c**) $[3, -4, 0, 3/2]^T$.

Solution: (a)

$$\|\mathbf{x}\|_1 = 12, \quad \|\mathbf{x}\|_2 = 7.0711, \quad \|\mathbf{x}\|_\infty = 6$$

(c)

$$\|\mathbf{x}\|_1 = 8.5, \quad \|\mathbf{x}\|_2 = 5.2202, \quad \|\mathbf{x}\|_\infty = 4$$

32. Find $\|.\|_1$, $\|.\|_\infty$ and $\|.\|_e$ for the following matrices.

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ 1 & -5 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 1 & 6 \\ -3 & 6 & 4 \\ 5 & 0 & 9 \end{pmatrix}, \quad C = \begin{pmatrix} 17 & 46 & 7 \\ 20 & 49 & 8 \\ 23 & 52 & 9 \end{pmatrix}.$$

Solution:

$$\|A\|_1 = 6, \quad \|A\|_\infty = 7, \quad \|A\|_e = 7.6158$$

$$\|B\|_1 = 19, \quad \|B\|_\infty = 14, \quad \|B\|_e = 14.8324$$

$$\|C\|_1 = 147, \quad \|C\|_\infty = 84, \quad \|C\|_e = 92.9139$$

33. Consider the following matrices

$$A = \begin{pmatrix} -11 & 7 & -8 \\ 5 & 9 & 6 \\ 6 & 3 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 2 & 7 \\ -12 & 10 & 8 \\ 3 & -15 & 14 \end{pmatrix}, \quad C = \begin{pmatrix} 5 & -6 & 4 \\ -7 & 8 & 5 \\ 3 & -9 & 12 \end{pmatrix}.$$

Find $\|.\|_1$ and $\|.\|_\infty$ for (**a**) A^3, (**b**) $A^2 + B^2$, (**c**) BC and (**d**) $C^2 + A^2$

Solution: (a)

$$\|A^3\|_1 = 2996, \qquad \|A^3\|_\infty = 23$$

(b)

$$\|A^2 + B^2\|_1 = 468, \qquad \|A^2 + B^2\|_\infty = 524$$

(c)

$$\|BC\|_1 = 427, \qquad \|BC\|_\infty = 531$$

(d)

$$\|C^2 + A^2\|_1 = 334, \qquad \|C^2 + A^2\|_\infty = 328$$

34. The $n \times n$ Hilbert matrix $H^{(n)}$ defined by

$$H_{ij}^{(n)} = \frac{1}{i+j-1}, \qquad 1 \le i,\ j \le n$$

Find the l_∞-norm of the 10×10 Hilbert matrix.

Solution: The 10×10 Hilbert matrix is

$$H = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 & 1/6 & 1/7 & 1/8 & 1/9 & 1/10 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1/6 & 1/7 & 1/8 & 1/9 & 1/10 & 1/11 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1/7 & 1/8 & 1/9 & 1/10 & 1/11 & 1/12 \\ 1/4 & 1/5 & 1/6 & 1/7 & 1/8 & 1/9 & 1/10 & 1/11 & 1/12 & 1/13 \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 & 1/10 & 1/11 & 1/12 & 1/13 & 1/14 \\ 1/6 & 1/7 & 1/8 & 1/9 & 1/10 & 1/11 & 1/12 & 1/13 & 1/14 & 1/15 \\ 1/7 & 1/8 & 1/9 & 1/10 & 1/11 & 1/12 & 1/13 & 1/14 & 1/15 & 1/16 \\ 1/8 & 1/9 & 1/10 & 1/11 & 1/12 & 1/13 & 1/14 & 1/15 & 1/16 & 1/17 \\ 1/9 & 1/10 & 1/11 & 1/12 & 1/13 & 1/14 & 1/15 & 1/16 & 1/17 & 1/18 \\ 1/10 & 1/11 & 1/12 & 1/13 & 1/14 & 1/15 & 1/16 & 1/17 & 1/18 & 1/19 \end{pmatrix}$$

and its l_∞-norm is

$$\|H\|_\infty = 2.9290$$

35. Solve the following linear systems using the Jacobi method, start with initial approximation $\mathbf{x^{(0)}} = \mathbf{0}$ and iterate until $\|\mathbf{x^{(k+1)}} - \mathbf{x^{(k)}}\|_\infty \le 10^{-5}$ for each system.

(a)

$$\begin{aligned} 4x_1 - x_2 + x_3 &= 7 \\ 4x_1 - 8x_2 + x_3 &= -21 \\ -2x_1 + x_2 + 5x_3 &= 15 \end{aligned}$$

(b)

$$\begin{aligned} 3x_1 + x_2 + x_3 &= 5 \\ 2x_1 + 6x_2 + x_3 &= 9 \\ x_1 + x_2 + 4x_3 &= 6 \end{aligned}$$

(c)

$$\begin{aligned} 4x_1 + 2x_2 + x_3 &= 1 \\ x_1 + 7x_2 + x_3 &= 4 \\ x_1 + x_2 + 20x_3 &= 7 \end{aligned}$$

(d)

$$\begin{aligned} 5x_1 + 2x_2 - x_3 &= 6 \\ x_1 + 6x_2 - 3x_3 &= 4 \\ 2x_1 + x_2 + 4x_3 &= 7 \end{aligned}$$

(e)

$$\begin{aligned} 6x_1 - x_2 + 3x_3 &= -2 \\ 3x_2 + x_3 &= 1 \\ -2x_1 + x_2 + 5x_3 &= 5 \end{aligned}$$

(f)

$$\begin{aligned} 4x_1 + x_2 &= -1 \\ 2x_1 + 5x_2 + x_3 &= 0 \\ -x_1 + 2x_2 + 4x_3 &= 3 \end{aligned}$$

(g)

$$\begin{array}{rcrcrcr} 5x_1 & - & x_2 & + & x_3 & = & 1 \\ & & 3x_2 & - & x_3 & = & -1 \\ x_1 & + & 2x_2 & + & 4x_3 & = & 2 \end{array}$$

(h)

$$\begin{array}{rcrcrcr} 9x_1 & + & x_2 & + & x_3 & = & 10 \\ 2x_1 & + & 10x_2 & + & 3x_3 & = & 19 \\ 3x_1 & + & 4x_2 & + & 11x_3 & = & 0 \end{array}$$

Solution: (a) The approximations of the given system are as follows Thus

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0
1	1.75000	2.62500	3.00000
2	1.65625	3.87500	3.17500
3	1.92500	3.85000	2.88750
4	1.99063	3.94844	3.00000
5	1.98711	3.99531	3.00656
6	1.99719	3.99438	2.99578
7	1.99965	3.99807	3.00000
8	1.99952	3.99982	3.00025
9	1.99990	3.99979	2.99984
10	1.99999	3.99993	3.00000
11	1.99998	3.99999	3.00001
12	2.00000	3.99999	2.99999
13	2.00000	4.00000	3.00000

$$\mathbf{x}^{(\mathbf{13})} = [2, 4, 3]^T.$$

(c) The approximations of the given system are as follows Thus

$$\mathbf{x}^{(\mathbf{12})} = [-0.10156, 0.53906, 0.32813]^T.$$

(e) The approximations of the given system are as follows Thus

$$\mathbf{x}^{(\mathbf{14})} = [-0.67308, 0.09616, 0.71154]^T$$

(g) The approximations of the given system are as follows Thus

$$\mathbf{x}^{(\mathbf{12})} = [0.05882, -0.14706, 0.55883]^T.$$

36. Consider the following system of equations

$$\begin{array}{rcrcrcr} 4x_1 & + & 2x_2 & + & x_3 & = & 1 \\ x_1 & + & 7x_2 & + & x_3 & = & 4 \\ x_1 & + & x_2 & + & 20x_3 & = & 7 \end{array}$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0
1	0.25000	0.57143	0.35000
2	-0.12321	0.48571	0.30893
3	-0.07009	0.54490	0.33188
4	-0.10542	0.53403	0.32626
5	-0.09858	0.53988	0.32857
6	-0.10208	0.53857	0.32794
7	-0.10127	0.53916	0.32818
8	-0.10163	0.53901	0.32811
9	-0.10153	0.53907	0.32813
10	-0.10157	0.53906	0.32812
11	-0.10156	0.53906	0.32813
12	-0.10156	0.53906	0.32813

(a) Show that the Jacobi method converges by using $\|T_J\|_\infty < 1$.
(b) Compute 2nd approximation $\mathbf{x}^{(2)}$, starting with $\mathbf{x}^{(0)} = [0,0,0]^T$.
(c) Compute an error estimate $\|\mathbf{x} - \mathbf{x}^{(2)}\|_\infty$ for your approximation.

Solution: **(a)** The Jacobi iterative matrix is

$$T_J = -D^{-1}(L+U) = \begin{pmatrix} 0 & -1/2 & -1/4 \\ -1/7 & 0 & -1/7 \\ -1/20 & -1/20 & 0 \end{pmatrix}$$

and its l_∞-norm is

$$\|T_J\| = 3/4 = 0.75$$

(b) Using the Jacobi method, we have

$$x^{(1)} = [0.25, 0.5714, 0.35]^T, \; x^{(2)} = [-0.1232, 0.4857, 0.3089]^T$$

(c) To compute the error bound for the approximation, we have

$$\|x - x^{(2)}\| \leq \frac{(0.75)^2}{1-0.75}(0.5714) \leq 1.2857$$

37. Solve the Problem 35 using the Gauss-Seidel method.

Solution: Using the Gauss-Seidel method for each system, we get

(a) $\mathbf{x}^{(8)} = [\mathbf{2}, \mathbf{4}, \mathbf{3}]^{\mathbf{T}}$

(c) $\mathbf{x}^{(6)} = [-0.102, 0.539, 0.328]^T$

(e) $\mathbf{x}^{(8)} = [-0.673, 0.096, 0.711]^T$

(g) $\mathbf{x}^{(7)} = [0.588, -0.147, 0.559]^T$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0
1	-0.33333	0.33333	1.00000
2	-0.77778	0.00000	0.80000
3	-0.73333	0.06667	0.68889
4	-0.66667	0.10370	0.69333
5	-0.66272	0.10222	0.71259
6	-0.67259	0.09580	0.71447
7	-0.67460	0.09518	0.71180
8	-0.67337	0.09607	0.71112
9	-0.67289	0.09629	0.71144
10	-0.67300	0.09619	0.71159
11	-0.67310	0.09614	0.71156
12	-0.67309	0.09615	0.71153
13	-0.67307	0.09616	0.71153
14	-0.67308	0.09616	0.71154

38. Consider the following system of equations

$$\begin{array}{rcrcrcr} 4x_1 & + & 2x_2 & + & x_3 & = & 11 \\ -x_1 & + & 2x_2 & & & = & 3 \\ 2x_1 & + & x_2 & + & 4x_3 & = & 16 \end{array}$$

(a) Show that the Gauss-Seidel method converges by using $\|T_G\|_\infty < 1$.
(b) Compute the second approximation $\mathbf{x^{(2)}}$, starting with $\mathbf{x^{(0)}} = [1,1,1]^T$.
(c) Compute an error estimate $\|\mathbf{x} - \mathbf{x^{(2)}}\|_\infty$ for your approximation.

Solution: **(a)** The Gauss-Seidel iterative matrix is

$$T_G = -(L+D)^{-1}U = \begin{pmatrix} 0 & -1/2 & -1/4 \\ 0 & -1/4 & -1/8 \\ 0 & 5/16 & 5/32 \end{pmatrix}$$

and its l_∞-norm is

$$\|T_G\| = 3/4 = 0.75$$

(b) Using the Gauss-Seidel method, we have

$$x^{(1)} = [2.75, 2.875, 1.906]^T,\ x^{(2)} = [0.836, 1.918, 3.103]^T$$

(c) To compute the error bound for the approximation, we have

$$\|x - x^{(2)}\| \leq \frac{(0.75)^2}{1-0.75}(2.875) \leq 6.469$$

39. Which of the following matrix is convergent ?

$$(\mathbf{a}) \quad \begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix}, \quad (\mathbf{b}) \quad \begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 3 & 2 & -2 \end{pmatrix}$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0
1	0.20000	-0.33333	0.50000
2	0.03333	-0.16667	0.61667
3	0.04333	-0.12778	0.57500
4	0.05944	-0.14167	0.55306
5	0.06106	-0.14898	0.55597
6	0.05901	-0.14801	0.55923
7	0.05855	-0.14692	0.55925
8	0.05877	-0.14692	0.55882
9	0.05885	-0.14706	0.55877
10	0.05884	-0.14708	0.55882
11	0.05882	-0.14706	0.55883
12	0.05882	-0.14706	0.55883

Solution: **(a)** Solving the following equation to find the eigenvalues of the given matrix

$$|A - \lambda\mathbf{I}| = -\lambda^3 + 5\lambda^2 - 2\lambda - 8 = -(\lambda - 2)(\lambda - 4)(\lambda + 1) = 0$$

gives

$$\rho(A) = 4 > 1$$

which shows that A is divergent.

(b) Solving the following equation to find the eigenvalues of the given matrix

$$|A - \lambda\mathbf{I}| = (1 - \lambda)(3 - \lambda)(-2 - \lambda) = 0$$

gives

$$\rho(A) = \max_i |\lambda_i| = 3 > 1$$

which shows that A is divergent.

40. Find the eigenvalues and their associated eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

Also, show that $\|A\|_2 > \rho(A)$.

Solution: Solving the following equation to find the eigenvalues of the given matrix

$$|A - \lambda\mathbf{I}| = -\lambda^3 + 7\lambda^2 - 14\lambda + 8 = -(\lambda - 2)(\lambda - 1)(\lambda - 4) = 0$$

gives

$$\lambda_1 = 4, \quad \lambda_2 = 2, \quad \lambda_3 = 1$$

the eigenvalues of the matrix. The corresponding eigenvectors are

$$x_1 = [-1\ 1\ 1/2]^T, \quad x_2 = [1\ 0\ 0]^T, \quad x_3 = [-1\ 1\ 1]^T$$

Also

$$A^T A = \begin{pmatrix} 4 & -4 & 6 \\ -4 & 14 & -14 \\ 6 & -14 & 17 \end{pmatrix}$$

and the characteristic equation of $A^T A$ is

$$|A^T A - \lambda \mathbf{I}| = -\lambda^3 + 35\lambda^2 - 114\lambda + 64 = 0$$

which gives eigenvalues of $A^T A$

$$\lambda_1 = 31.4386, \quad \lambda_2 = 2.8461, \quad \lambda_3 = 0.7153$$

Thus

$$\|A\|_2 = \sqrt{\rho(A^T A)} = \sqrt{31.4386} = 5.6070 > 4 = \rho(A)$$

41. Solve the Problem 35 using the SOR method by taking $\omega = 1.02$ for each system.

Solution: Using SOR method by taking $\omega = 1.02$ within accuracy 10^{-6} for the following systems, we have:

(a) $\mathbf{x}^{(8)} = [2, 4, 3]^T$

(c) $\mathbf{x}^{(6)} = [-0.102, 0.539, 0.328]^T$

(e) $\mathbf{x}^{(8)} = [-0.673, 0.096, 0.711]^T$

(g) $\mathbf{x}^{(8)} = [0.588, -0.147, 0.559]^T$

42. Use the parameter $\omega = 1.543$ to solve the linear system by the SOR method within accuracy 10^{-6} in the l_∞-norm, starting $\mathbf{x}^{(0)} = \mathbf{0}$

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & & & & & = & -3 \\ -2x_1 & + & 5x_2 & - & x_3 & & & = & 5 \\ & - & x_2 & + & 2x_3 & - & 0.5x_4 & = & 2 \\ & & & - & 0.5x_3 & + & 1.25x_4 & = & 3.5 \end{array}$$

Solution: Using SOR method within accuracy 10^{-6}, we need 33 iterations and the 33th iteration gives

$$x^{(33)} = [0.999999, 2.000000, 3.000000, 4.000000]^T$$

43. Find the optimal choice for ω and use it to solve the linear system by the SOR method within accuracy 10^{-4} in the l_∞-norm, starting $\mathbf{x}^{(0)} = \mathbf{0}$. Also, find how many iterations need by using the Jacobi method and the Gauss-Seidel method.

Since

$$T_J = -D^{-1}(L+U) = \begin{pmatrix} 0 & -1/2 & 0 \\ -1/2 & 0 & 1/2 \\ 0 & 1/4 & 0 \end{pmatrix}$$

we have

$$T_J - \lambda\mathbf{I} = \begin{pmatrix} -\lambda & -1/2 & 0 \\ -1/2 & -\lambda & 1/2 \\ 0 & 1/4 & -\lambda \end{pmatrix}$$

so

$$det(T_J - \lambda\mathbf{I}) = -\lambda(\lambda - 3/8)$$

Thus

$$\rho(T_J) = \sqrt{3/8} = 0.6124$$

and

$$\omega = \frac{2}{1+\sqrt{1-[\rho(T_J)]^2}} \approx 1.117$$

Using SOR method, we have Thus using SOR method we need 8 iterations

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0
1	1.3963	0.0579	1.6917
2	1.2005	1.1053	1.7862
3	0.6385	1.3494	1.8433
4	0.5679	1.3922	1.8486
5	0.5523	1.3989	1.8498
6	0.5504	1.3998	1.8500
7	0.5500	1.4000	1.8500
8	0.5500	1.4000	1.8500

$$\mathbf{x}^{(\mathbf{8})} = [0.5500, 1.4000, 1.8500]^T$$

and using Gauss-Seidel method we need 12 iterations

$$\mathbf{x}^{(\mathbf{12})} = [0.5500, 1.4000, 1.8500]^T$$

and using Jacobi method we need 20 iterations

$$\mathbf{x}^{(\mathbf{20})} = [0.5500, 1.3900, 1.8500]^T$$

44. Consider the following system

$$\begin{array}{rcrcrcrcl} 4x_1 & - & 2x_2 & - & x_3 & & & = & 1 \\ -x_1 & + & 4x_2 & & & - & x_4 & = & 2 \\ -x_1 & & & + & 4x_3 & - & x_4 & = & 0 \\ & - & x_2 & - & x_3 & + & 4x_4 & = & 1 \end{array}$$

Using $\mathbf{x^{(0)}} = \mathbf{0}$, how many iterations are required to approximate the solution to within five decimal places using: **(a)** Jacobi method, **(b)** Gauss-Seidel method, **(c)** SOR method (take $\omega = 1.1$).

Solution: **(a)** Using Jacobi method within five decimal places, we need 19 iterations and get

$$\mathbf{x^{(19)}} = [0.73862, 0.81817, 0.31817, 0.53408]^T$$

(b) Using Gauss-Seidel method for same accuracy, we need 12 iterations and get

$$\mathbf{x^{(12)}} = [0.73864, 0.81818, 0.31818, 0.53409]^T$$

(c) Using SOR method with $\omega = 1.1$, we need only 7 iterations and get

$$\mathbf{x^{(7)}} = [0.73864, 0.81818, 0.31818, 0.53409]^T$$

45. Find the spectral radius of the Jacobi, the Gauss-Seidel and the SOR ($\omega = 1.25962$) iteration matrices when

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Solution: The spectral radius of the three iteration matrices are as follows

$$\rho(T_J) = 0.809017, \qquad \rho(T_G) = 0.654508, \qquad \rho(T_\omega) = 0.25962$$

46. Perform only two steps of the conjugate gradient method for the following linear system, taking $\mathbf{x^{(0)}} = \mathbf{0}$

(a)

$$\begin{array}{rcrcrcr} 3x_1 & - & x_2 & + & x_3 & = & 7 \\ -x_1 & + & 3x_2 & + & 2x_3 & = & 1 \\ x_1 & + & 2x_2 & + & 5x_3 & = & 5 \end{array}$$

(b)

$$\begin{array}{rcrcrcr} 3x_1 & - & 2x_2 & + & x_3 & = & 5 \\ -2x_1 & + & 6x_2 & - & x_3 & = & 9 \\ x_1 & - & x_2 & + & 4x_3 & = & 6 \end{array}$$

(c)

$$\begin{array}{rcrcrcr} 4x_1 & - & 2x_2 & + & x_3 & = & 1 \\ -2x_1 & + & 7x_2 & + & x_3 & = & 4 \\ x_1 & + & x_2 & + & 20x_3 & = & 1 \end{array}$$

(d)

$$\begin{array}{rcrcrcr} 5x_1 & - & 3x_2 & - & x_3 & = & 6 \\ -3x_1 & + & 6x_2 & - & 3x_3 & = & 4 \\ -x_1 & - & 3x_2 & + & 4x_3 & = & 7 \end{array}$$

Solution: **(a)** The first two steps are as follows

$$\mathbf{x^{(1)}} = [1.4957, 0.2137, 1.0683]^T, \quad \text{and} \quad \mathbf{x^{(2)}} = [2.2261, 0.1602, 0.5408]^T$$

(b) The first two steps are as follows

$$\mathbf{x^{(1)}} = [1.4885, 2.6793, 1.7862]^T, \quad \text{and} \quad \mathbf{x^{(2)}} = [2.4796, 2.4959, 1.9858]^T$$

(c) The first two steps are as follows

$$\mathbf{x^{(1)}} = [0.1385, 0.5539, 0.1385]^T, \quad \text{and} \quad \mathbf{x^{(2)}} = [0.3010, 0.7339, -0.0365]^T$$

(d) The first two steps are as follows

$$\mathbf{x^{(1)}} = [7.9737, 5.3158, 9.3026]^T, \quad \text{and} \quad \mathbf{x^{(2)}} = [16.8605, 19.4381, 0.7106]^T$$

47. Perform only two steps of the conjugate gradient method for the following linear system, taking $\mathbf{x^{(0)}} = \mathbf{0}$

(a)

$$\begin{array}{rcrcrcr} 6x_1 & + & 2x_2 & + & x_3 & = & 1 \\ 2x_1 & + & 3x_2 & - & x_3 & = & 0 \\ x_1 & - & x_2 & + & 2x_3 & = & -2 \end{array}$$

(b)

$$\begin{array}{rcrcrcr} 5x_1 & - & 2x_2 & + & x_3 & = & 3 \\ -2x_1 & + & 4x_2 & - & x_3 & = & 2 \\ x_1 & - & x_2 & + & 3x_3 & = & 1 \end{array}$$

(c)

$$\begin{array}{rcrcrcrcr} 6x_1 & - & x_2 & - & x_3 & + & 5x_4 & = & 1 \\ -x_1 & + & 7x_2 & + & x_3 & - & x_4 & = & 2 \\ -x_1 & + & x_2 & + & 3x_3 & - & 3x_4 & = & 0 \\ 5x_1 & - & x_2 & - & 3x_3 & + & 6x_4 & = & -1 \end{array}$$

(d)

$$\begin{array}{rcrcrcrcr} 3x_1 & - & 2x_2 & - & x_3 & + & 3x_4 & = & 1 \\ -2x_1 & + & 7x_2 & + & x_3 & - & x_4 & = & 0 \\ -x_1 & + & x_2 & + & 3x_3 & - & 3x_4 & = & 0 \\ 3x_1 & - & x_2 & - & 3x_3 & + & 6x_4 & = & 0 \end{array}$$

Solution: (a) The first two steps are as follows

$$\mathbf{x^{(1)}} = [0.5000, 0.0000, -1.0000]^T \quad \text{and} \quad \mathbf{x^{(2)}} = [0.5181, -0.7236, -1.9430]^T$$

(b) The first two steps are as follows

$$\mathbf{x^{(1)}} = [1.0000, 0.6667, 0.3333]^T \quad \text{and} \quad \mathbf{x^{(2)}} = [0.9701, 1.0659, 0.3234]^T$$

(c) The first two steps are as follows

$$\mathbf{x^{(1)}} = [0.2000, 0.4000, 0, -0.2000]^T \quad \text{and} \quad \mathbf{x^{(2)}} = [0.9065, 0.4673, -0.3365, -0.5701]^T$$

(d) The first two steps are as follows

$$\mathbf{x^{(1)}} = [0.3333, 0, 0, 0]^T \quad \text{and} \quad \mathbf{x^{(2)}} = [0.7391, 0.1739, 0.0870, -0.2609]^T$$

48. Compute the condition numbers of each of the matrix in the Problem 47 relative to $\|.\|_\infty$.

Solution: (a) Since the inverse of the given matrix is

$$A^{-1} = \begin{pmatrix} 0.3333 & -0.3333 & -0.3333 \\ -0.3333 & 0.7333 & 0.5333 \\ -0.3333 & 0.5333 & 0.9333 \end{pmatrix}$$

The condition number of the matrix is

$$K(A) = \|A\|_\infty \|A^{-1}\|_\infty = (9.0)(1.8000) = 16.2000$$

(b) Since the inverse of the given matrix is

$$A^{-1} = \begin{pmatrix} 0.2558 & 0.1163 & -0.0465 \\ 0.1163 & 0.3256 & 0.0698 \\ -0.0465 & 0.0698 & 0.3721 \end{pmatrix}$$

The condition number of the matrix is

$$K(A) = \|A\|_\infty \|A^{-1}\|_\infty = (8.0)(0.5116) = 4.0930$$

(c) Since the inverse of the given matrix is

$$A^{-1} = \begin{pmatrix} 3.7500 & 0.3750 & -3.8750 & -5.0000 \\ 0.3750 & 0.1875 & -0.4375 & -0.5000 \\ -3.8750 & -0.4375 & 4.6875 & 5.5000 \\ -5.0000 & -0.5000 & 5.5000 & 7.0000 \end{pmatrix}$$

The condition number of the matrix is

$$K(A) = \|A\|_\infty \|A^{-1}\|_\infty = (15.0)(18.0000) = 270.0000$$

(d) Since the inverse of the given matrix is

$$A^{-1} = \begin{pmatrix} 1.0909 & 0.2727 & -0.4545 & -0.7273 \\ 0.2727 & 0.2182 & -0.1636 & -0.1818 \\ -0.4545 & -0.1636 & 0.8727 & 0.6364 \\ -0.7273 & -0.1818 & 0.6364 & 0.8182 \end{pmatrix}$$

The condition number of the matrix is

$$K(A) = \|A\|_\infty \|A^{-1}\|_\infty = (13.0)(2.5455) = 33.0909$$

49. Compute the condition numbers of the following matrices relative to $\|.\|_\infty$

$$(\mathbf{a}) \quad \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{5} & \frac{1}{3} \\ \frac{1}{5} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}, \quad (\mathbf{b}) \quad \begin{pmatrix} 0.03 & 0.01 & -0.02 \\ 0.15 & 0.51 & -0.11 \\ 1.11 & 2.22 & 3.33 \end{pmatrix}, \quad (\mathbf{c}) \quad \begin{pmatrix} 1.11 & 1.98 & 2.01 \\ 1.01 & 1.05 & 2.05 \\ 0.85 & 0.45 & 1.25 \end{pmatrix}$$

Solution: **(a)** Since the inverse of the given matrix is

$$\begin{pmatrix} 300/1891 & 4950/1891 & -2069/1144 \\ 4950/1891 & -1351/747 & 300/1891 \\ -2069/1144 & 300/1891 & 4950/1891 \end{pmatrix}$$

The condition number of the matrix is

$$K(A) = (31/30)(2971/648) = 289/61 = 4.7377$$

(b) Since the inverse of the given matrix is

$$\begin{pmatrix} 34.2466 & -1.3699 & 0.1604 \\ -10.9589 & 2.1526 & 0.0053 \\ -4.1096 & -0.9785 & 0.2433 \end{pmatrix}$$

The condition number of the matrix is

$$K(A) = (6.6600)(35.7769) = 238.2740$$

(c) Since the inverse of the given matrix is

$$\begin{pmatrix} 0.7755 & -3.1228 & 3.8744 \\ 0.9544 & -0.6383 & -0.4880 \\ -0.8709 & 2.3533 & -1.6589 \end{pmatrix}$$

The condition number of the matrix is

$$K(A) = (5.1000)(7.7726) = 39.6403$$

50. The following linear systems have $\mathbf{x}$ as the exact solution and $\mathbf{x}^*$ is an approximate solution. Compute $\|\mathbf{x} - \mathbf{x}^*\|_\infty$ and $K(A)\dfrac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty}$, where $\mathbf{r} = \mathbf{b} - A\mathbf{x}^*$ is the residual vector. **(a)**

$$\begin{array}{rcrcl} 0.89x_1 & + & 0.53x_2 & = & 0.36 \\ 0.47x_1 & + & 0.28x_2 & = & 0.19 \end{array}$$

$$\begin{array}{rcl} \mathbf{x} & = & [1, -1]^T \\ \mathbf{x}^* & = & [0.702, -0.500]^T \end{array}$$

(b)

$$\begin{array}{rcrcl} 0.986x_1 & + & 0.579x_2 & = & 0.235 \\ 0.409x_1 & + & 0.237x_2 & = & 0.107 \end{array}$$

$$\begin{array}{rcl} \mathbf{x} & = & [2, -3]^T \\ \mathbf{x}^* & = & [2.110, -3.170]^T \end{array}$$

(c)

$$\begin{array}{rcrcl} 1.003x_1 & + & 58.090x_2 & = & 68.12 \\ 5.550x_1 & + & 321.8x_2 & = & 377.3 \end{array}$$

$$\begin{array}{rcl} \mathbf{x} & = & [10, 1]^T \\ \mathbf{x}^* & = & [-10, 1]^T \end{array}$$

Solution: (a) Since

$$\mathbf{r} = \mathbf{b} - A\mathbf{x}^* = [0.00022, 0.00006]^T$$

and

$$K(A) = 19312.0$$

therefore,

$$\|\mathbf{x} - \mathbf{x}^*\|_\infty = 0.5$$

and

$$K(A)\frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty} = (19312)(0.00022)/(0.36) = 11.8018$$

(b) Since

$$\mathbf{r} = \mathbf{b} - A\mathbf{x}^* = [-0.1003, -0.0047]^T$$

and

$$K(A) = 697.7229$$

therefore,

$$\|\mathbf{x} - \mathbf{x}^*\|_\infty = 0.1700$$

and

$$K(A)\frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty} = (697.7229)(0.1003)/(0.235) = 297.7941$$

(c) Since

$$\mathbf{r} = \mathbf{b} - A\mathbf{x}^* = [20.0600, 111.0000]^T$$

and

$$K(A) = 339866.0604$$

therefore,

$$\|\mathbf{x} - \mathbf{x}^*\|_\infty = 20.0$$

and

$$K(A)\frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty} = (339866.0604)(111.00)/(377.3) = 99987.0997$$

51. Discuss the ill-conditioning (stability) of the linear system

$$\begin{array}{rcrcl} 1.01x_1 & + & 0.99x_2 & = & 2 \\ 0.99x_1 & + & 1.01x_2 & = & 2 \end{array}$$

If $\mathbf{x}^* = [2, 0]^T$ be an approximate solution of the system, then find the residual vector $\mathbf{r}$ and estimate the relative error.

Solution: Since the inverse of the given matrix A is

$$A^{-1} = \begin{pmatrix} 25.2500 & -24.7500 \\ -24.7500 & 25.2500 \end{pmatrix}$$

and the l_∞-norms of A and A^{-1} are

$$\|A\| = 2 \quad \text{and} \quad \|A^{-1}\| = 50$$

Thus

$$K(A) = (2)(50) = 100 >> 1 \quad (Ill - Conditioned)$$

The residual vector is

$$\mathbf{r} = \mathbf{b} - A\mathbf{x}^* = [-0.02, 0.02]^T$$

and the relative error is

$$Rel.\ Error \leq \frac{(100)(0.02)}{2} = 1$$

53. Show that if B is singular, then

$$\frac{1}{K(A)} \leq \frac{\|A - B\|}{\|A\|}$$

Solution: Since

$$\|\mathbf{x}\| = \|A^{-1}A\mathbf{x}\| \leq \|A^{-1}\|\|A\mathbf{x}\|$$

which gives

$$\|A\mathbf{x}\| \geq \frac{\|\mathbf{x}\|}{\|A^{-1}\|}$$

With $\mathbf{x} \neq \mathbf{0}$ such that $\|\mathbf{x}\| = 1$ and $B\mathbf{x} = \mathbf{0}$, gives

$$\|A\mathbf{x} - B\mathbf{x}\| = \|(A - B)\mathbf{x}\| = \|A\mathbf{x}\| \geq \frac{\|\mathbf{x}\|}{\|A^{-1}\|}$$

Thus

$$\frac{\|(A - B)\mathbf{x}\|}{\|A\|} \geq \frac{1}{\|A\|\|A^{-1}\|}$$

or

$$\frac{1}{K(A)} \leq \frac{\|(A - B)\mathbf{x}\|}{\|A\|}$$

Since

$$\|(A - B)\mathbf{x}\| \leq \|A - B\|\|\mathbf{x}\| = \|A - B\|$$

therefore,

$$\frac{1}{K(A)} \leq \frac{\|A - B\|}{\|A\|}$$

53. Consider the following matrices

$$A = \begin{pmatrix} 0.06 & 0.01 & 0.02 \\ 0.13 & 0.05 & 0.11 \\ 1.01 & 2.02 & 3.03 \end{pmatrix}, \qquad B = \begin{pmatrix} -0.04 & 0.01 & -0.05 \\ 0.11 & 0.02 & -0.03 \\ 0.89 & 1.94 & 2.99 \end{pmatrix}.$$

Using the Problem 52, compute the approximation of the condition number of the matrix A relative to $\|.\|_\infty$.

Solution: Since $\|A\| = 6.06$ and the norm of the following matrix

$$A - B = \begin{pmatrix} 0.1000 & 0 & 0.0700 \\ 0.0200 & 0.0300 & 0.1400 \\ 0.1200 & 0.0800 & 0.0400 \end{pmatrix}$$

is 0.24. The from the Problem 52, we get

$$K(A) \geq \frac{6.06}{0.24} = 25.25$$

54. Let A and B are nonsingular $n \times n$ matrices. Show that

$$K(AB) \leq K(A)K(B)$$

Solution: By using definition of the condition number of matrix, we have

$$K(AB) = \|AB\|\|(AB)^{-1}\|$$

Since $(AB)^{-1} = B^{-1}A^{-1}$ and the norm can be written as

$$\|(AB)^{-1}\| = \|B^{-1}A^{-1}\| \leq \|A^{-1}\|\|B^{-1}\|$$

Also

$$\|AB\| \leq \|A\|\|B\|$$

Thus using these two inequalities, we get the required result.

55. The exact solution of the following linear system

$$\begin{array}{ccccc} x_1 & + & x_2 & = & 1 \\ x_1 & + & 1.01x_2 & = & 2 \end{array}$$

is $\mathbf{x} = [-99, 100]^T$. Change the coefficient matrix slightly to

$$\delta A = \begin{pmatrix} 1 & 1 \\ 1 & 0.99 \end{pmatrix}$$

and consider the linear system

$$\begin{array}{ccccc} x_1 & + & x_2 & = & 1 \\ x_1 & + & 0.99x_2 & = & 2 \end{array}$$

Compute the change solution $\delta\mathbf{x}$ of this system. Is the matrix A ill-conditioned ?

Solution: Solving the linear system using simple Gaussian elimination method, we get

$$\delta\mathbf{x} = [101, 100]^T$$

and the condition number of the matrix is

$$K(A) = 404.01$$

which shows that A is ill-Conditioned.

56. Using the Problem 15, compute the relative error and the relative residual.

Solution: (a)

$$Rel.Error = 0, \qquad Rel.Residual = 0$$

(c)

$$Rel.Error = 0, \qquad Rel.Residual = 0$$

57. The exact solution of the following linear system

$$\begin{array}{rcrcl} x_1 & + & 3x_2 & = & 4 \\ 1.0001x_1 & + & 3x_2 & = & 4.0001 \end{array}$$

is $\mathbf{x} = [1, 1]^T$. Change the right-hand vector $\mathbf{b}$ slightly to $\delta\mathbf{b} = [4.0001, 4.0003]^T$ and consider the linear system

$$\begin{array}{rcrcl} x_1 & + & 3x_2 & = & 4.0001 \\ 1.0001x_1 & + & 3x_2 & = & 4.0003 \end{array}$$

Find the change solution $\delta\mathbf{x}$ of this system. Is the matrix A ill-conditioned ?

Solution: Solving the linear system using simple Gaussian elimination method, we get

$$\delta\mathbf{x} = [1.9976, 0.6675]^T$$

and the condition number of the matrix is

$$K(A) = 80002$$

which shows that A is ill-Conditioned.

58. If $\|A\| < 1$, then show that the matrix $(\mathbf{I} - A)$ is nonsingular and

$$\|(\mathbf{I} - A)^{-1}\| \leq \frac{1}{1 - \|A\|}$$

Solution: Since $\|A\| < 1$ and using matrix geometric series, we have

$$\frac{1}{\mathbf{I} - A^{-1}} = \mathbf{I} + A + A^2 + A^3 + \cdots$$

Taking norm and applying the triangle inequality, we get

$$\|(\mathbf{I} - A)^{-1}\| \leq 1 + \|A\| + \|A\|^2 + \|A\|^3 + \cdots = \frac{1}{(1 - \|A\|)}$$

59. The exact solution of the following linear system

$$\begin{array}{rcrcl} x_1 & + & x_2 & = & 3 \\ x_1 & + & 1.0005x_2 & = & 3.0010 \end{array}$$

is $\mathbf{x} = [1, 2]^T$. Change the coefficient matrix and the right-hand vector $\mathbf{b}$ slightly to

$$\delta A = \begin{pmatrix} 1 & 1 \\ 1 & 1.001 \end{pmatrix} \quad \text{and} \quad \delta\mathbf{b} = \begin{pmatrix} 2.99 \\ 3.01 \end{pmatrix}$$

and consider the linear system

$$\begin{array}{rcrcl} x_1 & + & x_2 & = & 2.99 \\ x_1 & + & 1.001x_2 & = & 3.01 \end{array}$$

Find the change solution $\delta\mathbf{x}$ of this system. Is the matrix A ill-conditioned ?

Solution: Solving the linear system using simple Gaussian elimination method, we get

$$\delta\mathbf{x} = [-17, 20]^T$$

and the condition number of the matrix is

$$K(A) = 8004$$

which shows that A is ill-Conditioned.

60. Find the condition number of the following matrix

$$A_n = \begin{pmatrix} 1 & 1 \\ 1 & 1 - \dfrac{1}{n} \end{pmatrix}$$

Solve the linear system $A_4\mathbf{x} = [2, 2]^T$ and compute the relative residual.

Solution: Since the inverse of the given matrix for $n > 0$, we have

$$A_n^{-1} = \begin{pmatrix} -n+1 & n \\ n & -n \end{pmatrix}$$

Then condition number of the given matrix is

$$K(A) = \|A_n\| \|A_n^{-1}\| = 4n, \qquad n > 0$$

Solving the linear system $A_4\mathbf{x} = [2, 2]^T$, we have

$$\mathbf{x}^* = [2, 0]^T$$

To compute the relative residual, we first compute the residual vector as

$$\mathbf{r} = \mathbf{b} - A\mathbf{x}^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus the relative residual is

$$\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} = 0$$

61. The following linear system has exact solution $\mathbf{x} = [10, 1]^T$, find the approximate solution of the system

$$\begin{array}{rcrcr} 0.03x_1 & + & 58.9x_2 & = & 59.2 \\ 5.31x_1 & - & 6.10x_2 & = & 47.0 \end{array}$$

by using the simple Gaussian elimination and then use the residual correction method (one iteration only) to improve the approximate solution.

Solution: Using simple Gaussian elimination method, we get

$$\mathbf{x^{(1)}} = [10, 1]^T$$

Now using residual corrector method, we got

$$\mathbf{x^{(2)}} = \mathbf{x^{(1)}} + \mathbf{y^{(1)}} = [10, 1]^T + [0, 0]^T = [10, 1]^T$$

Chapter 4 Approximating Functions

1. Find the second-degree Taylor polynomial for the function $f(x) = (e^x + 2)^{1/2}$ expanded about $x_0 = 0$. Then use it to approximate $f(1)$.

Solution: The second degree Taylor polynomial expanded about x_0 is

$$p_2(x) = f(x_0) + \frac{(x - x_0)}{1!} f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0)$$

Given $f(x) = (e^x + 2)^{1/2}$, then $f(x_0) = f(0) = \sqrt{3}$. Now calculating the derivatives require for the desired polynomial $p_2(x)$, we get

$$\begin{array}{lcll} f'(x) & = & e^x/2(e^x + 2)^{1/2}, & f'(x_0) = f'(0) = 0.2887 \\ f''(x) & = & -(e^x)^2/4(e^x + 2)^{3/2} + e^x/2(e^x + 2)^{1/2}, & f''(x_0) = f''(0) = 0.2406 \end{array}$$

using these values, we have

$$p_2(x) = 1.7321 + 0.2887x + 0.1203x^2$$

and the approximation of $f(x)$ at $x = 1$ is

$$f(1) \approx p_2(1) = 1.7321 + 0.2887(1) + 0.1203(1)^2 = 2.1411$$

2. Use the Taylor polynomial of third-degree about $x_0 = 1$ to approximate the function $f(x) = x^2 + 4e^x$ at $x = 1$ and $x = 2$. Find the error bounds for these approximations and compare your results.

Solution: The third-degree Taylor polynomial expanded about x_0 is

$$p_3(x) = f(x_0) + \frac{(x - x_0)}{1!} f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \frac{(x - x_0)^3}{3!} f'''(x_0)$$

Since $f(x) = x^2 + 4e^x$, and the derivatives require for the desired polynomial $p_3(x)$ can be obtained as

$$\begin{array}{lcll} f'(x) & = & 2x + 4e^x, & f'(x_0) = f'(1) = 2 + 4e^1 \\ f''(x) & = & 2 + 4e^x, & f''(x_0) = f''(1) = 2 + 4e^1 \\ f'''(x) & = & 4e^x, & f'''(x_0) = f'''(1) = 4e^1 \end{array}$$

using all these values, we have

$$p_3(x) = 1 + 4e^1 + (x - 1)(2 + 4e^1) + (x - 1)^2(1 + 2e^1) + (x - 1)^3(2/3e^1)$$

Now take $x = 1$, we get

$$f(1) \approx p_3(1) = 1 + 4e^1 = 11.8731$$

The error bound for the approximation of $f(1)$ is

$$|R_4(1)| = 0$$

Note that the exact value of $f(1)$ is 11.8731.

Similarly, at $x = 2$, we have

$$f(2) \approx p_3(2) = 1 + 4e^1 + (2 + 4e^1) + (1 + 2e^1) + 2/3e^1 = 32.9950$$

To compute error bound for the approximation of $f(2)$, we have

$$|R_4(x)| \leq \frac{1}{24}|x - x_0|^4 M$$

where

$$M = \max_{1.0 \leq x \leq 2.0} |4e^x| = 29.5562$$

at $x = 2.0$ and $|f^{(4)}(\eta(x))| \leq M$. Hence

$$|R_4(x)| \leq \frac{1}{24}(29.5562) = 1.2315$$

Note that the exact value of $f(2)$ is 33.5562.

3. Find the third-degree Taylor polynomial for the function $f(x) = \ln(x^2 + 2x + 2)$ expanded about $x_0 = 1$. Use this polynomial to approximate $f(1.1)$. Compute the actual error of the problem.

Solution: The third degree Taylor polynomial expanded about $x_0 = 1$ is

$$p_3(x) = f(1) + \frac{(x-1)}{1!}f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1)$$

Since $f(x) = \ln(x^2 + 2x + 2)$, and the derivatives require for the desired polynomial $p_3(x)$ can be obtained as

$$f'(x) = \frac{2x+2}{x^2+2x+2}, \qquad f'(x_0) = f'(1) = 0.8000$$

$$f''(x) = \frac{2}{x^2+2x+2} - \frac{(2x+2)^2}{(x^2+2x+2)^2}, \qquad f''(1) = -0.2400$$

$$f'''(x) = -\frac{6(2x+2)}{(x^2+2x+2)^2} + \frac{2(2x+2)^3}{(x^2+2x+2)^3}, \qquad f'''(1) = 0.0640$$

using all these values, we have

$$f(1.1) \approx p_3(1.1) = 1.6094 + 0.8(1.1-1) - 0.12(1.1-1)^2 + 0.0107(1.1-1)^3 = 1.6882$$

The actual error is

$$Error = f(1.1) - p_3(1.1) = 1.6882 - 1.6882 = 0.0000$$

4. Find the fourth-degree Taylor polynomial for the function $f(x) = (x^3 + 1)^{-1}$ expanded about $x_0 = 0$, and use it to approximate $f(0.2)$. Also, find a bound for the error in this approximation.

Solution: The fourth degree Taylor polynomial expanded about $x_0 = 0$ is

$$p_4(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{6}f'''(0) + \frac{x^4}{24}f^{(4)}(0)$$

Since $f(x) = 1/(x^3 + 1)$, and the derivatives require for the desired polynomial $p_4(x)$ can be obtained as

$$f'(x) = \frac{-3x^2}{(x^3+1)^2}, \qquad f'(x_0) = f'(0) = 0.0000$$

$$f''(x) = \frac{18x^4}{(x^3+1)^3} - \frac{6x}{(x^3+1)^2}, \qquad f''(0) = 0.0000$$

$$f'''(x) = -\frac{162x^6}{(x^3+1)^4} + \frac{108x^3}{(x^3+1)^3} - \frac{6}{(x^3+1)^2} \qquad f'''(0) = -6.0$$

$$f^{(4)}(x) = \frac{1944x^8}{(x^3+1)^5} - \frac{1944x^5}{(x^3+1)^4} - \frac{360x^2}{(x^3+1)^3}, \qquad f^{(4)}(0) = 0$$

using all these values, we have

$$f(0.2) \approx p_4(0.2) = 1 - (0.2 - 0)^3 = 0.9920$$

To compute error bound for the approximation of $f(0.2)$, we have

$$|R_5(x)| \leq \frac{1}{120}|x - x_0|^5 M$$

where

$$M = \max_{0.0 \leq x \leq 0.2} |f^{(5)}(x)| = 120.9884$$

at $x = 0.2$ and

$$f^{(5)}(x) = -\frac{29160x^{10}}{(x^3+1)^6} + \frac{38880x^7}{(x^3+1)^5} - \frac{12960x^4}{(x^3+1)^4} + \frac{720x}{(x^3+1)^3}$$

Hence for $|f^{(5)}(\eta(x))| \leq M$, we have

$$|R_5(x)| \leq \frac{1}{120}(0.2)^5(120.9884) = 0.0003$$

5. Use the Lagrange interpolation formula based on the points $x_0 = 0, x_1 = 1, x_2 = 2.5$ to find the equation of the quadratic polynomial to approximate $f(x) = \dfrac{2}{x+2}$ at $x = 2.3$.

Solution: The values of $f(x) = \dfrac{2}{x+2}$ at the given points are

$$f(0) = 1, \quad f(1) = 2/3, \quad f(2.5) = 2/4.5$$

and using these values in the quadratic Lagrange interpolating formula, we have the approximation of $f(2.3)$ as follows

$$\begin{aligned} p_2(2.3) &= L_0(2.3)(1) + L_1(2.3)(2/3) + L_2(2.3)(2/4.5) \\ &= \frac{(2.3-1)(2.3-2.5)}{(0-1)(0-2.5)}(1) \\ &+ \frac{(2.3-0)(2.3-2.5)}{(1-0)(1-2.5)}(2/3) \\ &+ \frac{(2.3-0)(2.3-1)}{(2.5-0)(2.5-1)}(2/4.5) \\ &= 0.4548 \end{aligned}$$

6. Let $f(x) = \cos(x\pi/4)$, where x is in radian. Use the quadratic Lagrange interpolation formula based on the points $x_0 = 0, x_1 = 1, x_2 = 2$, and $x_3 = 4$ to find the polynomial $p_2(x)$ to approximate the function $f(x)$ at $x = 0.5$ and $x = 3.5$.

Solution: The values of $f(x) = \cos(x\pi/4)$ at the given points are

$$f(0) = 1, \quad f(1) = 0.9999, \quad f(2) = 0.9996$$

and using these values in the quadratic Lagrange interpolating formula, we have the approximation of $f(0.5)$ as follows

$$\begin{aligned} p_2(0.5) &= L_0(0.5)(1) + L_1(0.5)(0.9999) + L_2(0.5)(0.9996) \\ &= \frac{(0.5-1)(0.5-2)}{(0-1)(0-2)}(1) \\ &+ \frac{(0.5-0)(0.5-2)}{(1-0)(1-2)}(0.9999) \\ &+ \frac{(0.5-0)(0.5-1)}{(2-0)(2-1)}(0.9996) \\ &= 1.0000 \end{aligned}$$

Similarly, using the data points $(1, 0.9999), (2, 0.9996), (4, 0.9985)$ in the quadratic

Lagrange interpolating formula, we have the approximation of $f(3.5)$ as follows

$$\begin{aligned} p_2(3.5) &= L_0(3.5)(0.9999) + L_1(3.5)(0.9996) + L_2(3.5)(0.9985) \\ &= \frac{(3.5-2)(3.5-4)}{(1-2)(1-4)}(0.9999) \\ &+ \frac{(3.5-1)(3.5-4)}{(2-1)(2-4)}(0.9996) \\ &+ \frac{(3.5-1)(3.5-2)}{(4-1)(4-2)}(0.9985) \\ &= 0.9988 \end{aligned}$$

7. Let $f(x) = (x+2)\ln(x+2)$. Use the quadratic Lagrange interpolation formula based on the points $x_0 = 0, x_1 = 1, x_2 = 2$, and $x_3 = 3$ to approximate $f(0.5)$ and $f(2.8)$. Also, compute the error bounds for your approximations.

Solution: The values of $f(x) = (x+2)\ln(x+2)$ at the given points are

$$f(0) = 1.3863, \quad f(1) = 3.2958, \quad f(2) = 5.5452$$

and using these values in the quadratic Lagrange interpolating formula, we have the approximation of $f(0.5)$ as follows

$$\begin{aligned} p_2(0.5) &= L_0(0.5)(1.3863) + L_1(0.5)(3.2958) + L_2(0.5)(5.5452) \\ &= \frac{(0.5-1)(0.5-2)}{(0-1)(0-2)}(1.3863) \\ &+ \frac{(0.5-0)(0.5-2)}{(1-0)(1-2)}(3.2958) \\ &+ \frac{(0.5-0)(0.5-1)}{(2-0)(2-1)}(5.5452) \\ &= 2.2986 \end{aligned}$$

Similarly, using the data points $(1, 3.2958), (2, 5.5452), (3, 8.0472)$ in the quadratic

Lagrange interpolating formula, we have the approximation of $f(3.5)$ as follows

$$\begin{aligned} p_2(2.8) &= L_0(2.8)(3.2958) + L_1(2.8)(5.5452) + L_2(2.8)(8.0472) \\ &= \frac{(2.8-2)(2.8-3)}{(1-2)(1-3)}(3.2958) \\ &+ \frac{(2.8-1)(2.8-3)}{(2-1)(2-3)}(5.5452) \\ &+ \frac{(2.8-1)(2.8-2)}{(3-1)(3-2)}(8.0472) \\ &= 7.5266 \end{aligned}$$

Now to compute the error bound, we have

$$|E_{p_2(0.5)}| \leq |(0.5-0)(0.5-1)(0.5-2)|\frac{0.25}{6} = 0.0156$$

and

$$|E_{p_2(2.8)}| \leq |(2.8-1)(2.8-2)(2.8-3)|\frac{0.1111}{6} = 0.0053$$

8. Consider the function $f(x) = e^{x^2}$ and $x = 0, 0.25, 0.5, 1$. Then use the suitable Lagrange interpolating polynomial to approximate $f(0.75)$. Also, compute an error bound for your approximation.

Solution: The values of $f(x) = e^{x^2}$ at the given points are

$$f(0) = 1.0000, \quad f(0.25) = 1.0645, \quad f(0.5) = 1.2840, \quad f(1) = 2.7183$$

and using these values in the cubic Lagrange interpolating formula, we have the approximation of $f(0.5)$ as follows

$$\begin{aligned} p_3(0.75) &= L_0(0.75)(1) + L_1(0.75)(1.0645) + L_2(0.75)(1.284) + L_3(0.75)(5.5452) \\ &= \frac{(0.75-0.25)(0.75-0.5)(0.75-1)}{(0-0.25)(0-0.5)(0-1)}(1) \\ &+ \frac{(0.75-0)(0.75-0.5)(0.75-1)}{(0.25-0)(0.25-0.5)(0.25-1)}(1.0645) \\ &+ \frac{(0.75-0)(0.75-0.25)(0.75-1)}{(0.5-0)(0.5-0.25)(0.5-1)}(1.2840) \\ &+ \frac{(0.75-0)(0.75-0.25)(0.75-0.5)}{(1-0)(1-0.25)(1-0.5)}(2.7183) \\ &= 1.7911 \end{aligned}$$

Now to compute the error bound, we have

$$|E_{p3(0.75)}| \leq |(0.75 - 0)(0.75 - 0.25)(0.75 - 0.5)(0.75 - 1)| \frac{206.5894}{24} = 0.2017$$

Note that

$$|f^{(4)}(\eta) \leq M = \max_{0 \leq x \leq 1} |f^{(4)}(x)| = 206.5894$$

where

$$f^{(4)}(x) = 4e^{x^2}(3 + 16x^2 + 4x^4)$$

9. Let $f(x) = x^4 - 2x + 1$. Use cubic Lagrange interpolation formula based on the points $x_0 = -1, x_1 = 0, x_2 = 2$, and $x_3 = 3$ to find the polynomial $p_3(x)$ to approximate the function $f(x)$ at $x = 1.1$. Also, compute an error bound for your approximation.

Solution: The values of $f(x) = x^4 - 2x + 1$ at the given points are

$$f(-1) = 4.0000, \quad f(0) = 1.0000, \quad f(2) = 13.0000, \quad f(3) = 76.0000$$

and using these values in the cubic Lagrange interpolating formula, we have the approximation of $f(0.5)$ as follows

$$\begin{aligned} p_3(1.1) &= L_0(1.1)(4) + L_1(1.1)(1) + L_2(1.1)(13) + L_3(1.1)(76) \\ &= \frac{(1.1-0)(1.1-2)(1.1-3)}{(-1-0)(-1-2)(-1-3)}(4) \\ &+ \frac{(1.1+1)(1.1-2)(1.1-3)}{(0+1)(0-2)(0-3)}(1) \\ &+ \frac{(1.1+1)(1.1-0)(1.1-3)}{(2+1)(2-0)(2-3)}(13) \\ &+ \frac{(1.1+1)(1.1-0)(1.1-2)}{(3+1)(3-0)(3-2)}(76) \\ &= -3.6860 \end{aligned}$$

Now to compute the error bound, we have

$$|E_{p3(1.1)}| \leq |(1.1 + 1)(1.1 - 0)(1.1 - 2)(1.1 - 3)| \frac{24}{24} = 3.9501$$

Note that

$$|f^{(4)}(\eta) \leq M = \max_{-1 \leq x \leq 3} |f^{(4)}(x)| = 24$$

10. Construct the Lagrange interpolation polynomials for the following functions and compute the error bounds for the approximations:

(a) $f(x) = x + 2^{x+1}, \qquad x_0 = 0, x_1 = 1, x_2 = 2.5, x_3 = 3.$

(b) $f(x) = 3x^3 + 2x^2 + 1, \qquad x_0 = 1, x_1 = 2, x_2 = 3.$

(c) $f(x) = \cos x - \sin x, \qquad x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 1.$

Solution: (a) The cubic Lagrange interpolating polynomial the approximation of $f(x) = x + 2^{x+1}$ is

$$p_3(x) = 0.3661x^3 - 1.3079x^2 + 0.2764x + 2$$

The error bound for the approximation is

$$|E_{p_3(x)}| \leq |(x-0)(x-1)(x-2.5)(x-3)|\frac{3.6934}{24}$$

11. Consider the following table:

x	0	1	2	3
$f(x)$	2.0	3.72	8.39	21.06

(a) Construct divided difference table for the tabulated function.
(b) Compute the Newton interpolating polynomials $p_2(x)$ and $p_3(x)$ at $x = 2.2$.

Solution: (a) The divided difference table for the given tabulated function is as follows:

2.0000	0	0	0
3.7200	1.7200	0	0
8.3900	4.6700	1.4750	0
21.0600	12.6700	4.0000	0.8417

(b) Firstly, we construct the second degree polynomial $p_2(x)$ by using the quadratic Newton interpolation formula as follows

$$p_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

then with the help of the divided differences table, we get approximation of $f(2.2)$ as

$$f(2.2) \approx p_2(2.2) = 3.7200 + 4.6700(2.2 - 1) + 4.0000(2.2 - 1)(2.2 - 2) = 10.1224$$

Now to construct the cubic interpolatory polynomial $p_3(x)$ that fits at all four given points, we use

$$\begin{aligned} p_3(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &+ f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) \end{aligned}$$

which gives the approximation of $f(2.2)$ as

$$\begin{aligned} f(2.2) &\approx p_3(2.2) = 2 + 1.72(2.2) + 1.475(2.2)(1.2) + 0.8417(2.2)(1.2)(0.2) \\ &\approx p_3(2.2) = 10.2840 \end{aligned}$$

12. Consider the following table:

x	1	2	3	4	5
$f(x)$	3.60	1.80	1.20	0.90	0.72

(a) Construct divided difference table for the tabulated function.
(b) Compute the Newton interpolating polynomials $p_3(x)$ and $p_4(x)$ at $x = 2.5, 3.5$.

Solution: **(a)** The divided difference table for the given tabulated function is as follows:

3.6000	0	0	0	0
1.8000	−1.8000	0	0	0
1.2000	−0.6000	0.6000	0	0
0.9000	−0.3000	0.1500	−0.1500	0
0.7200	−0.1800	0.0600	−0.0300	0.0300

(b) Firstly, we construct the third degree polynomial $p_3(x)$ by using the cubic Newton interpolation formula as follows

$$\begin{aligned} p_3(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &+ f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) \end{aligned}$$

which gives the approximation of $f(2.5)$ as

$$\begin{aligned} f(2.5) &\approx p_3(2.5) = 3.6 - 1.8(1.5) + 0.6(1.5)(0.5) - 0.15(1.5)(0.5)(-0.5) \\ &\approx p_3(2.5) = 1.4063 \end{aligned}$$

Now to construct the fourth degree interpolatory polynomial $p_4(x)$ that fits at all four points. We only have to add one more term to the polynomial $p_3(x)$ and get

$$p_4(x) = p_3(x) + f[x_0, x_1, x_2, x_3, x_4](x - x_0)(x - x_1)(x - x_2)(x - x_3)$$

Using it to get the approximation of $f(2.5)$ as

$$\begin{aligned} f(2.5) &\approx p_4(2.5) = p_3(2.5) + 0.03(1.5)(0.5)(-0.5)(-1.5) \\ &\approx p_4(2.5) = 1.4063 + 0.0169 = 1.4232 \end{aligned}$$

Similarly, the approximations of $f(3.5)$ using $p_3(3.5)$ and $p_4(3.5)$ are

$$f(3.5) \approx p_3(3.5) = 1.0687 \quad \text{and} \quad f(3.5) \approx p_4(3.5) = 1.0406$$

13. Consider the following table of the $f(x) = \sqrt{x}$:

x	4	5	6	7	8
$f(x)$	2.0000	2.2361	2.4495	2.6458	2.8284

(a) Construct the divided difference table for the tabulated function.
(b) Find the Newton interpolating polynomials $p_3(x)$ and $p_4(x)$ at $x = 5.9$.
(c) Compute error bounds for your approximations in part (b).

Solution: **(a)** The divided difference table for the given tabulated function is as follows:

2.0000	0	0	0	0
2.2361	0.2361	0	0	0
2.4495	0.2134	−0.0113	0	0
2.6458	0.1963	−0.0086	0.0009	0
2.8284	0.1826	−0.0069	0.0006	−0.0001

(b) To find the approximation of $f(5.9)$ using $p_3(5.9)$, we use

$$\begin{aligned} f(5.9) &\approx p_3(5.9) = 2 + 0.2361(1.9) - 0.0113(1.9)(0.9) + 0.0009(1.9)(0.9)(-0.1) \\ &\approx p_3(5.9) = 2.4290 \end{aligned}$$

Now to find the approximation of $f(5.9)$ using $p_4(5.9)$, we use

$$\begin{aligned} f(5.9) &\approx p_4(5.9) = p_3(5.9) + 0.03(1.5)(0.5)(-0.5)(-1.5) \\ &\approx p_4(5.9) = 2.4290 - 0.0001(1.9)(0.9)(-0.1)(-1.1) \\ &\approx p_4(5.9) = 2.4290 - 0.000019 = 2.4290 \end{aligned}$$

(c) To compute the error bound, we have

$$|E_{p_3}| \leq |(5.9-4)(5.9-5)(5.9-6)(5.9-7)|\frac{0.0073}{24} = 0.000057$$

Note that

$$|f^{(4)}(\eta) \leq M = \max_{4 \leq x \leq 7} |f^{(4)}(x)| = 0.0073$$

where

$$f^{(4)}(x) = -15/16x^(7/2)$$

Also

$$|E_{p_4}| \leq |(5.9-4)(5.9-5)(5.9-6)(5.9-7)(5.9-8)|\frac{0.0064}{120} = 0.000021$$

where

$$|f^{(5)}(\eta) \leq M = \max_{4 \leq x \leq 8} |f^{(5)}(x)| = 0.0064$$

and

$$f^{(5)}(x) = 105/32x^(9/2)$$

14. Let $f(x) = e^x sinx$, with $x_0 = 0, x_1 = 2, x_2 = 2.5, x_3 = 4, x_4 = 4.5$. Then
(a) Construct the divided-difference table for the given data points.
(b) Find the Newton divided difference polynomials $p_2(x), p_3(x)$ and $p_4(x)$ at $x = 2.4$.
(c) Compute error bounds for your approximations in part (b).
(d) Compute the actual error.

Solution: **(a)** The divided difference table for the given tabulated function is as follows:

0	0	0	0	0
6.7188	3.3594	0	0	0
7.2909	1.1441	−0.8861	0	0
−41.3200	−32.4073	−16.7757	−3.9724	0
−87.9945	−93.3489	−30.4708	−5.4781	−0.3346

(b) The approximation of $f(2.4)$ using $p_2(2.4), p_3(2.4)$, and $p_4(2.4)$ are as follows:

$$p_2(2.4) = 7.8475, \quad p_3(2.4) = 7.4969, \quad p_4(2.4) = 7.5419$$

(c) Since the higher-order derivatives of the functions are

$$f'''(x) = 2e^x(\cos x - \sin x), \quad f^{(4)}(x) = -4e^x \sin x, \quad f^{(5)}(x) = -4e^x(\cos x + \sin x)$$

Then the possible error bounds for each approximations are

$$EB_2 = 0.5456, \quad EB_3 = 1.0578, \quad EB_4 = 0.7421$$

(d) The exact value of $f(2.4)$ is 7.4457.

15. Show that if x_0, x_1, and x_2 are distinct then

$$f[x_0, x_1, x_2] = f[x_1, x_2, x_0] = f[x_2, x_0, x_1]$$

Solution: All three divided differences can be expanded as

$$\frac{(x_2 - x_1)f_0 - (x_2 - x_0)f_1 + (x_1 - x_0)f_2}{(x_2 - x_1)(x_2 - x_0)(x_1 - x_0)}$$

16. The divided difference form of the interpolating polynomial $p_3(x)$ is

$$\begin{aligned} p_3(x) &= f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2, x_0] \\ &+ (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3] \end{aligned}$$

By expressing these divided differences in terms of the function values $f(x_i)(i = 0, 1, 2, 3)$, verify that $p_3(x)$ does pass through the points $(x_i, f(x_i))(i = 0, 1, 2, 3)$.

Solution: Let $p_3(x_0) = f[x_0] = f(x_0)$ and
$p_3(x_1) = f[x_0] + (x_1 - x_0)f[x_0, x_1] = f(x_0)(x_1 - x_0)\left[\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}\right] = f(x_1)$
Similarly, $p_3(x_2) = f(x_2)$ and $p_3(x_3) = f(x_3)$

17. Let $f(x) = x^2 + e^x$ and $x_0 = 0, x_1 = 1$. Use the divided differences to find the value of the second divided difference $f[x_0, x_1, x_0]$.

Solution: Since

$$f[x_0, x_0] = \frac{1}{1!}f'(x_0) = f'(x_0)$$

therefore

$$f[x_0, x_1, x_0] = f[x_0, x_0, x_1] = \frac{f[x_0, x_1] - f[x_0, x_0]}{x_1 - x_0}$$

or

$$f[x_0, x_1, x_0] = \frac{f[x_0, x_1] - f'(x_0)}{x_1 - x_0}$$

Using definition of the first-order divided difference of $f(x)$ at points x_0 and x_1, we have

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

and it gives

$$f[0, 1] = \frac{3.7183 - 1}{1 - 0} = 2.7183$$

Also

$$f'(x_0) = 2x_0 + e^{x_0} \quad \text{and} \quad f'(0) = 1$$

Using these values, we obtain the value of the second divided difference as

$$f[0,1,0] = \frac{2.7183 - 1}{1 - 0} = 1.7183$$

Chapter 5 Differentiation and Integration

1. Let $f(x) = (x-1)e^x$ and take $h = 0.01$.
(a) Calculate approximation to $f'(2.3)$ using the two-point forward-difference formula. Also, compute the actual error and an error bound for you approximation.
(b) Solve part *(a)* using the two-point backward-difference formula.

Solution: **(a)** Given $f(x) = (x-1)e^x$ and $h = 0.01$, then using the two-point forward-difference formula with $x_0 = 2.3$, we have

$$f'(2.3) \approx \frac{f(2.3+h) - f(2.3)}{h}$$

Then for $h = 0.01$, we get

$$\begin{aligned} f'(2.3) &\approx \frac{f(2.31) - f(2.3)}{0.01} \\ &\approx \frac{(2.31-1)e^{2.31} - (2.3-1)e^{2.3}}{0.01} = 23.10591068 \end{aligned}$$

The actual error is

$$Error = f'(2.3) - 23.10591068 = 2.3e^{2.3} - 23.10591068 = -0.16529103$$

To find the error bound, we use the following formula

$$E_F(f,h) = -\frac{h}{2} f''(\eta(x)), \quad \text{where} \quad \eta(x) \in (x_0, x_0 + h)$$

which can be written as

$$|E_F(f,h)| = \Big| -\frac{0.01}{2}\Big| |f''(\eta(x))|, \quad \text{for} \quad \eta \in (2.3, 2.31)$$

The second derivative $f''(x)$ of the function can be found as

$$f'(x) = xe^x, \quad \text{and} \quad f''(x) = (1+x)e^x$$

The value of the second derivative $f''(\eta(x))$ cannot be computed exactly because $\eta(x)$ is not known. But one can bound the error by computing the largest possible value for $|f''(\eta(x))|$. So bound $|f''|$ on $[2.3, 2.31]$ can be obtain

$$M = \max_{2.3 \leq x \leq 2.31} |(1+x)e^x| = 33.346346$$

at $x = 2.31$. Since $|f''(\eta(x))| \leq M$, therefore, for $h = 0.01$, we have

$$|E_F(f,h)| \leq \frac{0.01}{2} M = 0.005(33.346346) = 0.16673173$$

which is the possible maximum error in our approximation.

(b) Using the two-point backward-difference formula with $x_0 = 2.3$, we have

$$f'(2.3) \approx \frac{f(2.3) - f(2.3-h)}{h}$$

Then for $h = 0.01$, we get

$$\begin{aligned} f'(2.3) &\approx \frac{f(2.3) - f(2.29)}{0.01} \\ &\approx \frac{(2.3-1)e^{2.3} - (2.29-1)e^{2.29}}{0.01} = 22.77675826 \end{aligned}$$

The actual error is

$$Error = f'(2.3) - 22.77675826 = 2.3e^{2.3} - 22.77675826 = 0.16386139$$

To find the error bound, we use the following formula

$$E_F(f,h) = -\frac{h}{2} f''(\eta(x)), \quad \text{where} \quad \eta(x) \in (x_0 - h, x_0)$$

which can be written as

$$|E_F(f,h)| = \left| -\frac{0.01}{2} \right| |f''(\eta(x))|, \quad \text{for} \quad \eta \in (2.29, 2.3)$$

The second derivative $f''(x)$ of the function can be found as

$$f'(x) = xe^x, \quad \text{and} \quad f''(x) = (1+x)e^x$$

The value of the second derivative $f''(\eta(x))$ cannot be computed exactly because $\eta(x)$ is not known. But one can bound the error by computing the largest possible value for $|f''(\eta(x))|$. So bound $|f''|$ on $[2.29, 2.3]$ can be obtain

$$M = \max_{2.29 \leq x \leq 2.3} |(1+x)e^x| = 32.9148021$$

at $x = 2.3$. Since $|f''(\eta(x))| \leq M$, therefore, for $h = 0.01$, we have

$$|E_F(f,h)| \leq \frac{0.01}{2} M = 0.005(32.9148021) = 0.16457401$$

which is the possible maximum error in our approximation.

2. Solve the Problem 1 for the $f(x) = (x^2 + x + 1)e^{2x}$ with $h = 0.05$.

Solution: (a) Given $f(x) = (x^2+x+1)e^{2x}$ and $h = 0.05$, then using the two-point forward-difference formula with $x_0 = 2.3$, we have

$$\begin{aligned} f'(2.3) &\approx \frac{f(2.35) - f(2.3)}{0.05} \\ &\approx \frac{[(2.35)^2 + 2.35 + 1]e^{4.7} - [(2.3)^2 + 2.3 + 1]e^{4.6}}{0.05} = 2418.720325 \end{aligned}$$

The actual error is

$$Error = [2(2.3)^2 + 4(2.3) + 3]e^{4.6} - 2418.720325 = -152.4676147$$

To find the error bound, we use the following formula

$$E_F(f,h) = -\frac{h}{2} f''(\eta(x)), \quad \text{where} \quad \eta(x) \in (x_0, x_0 + h)$$

which can be written as

$$|E_F(f,h)| = \Big| -\frac{0.05}{2} \Big| |f''(\eta(x))|, \quad \text{for} \quad \eta \in (2.3, 2.35)$$

The second derivative $f''(x)$ of the function can be found as

$$f'(x) = (2x^2 + 4x + 3)e^{2x} \quad \text{and} \quad f''(x) = (4x^2 + 12x + 10)e^{2x}$$

The value of the second derivative $f''(\eta(x))$ cannot be computed exactly because $\eta(x)$ is not known. But one can bound the error by computing the largest possible value for $|f''(\eta(x))|$. So bound $|f''|$ on $[2.3, 2.35]$ can be obtain

$$M = \max_{2.3 \leq x \leq 2.35} |(4x^2 + 12x + 10)e^{2x}| = 6628.715027$$

at $x = 2.35$. Since $|f''(\eta(x))| \leq M$, therefore, for $h = 0.01$, we have

$$|E_F(f,h)| \leq \frac{0.01}{2} M = 0.025(6628.715027) = 165.7178757$$

which is the possible maximum error in our approximation.

(b) Similarly, approximation of $f'(2.3)$ by using the two-point backward-difference formula is

$$f'(2.3) \approx 2126.1$$

with actual error

$$Error = 140.1954$$

and the maximum error (or the error bound) is

$$|E_F(f,h)| \leq 146.1425$$

3. By using the following data

$$(1.2, 11.6), (1.29, 13.8), (1.3, 14), (1.31, 14.3), (1.4, 16.8)$$

compute the best approximations of $f'(1.3)$ using the two-point forward-difference and backward-difference formulas.

Solution: Given $x_0 = 1.3$ and take $x_1 = 1.31$, gives $h = 0.01$, then using the two-point forward-difference formula, we have

$$\begin{aligned} f'(1.3) &\approx \frac{f(1.31) - f(1.3)}{0.01} \\ &\approx \frac{14.3 - 14.0}{0.01} = 30.0 \end{aligned}$$

Using the two-point backward-difference formula with $x_0 = 1.3,\ x_0 - h = 1.29$, and $h = 0.01$, we have

$$\begin{aligned} f'(1.3) &\approx \frac{f(1.3) - f(1.29)}{0.01} \\ &\approx \frac{14.0 - 13.8}{0.01} = 20.0 \end{aligned}$$

4. Let $f(x) = \sin(x+1)$. Compute the approximations of $f'(\frac{\pi}{4})$ using the two-point forward-difference and backward-difference formulas. Compute the actual errors and also, find error bounds using the error formulas.

Solution: Given $f(x) = \sin(x+1)$ and $h = 0.1$, then using the two-point forward-difference formula with $x_0 = \pi/4$, we have

$$f'(\pi/4) \approx \frac{f(\pi/4 + h) - f(\pi/4)}{h}$$

Then for $h = 0.1$, we get

$$\begin{aligned} f'(\pi/4) &\approx \frac{sin(\pi/4 + 0.1 + 1) - sin(\pi/4 + 1)}{0.1} \\ &\approx \frac{0.9509197 - 0.9770613}{0.1} = -0.261416 \end{aligned}$$

The actual error is

$$Error = f'(\pi/4) + 0.261416 = -0.212958 + 0.261416 = 0.048458$$

To find the error bound, we use the following formula

$$E_F(f,h) = -\frac{h}{2} f''(\eta(x)), \quad \text{where} \quad \eta(x) \in (x_0, x_0 + h)$$

which can be written as

$$|E_F(f,h)| = \Big| -\frac{0.1}{2}\Big| |f''(\eta(x))|, \quad \text{for} \quad \eta \in (0.7854, 0.8854)$$

The second derivative $f''(x)$ of the function can be found as

$$f'(x) = \cos(x+1) \quad \text{and} \quad f''(x) = -\sin(x+1)$$

The value of the second derivative $f''(\eta(x))$ cannot be computed exactly because $\eta(x)$ is not known. But one can bound the error by computing the largest possible value for $|f''(\eta(x))|$. So bound $|f''|$ on $[0.7854, 0.8854]$ can be obtain

$$M = \max_{0.7854 \leq x \leq 0.8854} |-\sin(x+1)| = 0.977061$$

at $x = 0.7854$. Since $|f''(\eta(x))| \leq M$, therefore, for $h = 0.1$, we have

$$|E_F(f,h)| \leq \frac{0.1}{2} M = 0.05(0.977061) = 0.048853$$

which is the possible maximum error in our approximation.

Now using the two-point backward-difference formula with $x_0 = \pi/4$, we have

$$f'(\pi/4) \approx \frac{f(\pi/4) - f(\pi/4 - h)}{h}$$

Then for $h = 0.1$, we get

$$\begin{aligned} f'(\pi/4) &\approx \frac{\sin(\pi/4 + 1) - \sin(\pi/4 + 1 - 0.1)}{0.1} \\ &\approx \frac{0.9770613 - 0.993440}{0.1} = -0.163787 \end{aligned}$$

The actual error is

$$Error = f'(\pi/4) + 0.163787 = -0.212958 + 0.163787 = -0.049171$$

To find the error bound, we use the following formula

$$E_F(f, h) = -\frac{h}{2} f''(\eta(x)), \quad \text{where} \quad \eta(x) \in (x_0 - h, x_0)$$

which can be written as

$$|E_F(f, h)| = \Big| - \frac{0.1}{2} \Big| |f''(\eta(x))|, \quad \text{for} \quad \eta \in (0.6854, 0.7854)$$

The second derivative $f''(x)$ of the function can be found as

$$f'(x) = \cos(x + 1) \quad \text{and} \quad f''(x) = -\sin(x + 1)$$

The value of the second derivative $f''(\eta(x))$ cannot be computed exactly because $\eta(x)$ is not known. But one can bound the error by computing the largest possible value for $|f''(\eta(x))|$. So bound $|f''|$ on $[0.6854, 0.7854]$ can be obtain

$$M = \max_{0.6854 \leq x \leq 0.7854} |-\sin(x + 1)| = 0.993440$$

at $x = 0.6854$. Since $|f''(\eta(x))| \leq M$, therefore, for $h = 0.1$, we have

$$|E_F(f, h)| \leq \frac{0.1}{2} M = 0.05(0.993440) = 0.049672$$

which is the possible maximum error in our approximation.

5. Use the three-point central-difference formula to compute the approximate value for $f'(5)$ with $f(x) = (x^2 + 1)\ln x$, and $h = 0.05$. Compute the actual error and the error bound for you approximation.

Solution: Given $f(x) = (x^2 + 1)\ln x$ and $x_1 = 5, h = 0.05$, then using the three-point formula, we have

$$f'(5) \approx \frac{f(5 + 0.05) - f(5 - 0.05)}{2(0.05)} = \frac{f(5.05) - f(4.95)}{0.1}$$

Then

$$\begin{aligned} f'(5) &\approx \frac{[(5.05)^2+1]\ln(5.05) - [(4.95)^2+1]\ln(4.95)}{0.1} \\ &\approx \frac{42.917837 - 40.788382}{0.1} = 21.294553 \end{aligned}$$

The actual error is

$$Error = f'(5) - 21.294553 = 21.294379 - 21.294553 = -0.000174$$

To compute the error bound for the approximation, we use the formula

$$E_C(f,h) = -\frac{(0.05)^2}{6} f'''(\eta(x_1)), \quad \text{for} \quad \eta(x_1) \in (4.95, 5.05)$$

or

$$|E_C(f,h)| = \Big| - \frac{(0.05)^2}{6}\Big| |f'''(\eta(x_1))|, \quad \text{for} \quad \eta(x_1) \in (4.95, 5.05)$$

The third derivative $f'''(x)$ of the function can be found as

$$f'(x) = 2x\ln x + (x^2+1)/x, \;\; f''(x) = 2\ln x - (x^2+1)/x^2 - 4, \;\; f'''(x) = 2(x^2+1)/x^3$$

The value of the third derivative $f'''(\eta(x_1))$ cannot be computed exactly because $\eta(x_1)$ is not known. But one can bound the error by computing the largest possible value for $|f'''(\eta(x_1))|$. So bound $|f'''|$ on $[4.95, 5.05]$ can be obtain

$$M = \max_{4.95 \le x \le 5.05} |2(x^2+1)/x^3| = 0.4205302$$

at $x = 4.95$. Since $|f''(\eta(x))| \le M$, therefore, for $h = 0.1$, we have

$$|E_F(f,h)| \le \frac{(0.05)^2}{6} M = 0.00042(0.4205302) = 0.000175$$

which is the possible maximum error in our approximation.

6. Use the three-point central-difference formula to compute the approximate value for $f'(2)$ with $f(x) = e^{x/2} + 2\cos x$, and $h = 0.01$. Compute the actual error and the error bound for you approximation.

Solution: Given $f(x) = e^{x/2} + 2\cos x$ and $x_1 = 2, h = 0.01$, then using the three-point formula, we have

$$f'(2) \approx \frac{f(2+0.01) - f(2-0.01)}{2(0.01)} = \frac{f(2.01) - f(1.99)}{0.02}$$

Then

$$\begin{aligned} f'(2) &\approx \frac{e^{2.01/2} + 2\cos(2.01) - e^{1.99/2} + 2\cos(1.99)}{0.02} \\ &\approx \frac{1.88146957 - 1.89065793}{0.02} = -0.45941795 \end{aligned}$$

The actual error is

$$Error = f'(2) + 0.45941795 = -0.45945394 + 0.45941795 = -0.0000360$$

To compute the error bound for the approximation, we use the formula

$$E_C(f,h) = -\frac{(0.01)^2}{6} f'''(\eta(x_1)), \quad \text{for} \quad \eta(x_1) \in (1.99, 2)$$

or

$$|E_C(f,h)| = \Big| -\frac{(0.01)^2}{6} \Big| |f'''(\eta(x_1))|, \quad \text{for} \quad \eta(x_1) \in (1.99, 2)$$

The third derivative $f'''(x)$ of the function can be found as

$$f'(x) = 1/2e^{x/2} - 2\sin x, \;\; f''(x) = 1/4e^{x/2} - 2\cos x, \quad f'''(x) = 1/8e^{x/2} + 2\sin x$$

The value of the third derivative $f'''(\eta(x_1))$ cannot be computed exactly because $\eta(x_1)$ is not known. But one can bound the error by computing the largest possible value for $|f'''(\eta(x_1))|$. So bound $|f'''|$ on $[1.99, 2]$ can be obtain

$$M = \max_{1.99 \leq x \leq 2} |1/8e^{x/2} + 2\sin x| = 2.164917$$

at $x = 1.99$. Since $|f''(\eta(x))| \leq M$, therefore, for $h = 0.01$, we have

$$|E_F(f,h)| \leq \frac{(0.01)^2}{6} M = 0.00002(2.164917) = 0.0000361$$

which is the possible maximum error in our approximation.

7. Solve the Problem 6 to find the best approximation of $f'(2)$ using the three-point forward-difference and backward-difference formulas.

Solution: Given $f(x) = e^{x/2} + 2\cos x$ and $x_0 = 2, h = 0.01$, then using the three-point forward-difference formula, we have

$$\begin{aligned} f'(2) &\approx \frac{-3f(2) + 4f(2.01) - f(2.02)}{2(0.01)} \\ &\approx \frac{-3(e^{2/2} + 2\cos 2) + 4(e^{2.01/2} + 2\cos 2.01) - (e^{2.02/2} + 2\cos 2.02)}{0.02} \\ &\approx \frac{-5.6579645 + 7.5258783 - 1.8771043}{0.02} = -0.4595257 \end{aligned}$$

Now taking $x_2 = 2, h = 0.01$, and then using the three-point backward-difference formula, we have

$$\begin{aligned} f'(2) &\approx \frac{f(1.98) - 4f(1.99) + 3f(2)}{2(0.01)} \\ &\approx \frac{(e^{1.98/2} + 2\cos 1.98) - 4(e^{1.99/2} + 2\cos 1.99) + 3(e^{2/2} + 2\cos 2)}{0.02} \\ &\approx \frac{1.8954767 - 7.5626317 + 5.6579645}{0.02} = -0.4595261 \end{aligned}$$

8. By using the following data

$$(1.0, 2.0), (1.5, 1.94), (2.0, 2.25), (3.0, 3.11)$$

find the best approximate values for $f'(1.5)$, $f'(1.0)$, and $f'(3.0)$ using suitable three-point formulas.

Solution: Taking $(1.0, 2.0), (2.0, 2.25)$, and $h = 0.5$, we get approximation of $f'(1.5)$ by using

$*$ Central-difference formula:

$$f'(1.5) \approx \frac{f(2.0) - f(1.0)}{2h} = \frac{2.25 - 2.0}{1.0} = 0.25$$

Taking $(1.0, 2.0), (1.5, 1.94), (2.0, 2.25)$, and $h = 0.5$, we get approximation of $f'(1.0)$ by using

$*$ Forward-difference formula:

$$\begin{aligned} f'(1.0) &\approx \frac{-3f(1.0) + 4f(1.5) - f(2.0)}{2h} \\ &\approx \frac{-3(2) + 4(1.94) - 2.25}{1.0} = -0.49 \end{aligned}$$

Taking $(1.0, 2.0), (2.0, 2.25), (3.0, 3.11)$, and $h = 1.0$, we get approximation of $f'(1.0)$ by using

$*$ Backward-difference formula:

$$\begin{aligned} f'(3.0) &\approx \frac{f(1.0) - 4f(2.0) + 3f(3.0)}{2h} \\ &\approx \frac{2.0 - 4(2.25) + 3(3.11)}{2.0} = 1.165 \end{aligned}$$

9. Use all three-point formulas to compute the approximate value for $f'(2)$ for the derivative of $f(x) = e^{x/2} + x^3$, taking $h = 0.1$. Also, compute the actual errors and error bounds for your approximation.

Solution: Given $f(x) = e^{x/2} + x^3$ and $h = 0.1$, then

$*$ Central-difference formula:

$$f'(2) \approx \frac{(f(2.1) - f(1.9))}{2h} = \frac{(e^{2.1/2} + (2.1)^3) - (e^{1.9/2} + (1.9)^3)}{0.2} = 13.36970729$$

$*$ Forward-difference formula:

$$\begin{aligned} f'(2) &\approx \frac{-3f(2) + 4f(2.1) - f(2.2)}{2h} \\ &\approx \frac{-3(e^{2/2} + (2)^3) + 4(e^{2.1/2} + (2.1)^3) - (e^{2.2/2} + (2.2)^3)}{0.2} = 13.33796482 \end{aligned}$$

∗ Backward difference formula:

$$f'(2) \approx \frac{f(1.8) - 4f(1.9) + 3f(2)}{2h}$$

$$\approx \frac{(e^{1.8/2} + (1.8)^3) - 4(e^{1.9/2} + (1.9)^3) + 3(e^{2/2} + (2)^3)}{0.2} = 13.3380498$$

Since the exact solution of the first derivative of the given function at $x = 2$ is 13.35914091, so the corresponding actual errors are, $-0.01056638, 0.02117609$, and 0.02109111 respectively. This shows that the approximate solution got by using the central-difference formula is closer to exact solution as compared with the other two difference formulas.

The error bounds for the approximations got by central-difference, forward-difference, and backward-difference formulas are as follows:

∗ Central-difference error formula

$$E_C(f, h) = -\frac{h^2}{6} f'''(\eta(x_1))$$

or can be written as

$$|E_C(f, h)| \leq \frac{h^2}{6} |f'''(\eta(x_1))|$$

Since $f'''(x) = 1/8e^{x/2} + 6$, therefore, taking

$$|f'''(\eta(x_1))| \leq M = \max_{1.9 \leq x \leq 2.1} |1/8e^{x/2} + 6| = 6.35720639$$

at $x = 2.1$ and $h = 0.1$, we obtain

$$|E_C(f, h)| \leq \frac{(0.1)^2}{6} 6.35720639 = 0.010595343$$

∗ Forward-difference error formula

$$E_F(f, h) = \frac{h^2}{3} f'''(\eta(x_0))$$

or can be written as

$$|E_F(f, h)| \leq \frac{h^2}{3} |f'''(\eta(x_0))|$$

Taking $|f'''(\eta(x_0))| \leq M = \max_{2 \leq x \leq 2.2} |1/8e^{x/2} + 6| = 6.37552075$ at $x = 2.2$ and $h = 0.1$, we obtain

$$|E_F(f, h)| \leq \frac{(0.1)^2}{3} 6.37552075 = 0.021251735$$

∗ Backward-difference error formula

$$E_B(f, h) = \frac{h^2}{3} f'''(\eta(x_2))$$

or can be written as

$$|E_B(f,h)| \leq \frac{h^2}{3}|f'''(\eta(x_2))|$$

Taking

$$|f'''(\eta(x_2))| \leq M = \max_{1.8\leq x\leq 2}|1/8e^{x/2}+6| = 6.33978523$$

at $x=2$ and $h=0.1$, we obtain

$$|E_B(f,h)| \leq \frac{(0.1)^2}{3}6.33978523 = 0.021132617$$

10. Use all three-point formulas to compute the approximate value for $f'(2.2)$ for the derivative of $f(x) = x^2e^x - x + 1$, taking $h = 0.2$. Also, compute the actual errors and error bounds for your approximation.

Solution: Given $f(x) = x^2e^x - x + 1$ and $h = 0.2$, then

∗ Central-difference formula:

$$\begin{aligned} f'(2.2) &\approx \frac{(f(2.4)-f(2.0))}{2h} \\ &\approx \frac{((2.4)^2e^{2.4}-2.4+1)-((2.0)^2e^{2.0}-2.0+1)}{0.4} \\ &\approx 83.84317889 \end{aligned}$$

∗ Forward-difference formula:

$$\begin{aligned} f'(2.2) &\approx \frac{-3f(2.2)+4f(2.4)-f(2.6)}{2h} \\ &\approx \frac{-3((2.2)^2e^{2.2}-2.2+1)+4((2.4)^2e^{2.4}-2.4+1)-((2.6)^2e^{2.6}-2.6+1)}{0.4} \\ &\approx 78.78979670 \end{aligned}$$

∗ Backward difference formula:

$$\begin{aligned} f'(2.2) &\approx \frac{f(1.8)-4f(2.0)+3f(2.2)}{2h} \\ &\approx \frac{((1.8)^2e^{1.8}-1.8+1)-4((2.0)^2e^{2}-2.0+1)+3((2.2)^2e^{2.2}-2.2+1)}{0.4} \\ &\approx 80.04789053 \end{aligned}$$

Since the exact solution of the first derivative of the given function at $x = 2.2$ is 82.39112475, so the corresponding actual errors are, $-1.45205432, 3.60132803$, and 2.3432342 respectively. This shows that the approximate solution got by using the central-difference formula is closer to exact solution as compared with the other two difference formulas.

The error bounds for the approximations got by central-difference, forward-difference, and backward-difference formulas are as follows:

* Central-difference error formula

$$E_C(f,h) = -\frac{h^2}{6} f'''(\eta(x_1))$$

or can be written as

$$|E_C(f,h)| \leq \frac{h^2}{6} |f'''(\eta(x_1))|$$

Since $f'''(x) = e^x(x^2 + 6x + 6)$, therefore, taking

$$|f'''(\eta(x_1))| \leq M = \max_{2.0 \leq x \leq 2.4} |e^x(x^2 + 6x + 6)| = 288.36629$$

at $x = 2.4$ and $h = 0.2$, we obtain

$$|E_C(f,h)| \leq \frac{(0.2)^2}{6} 288.36629 = 1.922442$$

* Forward-difference error formula

$$E_F(f,h) = \frac{h^2}{3} f'''(\eta(x_0))$$

or can be written as

$$|E_F(f,h)| \leq \frac{h^2}{3} |f'''(\eta(x_0))|$$

Taking $|f'''(\eta(x_0))| \leq M = \max_{2.2 \leq x \leq 2.6} |e^x(x^2 + 6x + 6)| = 381.83161$ at $x = 2.6$ and $h = 0.2$, we obtain

$$|E_F(f,h)| \leq \frac{(0.2)^2}{3} 381.83161 = 5.091088$$

* Backward-difference error formula

$$E_B(f,h) = \frac{h^2}{3} f'''(\eta(x_2))$$

or can be written as

$$|E_B(f,h)| \leq \frac{h^2}{3} |f'''(\eta(x_2))|$$

Taking

$$|f'''(\eta(x_2))| \leq M = \max_{1.8 \leq x \leq 2.2} |e^x(x^2 + 6x + 6)| = 216.96133$$

at $x = 2.2$ and $h = 0.2$, we obtain

$$|E_B(f,h)| \leq \frac{(0.2)^2}{3} 216.96133 = 2.892918$$

11. Use the most accurate formula to determine approximations that will complete the following table.

x	f(x)	$f'(x)$
2.1	-1.709847	
2.2	-1.373823	
2.3	-1.11921	
2.4	-0.916014	

Solution: We first find the approximation of $f'(2.1)$ using $h = 0.1$ with the help of forward-difference formula as follows

$$
\begin{aligned}
f'(2.1) &\approx \frac{-3f(2.1) + 4f(2.2) - f(2.3)}{2h} \\
&\approx \frac{-3(-1.709847) + 4(-1.373823) - (-1.11921)}{0.2} = 3.7672950
\end{aligned}
$$

Now we find the approximation of $f'(2.2)$ using $h = 0.1$ with the help of central-difference formula as follows

$$f'(2.2) \approx \frac{f(2.3) - f(2.1)}{2h} = \frac{-1.11921 + 1.709847}{0.2} = 2.9531850$$

To find the approximation of $f'(2.3)$ using $h = 0.1$, we use again central-difference formula as follows

$$f'(2.3) \approx \frac{f(2.4) - f(2.2)}{2h} = \frac{-0.916014 + 1.373823}{0.2} = 2.2890450$$

To find the approximation of $f'(2.4)$ using $h = 0.1$, we use backward-difference formula as follows

$$
\begin{aligned}
f'(2.4) &\approx \frac{f(2.2) - 4f(2.3) + 3f(2.4)}{2h} \\
&\approx \frac{-1.373823 - 4(-1.11921) + 3(-0.916014)}{0.2} = 1.7748750
\end{aligned}
$$

Thus

x	f(x)	$f'(x)$
2.1	-1.709847	3.7672950
2.2	-1.373823	2.9531850
2.3	-1.11921	2.2890450
2.4	-0.916014	1.7748750

12. The data in the Problem 11 were taken from the function $f(x) = \tan(x)$. Compute the actual errors in problem 11 and also, find error bounds using the error formulas.

Solution: The actual errors are as follows:

$$Error = f'(2.1) - 3.7672950 = 3.9235752 - 3.7672950 = 0.1562802$$

$$Error = f'(2.2) - 3.7672950 = 2.8873898 - 2.9531850 = -0.0657952$$

$$Error = f'(2.3) - 3.7672950 = 2.2526392 - 2.2890450 = -0.0364058$$

$$Error = f'(2.4) - 3.7672950 = 1.8390822 - 1.7748750 = 0.0642072$$

Now finding the error bound for the approximation of $f'(2.1)$, we do the following

$$f'''(x) = 2sec^2x(sec^2x + 2tan^2x) \quad \text{and} \quad |f'''(\eta(x_0))| \leq M$$

where

$$M = \max_{2.1 \leq x \leq 2.3} |2sec^2x(sec^2x + 2tan^2x)| = 76.6723533$$

at $x = 2.1$ and $h = 0.1$, we obtain

$$|E_F(f, h)| \leq \frac{(0.1)^2}{3} 76.6723533 = 0.2555745$$

The error bound for the approximation of $f'(2.2)$, we have

$$M = \max_{2.1 \leq x \leq 2.3} |2sec^2x(sec^2x + 2tan^2x)| = 76.6723533$$

at $x = 2.1$ and $h = 0.1$, we obtain

$$|E_C(f, h)| \leq \frac{(0.1)^2}{6} 76.6723533 = 0.1277873$$

The error bound for the approximation of $f'(2.3)$, we have

$$M = \max_{2.2 \leq x \leq 2.4} |2sec^2x(sec^2x + 2tan^2x)| = 38.4725597$$

at $x = 2.2$ and $h = 0.1$, we obtain

$$|E_C(f, h)| \leq \frac{(0.1)^2}{6} 38.4725597 = 0.0641209$$

Finally, the error bound for the approximation of $f'(2.4)$, we have

$$M = \max_{2.2 \leq x \leq 2.4} |2sec^2x(sec^2x + 2tan^2x)| = 38.4725597$$

at $x = 2.2$ and $h = 0.1$, we obtain

$$|E_B(f, h)| \leq \frac{(0.1)^2}{3} 38.4725597 = 0.1282419$$

13. Use the most accurate formula to determine approximations that will complete the following table.

x	f(x)	$f'(x)$
8.1	16.94410	
8.3	17.56492	
8.5	18.19056	
8.7	18.82091	

Solution: We first find the approximation of $f'(8.1)$ using $h = 0.2$ with the help of forward-difference formula as follows

$$\begin{aligned} f'(8.1) &\approx \frac{-3f(8.1) + 4f(8.3) - f(8.5)}{2h} \\ &\approx \frac{-3(16.94410) + 4(17.56492) - (18.19056)}{0.4} = 3.092050 \end{aligned}$$

Now we find the approximation of $f'(8.3)$ using $h = 0.2$ with the help of central-difference formula as follows

$$f'(8.3) \approx \frac{f(8.5) - f(8.1)}{2h} = \frac{18.19056 - 16.94410}{0.4} = 3.116150$$

To find the approximation of $f'(8.5)$ using $h = 0.2$, we use again central-difference formula as follows

$$f'(8.5) \approx \frac{f(8.7) - f(8.3)}{2h} = \frac{18.82091 - 17.56492}{0.4} = 3.1399750$$

To find the approximation of $f'(8.7)$ using $h = 0.2$, we use backward-difference formula as follows

$$\begin{aligned} f'(8.7) &\approx \frac{f(8.3) - 4f(8.5) + 3f(8.7)}{2h} \\ &\approx \frac{17.56492 - 4(18.19056) + 3(18.82091)}{0.4} = 3.1635250 \end{aligned}$$

Thus

x	f(x)	$f'(x)$
8.1	16.94410	3.092050
8.3	17.56492	3.116150
8.5	18.19056	3.1399750
8.7	18.82091	3.1635250

14. The data in the Problem 13 were taken from the function $f(x) = x \ln x$. Compute the actual errors in the Problem 13 and also, find error bounds using the error formulas.

Solution: The actual errors are as follows:

$$Error = f'(8.1) - 3.092050 = 3.0918641 - 3.092050 = -0.0001859$$

$$Error = f'(8.3) - 3.116150 = 3.1162555 - 3.116150 = 0.0001055$$

$$Error = f'(8.5) - 3.139975 = 3.1400662 - 3.139975 = 0.0000912$$

$$Error = f'(8.7) - 3.163525 = 3.163323 - 3.163525 = -0.000202$$

Now finding the error bound for the approximation of $f'(8.1)$, we do the following

$$f'''(x) = -1/x^2 \quad \text{and} \quad |f'''(\eta(x_0))| \leq M$$

where

$$M = \max_{8.1 \leq x \leq 8.3} |-1/x^2| = 0.0152416$$

at $x = 8.1$ and $h = 0.2$, we obtain

$$|E_F(f,h)| \leq \frac{(0.2)^2}{3} 0.0152416 = 0.000203$$

The error bound for the approximation of $f'(8.3)$, we have

$$M = \max_{8.1 \leq x \leq 8.5} |-1/x^2| = 0.0152416$$

at $x = 8.1$ and $h = 0.2$, we obtain

$$|E_C(f,h)| \leq \frac{(0.2)^2}{6} 0.0152416 = 0.000102$$

The error bound for the approximation of $f'(8.5)$, we have

$$M = \max_{8.3 \leq x \leq 8.7} |-1/x^2| = 0.0138408$$

at $x = 8.3$ and $h = 0.2$, we obtain

$$|E_C(f,h)| \leq \frac{(0.2)^2}{6} 0.0138408 = 0.0000923$$

Finally, the error bound for the approximation of $f'(8.7)$, we have

$$M = \max_{8.3 \leq x \leq 8.7} |-1/x^2| = 0.0138408$$

at $x = 8.3$ and $h = 0.2$, we obtain

$$|E_B(f,h)| \leq \frac{(0.1)^2}{3} 0.0138408 = 0.000185$$

15. Let $f(x) = x + \ln(x+2)$, with $h = 0.1$. Use the five-point formula to approximate $f'(2)$. Find error bound for your approximation and compare the actual error to the bound.

Solution: Given $f(x) = x + \ln(x+2)$, $x_1 = 2$ and $h = 0.1$, then using the five-point formula to get

$$f'(2) \approx \frac{f(2-2(0.1)) - 8f(2-0.1) + 8f(2+0.1) - f(2+2(0.1))}{12(0.1)}$$

or

$$\begin{aligned} f'(2) &\approx \frac{(1.8+\ln(3.8)) - 8(1.9+\ln(3.9)) + 8(2.1+\ln(4.1)) - (2.2+\ln(4.2))}{1.2} \\ &\approx 1.24999992 \end{aligned}$$

Now to compute the error bound for our approximation, we use the following error formula

$$E_C(f,h) = \frac{h^4}{30} f^{(5)}(\eta(x_1))$$

or it can be written as

$$|E_C(f,h)| = \Big|\frac{h^4}{30}\Big||f^{(5)}(\eta(x_1))|$$

Since $f^{(5)} = 24/(x+2)^5$ and $f^{(5)}(\eta(x_1))$ cannot be computed exactly because $\eta(x_1)$ is not known. But one can bound the error by computing the largest possible value for $|f^{(5)}(\eta(x_1))|$. So bound $|f^{(5)}|$ on $[1.8, 2.2]$ can be obtain

$$M = \max_{1.8 \leq 2.2} |f^{(5)}(x)| = \max_{1.8 \leq 2.2} |24/(x+2)^5| = 0.03028958$$

Since $|f^{(5)}(\eta(x_1))| \leq M$, therefore

$$|E_C(f,h)| \leq \frac{(0.1)^4}{30} M = \frac{(0.1)^4}{30}(0.03028958) = 0.000000101$$

The actual error is as follows:

$$Error = f'(2) - 1.24999992 = 1.250000000 - 1.24999992 = 0.0000001$$

16. Consider the following data

$$(1.1, 0.607), (1.2, 0.549), (1.4, 0.497), (1.5, 0.449), (1.7, 0.427), (1.8, 0.387)$$

Use the five-point formulas to compute the best approximations for $f'(1.3)$, and $f'(1.6)$.

Solution: To find the approximation of $f'(1.3)$ by five-point formula, we use

$$f'(1.3) \approx \frac{f(1.3-2h) - 8f(1.3-h) + 8f(2+h) - f(2+2h)}{12h}$$

Taking $h = 0.1$, gives

$$f'(1.3) \approx \frac{f(1.1) - 8f(1.2) + 8f(1.4) - f(1.5)}{12(0.1)}$$

or

$$f'(1.3) \approx \frac{(0.607 - 8(0.549) + 8(0.497) - 0.449)}{1.2} = -0.215000$$

Similarly, to find the approximation of $f'(1.6)$ with $h = 0.1$ using five-point formula, we get

$$f'(1.6) \approx \frac{f(1.4) - 8f(1.5) + 8f(1.7) - f(1.8)}{1.2}$$

or

$$f'(1.6) \approx \frac{(0.497 - 8(0.449) + 8(0.427) - 0.387)}{1.2} = -0.055000$$

17. Let $f(x) = x^2 + 1$, with $h = 0.2$. Use the five point formula to approximate $f'(1.8)$.

Solution: Given $f(x) = x^2 + 1,\ x_1 = 1.8$, and $h = 0.2$, then finding the approximation of $f'(1.8)$ by five-point formula, we use

$$f'(1.8) \approx \frac{f(1.8-0.4) - 8f(1.8-0.2) + 8f(1.8+0.2) - f(1.8+0.4)}{12(0.2)}$$

It gives

$$f'(1.8) \approx \frac{f(1.4) - 8f(1.6) + 8f(2.0) - f(2.2)}{12(2.4)}$$

or

$$f'(1.8) \approx \frac{(((1.4)^2+1) - 8((1.6)^2+1) + 8((2.0)^2+1) - ((2.2)^2+1))}{2.4} = 3.6000000$$

18. Let $f(x) = e^{-2x}$, with points $x = 0.25, 0.5, 0.75, 1.25, 1.50$. Use the three-point central-difference formula and the five point formula to approximate $f'(1.0)$. Also, compute the error bounds for your approximations.

Solution: Using the given data points to find the approximation of $f'(1.0)$ by the three-point formula central-difference formula with $h = 0.25$, we have

$$f'(1.0) \approx \frac{f(1.0+0.25) - f(1.0-0.25)}{2(0.25)} = \frac{f(1.25) - f(0.75)}{0.5}$$

Then

$$\begin{aligned} f'(1.0) &\approx \frac{e^{-2(1.25)} - e^{-2(0.75)}}{0.5} \\ &\approx \frac{0.082085 - 0.223130}{0.5} = -0.2820903 \end{aligned}$$

To compute the error bound for the approximation, we use the formula

$$E_C(f,h) = -\frac{(0.25)^2}{6} f'''(\eta(x_1)), \quad \text{for} \quad \eta(x_1) \in (0.75, 1.25)$$

or

$$|E_C(f,h)| = \Big| -\frac{(0.25)^2}{6} \Big| |f'''(\eta(x_1))|, \quad \text{for} \quad \eta(x_1) \in (0.75, 1.25)$$

The third derivative $f'''(x)$ of the function can be found as

$$f'(x) = -2e^{-2x}, \quad f''(x) = 4e^{-2x}, \quad f'''(x) = -8e^{-2x}$$

The value of the third derivative $f'''(\eta(x_1))$ cannot be computed exactly because $\eta(x_1)$ is not known. But one can bound the error by computing the largest possible value for $|f'''(\eta(x_1))|$. So bound $|f'''|$ on $[0.75, 1.25]$ can be obtain

$$M = \max_{0.75 \le x \le 1.25} |-8e^{-2x}| = 1.7850413$$

at $x = 0.75$. Since $|f''(\eta(x))| \le M$, therefore, for $h = 0.1$, we have

$$|E_F(f,h)| \le \frac{(0.25)^2}{6} M = 0.01042(1.7850413) = 0.0185942$$

which is the possible maximum error in our approximation.

Now using the five-point formula to get the approximation of $f'(1.0)$, we have

$$f'(1.0) \approx \frac{f(1.0-2(0.25)) - 8f(1.0-0.25) + 8f(1.0+0.25) - f(1.0+2(0.25))}{12(0.25)}$$

or

$$f'(1.0) \approx \frac{(e^{-2(0.5)} - 8(e^{-2(0.75)}) + 8(e^{-2(1.25)}) - e^{-2(1.5)})}{3.0} = -0.2700896$$

Now to compute the error bound for our approximation, we use the following error formula

$$E_C(f,h) = \frac{h^4}{30} f^{(5)}(\eta(x_1))$$

or it can be written as

$$|E_C(f,h)| = \Big|\frac{h^4}{30}\Big| |f^{(5)}(\eta(x_1))|$$

Since $f^{(5)} = -32e^{-2x}$ and $f^{(5)}(\eta(x_1))$ cannot be computed exactly because $\eta(x_1)$ is not known. But one can bound the error by computing the largest possible value for $|f^{(5)}(\eta(x_1))|$. So bound $|f^{(5)}|$ on $[0.5, 1.5]$ can be obtain

$$M = \max_{0.5 \leq 1.5} |f^{(5)}(x)| = \max_{0.5 \leq 1.5} |-32e^{-2x}| = 11.7721421$$

Since $|f^{(5)}(\eta(x_1))| \leq M$, therefore

$$|E_C(f,h)| \leq \frac{(0.25)^4}{30} M = \frac{(0.25)^4}{30}(11.7721421) = 0.0015328$$

19. Let $f(x) = e^{x^2}$. Use the five-point formula to compute the best approximations for $\dfrac{d^2 f}{dx^2}$ at 0.5 and $h = 0.1$. Find bound on the error for your approximation.

Solution: Given $f(x) = e^{x^2}$, $x_1 = 0.5$, $h = 0.1$, then using five-point formula for finding the approximation of $f''(0.5)$, we have

$$\begin{aligned} f''(0.5) &\approx \frac{-f(0.5-0.2) + 16f(0.5-0.1) - 30f(0.5) + 16f(0.5+0.1) - f(0.5+0.2)}{12(0.01)} \\ &\approx \frac{-f(0.3) + 16f(0.4) - 30f(0.5) + 16f(0.6) - f(0.7)}{0.12} \\ &\approx \frac{-e^{(0.3)^2} + 16e^{(0.4)^2} - 30e^{(0.5)^2} + 16e^{(0.6)^2} - e^{(0.7)^2}}{0.12} = 3.8515964 \end{aligned}$$

To compute the error bound for our approximation, we use the following error formula

$$E_C(f,h) = \frac{h^4}{30} f^{(5)}(\eta(x_1))$$

or it can be written as

$$|E_C(f,h)| = \Big|\frac{h^4}{30}\Big||f^{(5)}(\eta(x_1))|$$

Since $f^{(5)}(x) = 8xe^{x^2}(15+20x^2+32x^4)$ and $f^{(5)}(\eta(x_1))$ cannot be computed exactly because $\eta(x_1)$ is not known. But one can bound the error by computing the largest possible value for $|f^{(5)}(\eta(x_1))|$. So bound $|f^{(5)}|$ on $[0.3, 0.7]$ can be obtain

$$M = \max_{0.3\leq x\leq 0.7}|f^{(5)}(x)| = \max_{0.3\leq x\leq 0.7}|8xe^{x^2}(15+20x^2+32x^4)| = 44.2021901$$

Since $|f^{(5)}(\eta(x_1))| \leq M$, therefore

$$|E_C(f,h)| \leq \frac{(0.1)^4}{30}M = \frac{(0.1)^4}{30}(44.2021901) = 0.0001473$$

20. Let $f(x) = x+\ln(x+2)$, with $h = 0.1$. Use the three-point formula to approximate $f''(2)$. Find error bound for your approximation and compare the actual error to the bound.

Solution: Given $f(x) = x + \ln(x+2),\ x_1 = 2,\ h = 0.1$, then using three-point formula for finding the approximation of $f''(2)$, we have

$$f''(2) \approx \frac{f(2+0.1) - 2f(2) + f(2-0.1)}{(0.1)^2}$$

or

$$\begin{aligned} f''(2) &\approx \frac{f(2.1) - 2f(2) + f(1.9)}{0.01} \\ &\approx \frac{(2.1+\ln(4.1)) - 2(2+\ln(4.0)) + (1.9+\ln(3.9))}{0.01} \\ &\approx \frac{3.5109870 - 6.7725887 + 3.2609766}{0.01} = -0.0625195 \end{aligned}$$

To compute the error bound for our approximation, we use the following error formula

$$E_C(f,h) = -\frac{h^2}{12}f^{(4)}(\eta(x_1)), \quad \text{for} \quad \eta(x_1) \in (1.9, 2.1)$$

or

$$|E_C(f,h)| = \Big|-\frac{h^2}{12}\Big||f^{(4)}(\eta(x_1))|, \quad \text{for} \quad \eta(x_1) \in (1.9, 2.1)$$

The fourth derivative of the given function at $\eta(x_1)$ is

$$f^{(4)}(\eta(x_1)) = -6/(\eta(x_1)+2)^4$$

and it cannot be computed exactly because $\eta(x_1)$ is not known. But one can bound the error by computing the largest possible value for $|f^{(4)}(\eta(x_1))|$. So bound $|f^{(4)}|$ on the interval $(1.9, 2.1)$ is

$$M = \max_{1.9\leq x\leq 2.1}|-6/(x+2)^4| = 0.0259354$$

at $x = 1.9$, Thus, for $|f^{(4)}(\eta(x))| \leq M$, we have

$$|E_C(f,h)| \leq \frac{h^2}{12}M$$

Taking $M = 0.0259354$ and $h = 0.1$, we obtain

$$|E_C(f,h)| \leq \frac{0.01}{12}(0.0259354) = 0.0000216$$

which is the possible maximum error in our approximation.
The actual error is

$$Error = f''(2) + 0.0625195 = -0.0625000 + 0.0625195 = 0.0000195$$

21. Let $f(x) = x\sin x$, with $h = 0.1$. Use five-point formula to approximate $f''(1)$. Find error bound for your approximation.

Solution: Given $f(x) = x\sin x,\ x_1 = 1,\ h = 0.1$, then using five-point formula for finding the approximation of $f''(1)$, we have

$$\begin{aligned} f''(1) &\approx \frac{-f(1-0.2)+16f(1-0.1)-30f(1)+16f(1+0.1)-f(1+0.2)}{12(0.01)} \\ &\approx \frac{-f(0.8)+16f(0.9)-30f(1)+16f(1.1)-f(1.2)}{0.12} \\ &\approx \frac{-(0.8\sin 0.8)+16(0.9\sin 0.9)-30(1\sin 1)+16(1.1\sin 1.1)-(1.2\sin 1.2)}{0.12} \\ &\approx 0.23913096 \end{aligned}$$

To compute the error bound for our approximation, we use the following error formula

$$E_C(f,h) = \frac{h^4}{30}f^{(5)}(\eta(x_1))$$

or it can be written as

$$|E_C(f,h)| = \left|\frac{h^4}{30}\right||f^{(5)}(\eta(x_1))|$$

Since $f^{(5)}(x) = 5\sin x + x\cos x$ and $f^{(5)}(\eta(x_1))$ cannot be computed exactly because $\eta(x_1)$ is not known. But one can bound the error by computing the largest possible value for $|f^{(5)}(\eta(x_1))|$. So bound $|f^{(5)}|$ on $[0.8, 1.2]$ can be obtain

$$M = \max_{0.8\leq x\leq 1.2}|f^{(5)}(x)| = \max_{0.8\leq x\leq 1.2}|5\sin x + x\cos x| = 5.0950247$$

Since $|f^{(5)}(\eta(x_1))| \leq M$, therefore

$$|E_C(f,h)| \leq \frac{(0.1)^4}{30}M = \frac{(0.1)^4}{30}(5.0950247) = 0.0000170$$

22. Consider the following data

$$(0.2, 0.39), (0.4, 1.08), (0.6, 1.49), (0.8, 1.78), (1, 2), (1.2, 2.18), (1.4, 2.34)$$

Use the five-point formula to compute the best approximation for $f''(0.6)$ and $f''(1.0)$. The data in this problem were taken from the function $f(x) = \ln x + 2$. Compute the actual errors and also, find error bounds of your approximations.

Solution: Using five-point formula for finding the approximation of $f''(0.6)$ with $h = 0.2$, we have

$$\begin{aligned} f''(0.6) &\approx \frac{-f(0.2) + 16f(0.4) - 30f(0.6) + 16f(0.8) - f(1.0)}{12(0.04)} \\ &\approx \frac{-0.39 + 17.28 - 44.70 + 28.48 - 2.0}{0.48} = -2.7708333 \end{aligned}$$

Now using five-point formula for finding the approximation of $f''(1.0)$ with $h = 0.2$, we have

$$\begin{aligned} f''(1.0) &\approx \frac{-f(0.6) + 16f(0.8) - 30f(1.0) + 16f(1.2) - f(1.4)}{12(0.04)} \\ &\approx \frac{-1.49 + 28.48 - 60.0 + 34.88 - 2.34}{0.48} = -0.9791667 \end{aligned}$$

Since the exact solution of the second derivative $f''(x) = -1/x^2$ of the given function $f(x) = \ln x + 2$ at $x = 0.6$ is -2.7777778, so the corresponding actual error is, -0.0069445. Also, the exact solution of $f''(x)$ at $x = 1.0$ is -1.0, so the corresponding actual error is, -0.0208333.
To compute the error bound for the approximation of $f''(0.6)$, we have

$$|E_C(f, h)| \leq \frac{(0.2)^4}{30} M$$

where

$$M = \max_{0.2 \leq x \leq 1.0} |24/x^5| = 75000$$

So

$$|E_C(f, h)| \leq \frac{(0.2)^4}{30} 75000 = 4.0$$

Similarly, we have for the approximation of $f''(1.0)$

$$|E_C(f, h)| \leq \frac{(0.2)^4}{30} M$$

where

$$M = \max_{0.6 \leq x \leq 1.2} |24/x^5| = 308.64198$$

and

$$|E_C(f, h)| \leq \frac{(0.2)^4}{30} 308.64198 = 0.0164609$$

23. Let $f(x) = 1/x$, with $h = 0.2$. Use the five-point formula to approximate $f''(2)$. Find error bound for your approximation.

Solution: Given $f(x) = 1/x,\ x_1 = 2,\ h = 0.2$, then using five-point formula for finding the approximation of $f''(2.0)$, we have

$$\begin{aligned} f''(2) &\approx \frac{-f(2-0.4)+16f(2-0.2)-30f(2)+16f(2+0.2)-f(2+0.4)}{12(0.04)} \\ &\approx \frac{-f(1.6)+16f(1.8)-30f(2)+16f(2.2)-f(2.4)}{0.48} \\ &\approx \frac{-0.625+8.8889-15.0+7.2727-0.4167}{0.48} = 0.2497917 \end{aligned}$$

To compute the error bound for the approximation of $f''(2)$, we have

$$|E_C(f,h)| \leq \frac{(0.2)^4}{30} M$$

where

$$M = \max_{1.6 \leq 2.2} |-120/x^6| = 7.1525574$$

So

$$|E_C(f,h)| \leq \frac{(0.2)^4}{30} 7.1525574 = 0.0003812$$

24. For the three-point central-difference formula for $f'(x)$, perform an error analysis to show that the optimum stepsize h_{opt} is given by

$$h = h_{opt} = \sqrt[3]{\frac{3}{2M}} \times 10^{-t}$$

where $M = \max|f'''(x)|$.

Solution: Since

$$f'(x_1) \approx D_h^{(1)} = \frac{f(x_2) - f(x_)}{2h}$$

where $x_2 = x_1 + h$ and $x_0 = x_1 - h$. Taking

$$f(x_i) - \hat{f}_i = \epsilon_i, \quad i = 0, 1, 2$$

Then

$$f'(x_1) - \hat{D_h^{(1)}} f(x_1) \approx -\frac{h^2}{6} f'''(\eta) + \frac{\epsilon_2 - \epsilon_0}{2h}$$

and

$$|f'(x_1) - \hat{D_h^{(1)}} f(x_1)| \approx \frac{h^2}{6} M + \frac{2\delta}{2h}$$

where $M = max|f'''(x)|$, $\epsilon_0, \epsilon_1, \epsilon_2$ are generally random in some interval $[-\delta, \delta]$ and δ is a bound on the experimental error. Let

$$E(h) = \frac{h^2}{6} M + \frac{2\delta}{2h}$$

and $E'(h) = 0$ gives

$$h = h_{opt} = \sqrt[3]{\frac{3}{2M}} \times 10^{-t}$$

where $\delta = \dfrac{1}{2} \times 10^{-t}$.

25. Approximate the integral $\int_0^2 x^2 e^{-x^2} dx$, using composite Trapezoidal rules with $n = 4$, and $n = 6$.

Solution: Given $f(x) = x^2 e^{-x^2}$, then for $n = 4$ and $h = 0.5$, we have the Trapezoidal rule of the form

$$T_4(f) = \frac{0.5}{2}\Big[f(0) + 2[f(0.5) + f(1.0) + f(1.5)] + f(2)\Big] = 14.4980$$

which gives

$$T_4(f) = \frac{0.5}{2}\Big[0+2[(0.5)^2 e^{-(0.5)^2} + (1.0)^2 e^{-(1.0)^2} + (1.5)^2 e^{-(1.5)^2}] + (2.0)^2 e^{-(2.0)^2}\Big] = 0.4182$$

For $n = 6$ and $h = 1/3$, we get

$$\begin{aligned} T_6(f) &= \frac{1/3}{2}\Big[0 + 2[(1/3)^2 e^{-(1/3)^2} + (2/3)^2 e^{-(2/3)^2} + (1.0)^2 e^{-(1.0)^2} + (4/3)^2 e^{-(4/3)^2} \\ &+ (5/3)^2 e^{-(5/3)^2}] + (2.0)^2 e^{-(2.0)^2}\Big] = 0.4207 \end{aligned}$$

26. The following values of a function $f(x) = \tan x/x$ are given

x	f(x)	x	f(x)
1.00	1.5574	1.40	4.1342
1.10	1.7862	1.50	9.4009
1.20	2.1435	1.60	-21.3953
1.30	2.7709		

Find $\int_{1.0}^{1.6} f(x)dx$, using the Trapezoidal rule with $h = 0.1$.

Solution: Given $f(x) = \tan x/x$, then for $h = 0.1$ and $n = 6$, we have the Trapezoidal rule of the form

$$T_6(f) = \frac{0.1}{2}\Big[1.5574 + 2[1.7862 + 2.1435 + 2.7709 + 4.1342 + 9.4009] + -21.3953\Big] = 1.0316750$$

27. Use a suitable composite integration formula to approximate the integral $\int_0^1 \dfrac{dx}{2e^x - 1}$, with $n = 5$.

Solution: Given $f(x) = 1/(2e^x - 1)$, then for $n = 5$ and $h = 0.2$, we have the Trapezoidal rule of the form

$$\begin{aligned} T_5(f) &= \frac{0.2}{2}\Big[1/(2e^0 - 1) + 2[1/(2e^{0.2} - 1) + 1/(2e^{0.4} - 1) + 1/(2e^{0.6} - 1) \\ &+ 1/(2e^{0.8} - 1)] + 1/(2e^{1.0} - 1)\Big] = 0.49557207 \end{aligned}$$

28. Use a suitable composite integration formula for the approximation of the integral $\int_1^2 \frac{dx}{3-x}$, with $n = 5$. Compute an upper bound for your approximation.

Solution: Given $f(x) = 1/(3-x)$, then for $n = 5$ and $h = 0.2$, we have the Trapezoidal rule of the form

$$T_5(f) = \frac{0.2}{2}\Big[1/(3-1)+2[1/(3-1.2)+1/(3-1.4)+1/(3-1.6)+1/(3-1.8)]+1/(3-2)\Big]$$

it gives

$$T_5(f) = 0.6956349$$

To compute error bound for the approximation, we use the error formula

$$E_{T_5}(f) = -\frac{h^2}{12}(b-a)f''(\eta(x)), \quad \text{for} \quad \eta(x) \in (1,2)$$

or

$$|E_{T_5}(f)| = \frac{h^2}{12}|f''(\eta(x))|, \quad \text{for} \quad \eta(x) \in (1,2)$$

Since $f''(x) = 2/(3-x)^3$ and $f''(\eta(x))$ cannot be computed exactly because $\eta(x)$ is not known. But one can bound the error by computing the largest possible value for $|f''(\eta(x))|$. So bound $|f''|$ on $[1,2]$ can be obtain

$$M = \max_{1\le x\le 2}|f''(x)| = \max_{1\le x\le 2}|2/(3-x)^3| = 2.0$$

Since $|f''(\eta(x))| \le M$, therefore

$$|E_{T_5}(f)| \le \frac{(0.2)^2}{12}(2.0) = 0.006667$$

which is the possible maximum error in our approximation.

29. Use the composite Trapezoidal rule for the approximation of the integral $\int_1^3 \frac{dx}{7-2x}$ with $h = 0.5$. Also, compute an error term.

Solution: Given $f(x) = 1/(7-2x)$, then for $h = 0.5$ and $n = 4$, we have the Trapezoidal rule of the

$$T_5(f) = \frac{0.5}{2}\Big[1/(7-2)+2[1/(7-3)+1/(7-4)+1/(7-5)]+1/(7-6)\Big] = 0.8416667$$

To compute error bound for the approximation, we use

$$|E_{T_4}(f)| \le \frac{(0.5)^2}{12}(3-1)M$$

where

$$M = \max_{1\le x\le 3}|f''(x)| = \max_{1\le x\le 3}|8/(7-2x)^3| = 8.0$$

and

$$|E_{T_4}(f)| \le \frac{(0.5)^2}{12}(2)(8.0) = 0.3333333$$

which is the possible maximum error in our approximation.

30. Find the step size h so that the absolute value of the error for the composite Trapezoidal rule is less than 5×10^{-4} when it is used to approximate the integral $\int_2^7 \frac{dx}{x}$.

Solution: To find the step size h for the given accuracy, we use the composite Trapezoidal error formula as

$$|E_{T_n}(f)| \leq \frac{|-(b-a)|}{12} h^2 M \leq 5 \times 10^{-4}$$

As

$$M = \max_{2 \leq x \leq 7} |f''(x)| = \max_{2 \leq x \leq 7} |2/x^3| = 0.25$$

so solving for h^2, we obtain

$$h^2 \leq 0.0048, \quad \text{gives} \quad h \leq 0.06928 \quad \text{or} \quad h = 0.05$$

31. Estimate the integral $\int_{-1}^1 \frac{dx}{1+x^2}$ using the Simpson's rules with $n = 8$.

Solution: Given $f(x) = 1/(1+x^2)$,then for $n = 8$ and $h = 0.25$, we have the Simpson's rule of the form

$$\begin{aligned} \int_{x_0}^{x_8} f(x)dx &\approx \frac{h}{3}[f(x_0) + 4(f(x_1) + f(x_3) + f(x_5) + f(x_7)) \\ &+ 2(f(x_2) + f(x_4) + f(x_6)) + f(x_8)] \end{aligned}$$

or

$$\begin{aligned} \int_{-1}^1 1/(1+x^2)dx &\approx \frac{0.25}{3}[f(-1) + 4[f(-0.75) + f(-0.25) + f(-0.25) + f(0.75)] \\ &+ 2[f(-0.5) + f(0.0) + f(0.5)] + f(1)] \end{aligned}$$

It gives

$$\begin{aligned} \int_{-1}^1 1/(1+x^2)dx &\approx \frac{0.25}{3}\Big[0.5 + 4[0.64 + 0.9412 + 0.9412 + 0.64)] \\ &+ 2[0.8 + 1.0 + 0.8] + 0.5\Big] = 1.570800 \end{aligned}$$

32. Repeat the Problem 26 by using the composite Simpson's rule.

Solution: The Simpson's rule for $n = 6$ can be written as

$$\int_{x_0}^{x_6} f(x)dx \approx \frac{h}{3}[f(x_0) + 4(f(x_1) + f(x_3) + f(x_5))) + (f(x_2) + f(x_4)) + f(x_6)]$$

Using the given values, we have

$$\begin{aligned} \int_{1.0}^{1.6} \tan x/x \; dx &\approx \frac{0.1}{3}\Big[1.5574 + 4(1.7862 + 2.7709 + 9.4009) \\ &+ 2(2.1435 + 4.1342) - 21.3953\Big] = 1.6183167 \end{aligned}$$

33. Repeat the Problem 25 by using the composite Simpson's rule.

Solution: Given $f(x) = x^2 e^{-x^2}$, then for $n = 4$ and $h = 0.5$, we have the Simpson's rule of the form

$$S_4(f) = \frac{0.5}{3}\Big[f(0) + 4(f(0.5) + f(1.5)) + 2f(1.0) + f(2)\Big] = 14.4980$$

which gives

$$S_4(f) = \frac{0.5}{3}\Big[0+4((0.5)^2e^{-(0.5)^2}+(1.5)^2e^{-(1.5)^2})+2(1.0)^2e^{-(1.0)^2}+(2.0)^2e^{-(2.0)^2}\Big] = 0.4227$$

Similarly, for $n = 6$ and $h = 1/3$, we get

$$\begin{aligned} S_6(f) &= \frac{1/3}{3}\Big[0 + 4[(1/3)^2e^{-(1/3)^2} + (1.0)^2e^{-(1.0)^2} + (5/3)^2e^{-(5/3)^2}] + 2[(2/3)^2e^{-(2/3)^2} \\ &+ (4/3)^2e^{-(4/3)^2}] + (2.0)^2e^{-(2.0)^2}\Big] = 0.4227 \end{aligned}$$

34. Use the composite Trapezoidal and the Simpson's rules to approximate the integral $\int_1^2 \frac{dx}{4x+1}$ such that the error does not exceed 10^{-2}.

Solution: Given $f(x) = \frac{dx}{4x+1}$ $a = 1, b = 2$, and $\epsilon = 10^{-2}$, then using Trapezoidal and Simpson's rules for the given accuracy, we need $n = 2$. Thus for $n = 2$, we get the approximations from both rules as follows:

$$T_2(f) = 0.1492 \qquad \text{and} \quad S_2(f) = 0.1471$$

35. Evaluate $\int_0^1 e^{x^2} dx$ by the Simpson's rule choosing h small enough to guarantee five decimal accuracy. How large can h be ?

Solution: Given $f(x) = e^{x^2}$, $a = 0$, $b = 1$, and $\epsilon = 10^{-5}$, then using the error formula of the Simpson's rule, we have

$$|E| \leq \frac{1}{180}h^4 M \leq 10^{-5}$$

where

$$M = \max_{0.0 \leq x \leq 1.0} |4e^{x^2}(3 + 16x^2 + 4x^4)| = 206.5894$$

and $f^{(4)}(x) = 4e^{x^2}(3 + 16x^2 + 4x^4)$. So

$$\frac{1}{180}h^4(206.5894) \leq 10^{-5}$$

and solving for h, we get

$$h^4 \leq 8.7129 \times 10^{-6}, \quad \text{gives} \quad h \leq 0.0543$$

Thus for $h = 0.05$ and $n = 20$, we have approximate value of the given integral using Simpson's rule is

$$S_{20} = 1.4627$$

36. Use a suitable composite integration rule to find the best approximate value of the integral $\int_1^2 \sqrt{1+\sin x}\, dx$, with $h = 0.1$. Estimate the error bound.

Solution: Given $f(x) = \sqrt{1+\sin x},\ a = 1,\ b = 2,\ h = 0.1$, then for $n = 10$, the suitable composite integration rule is the Simpson's rule. Using Simpson's rule for $h = 0.1$, we get

$$S_{10}(f) = 1.3987$$

To compute the error bound, we use

$$|E| \leq \frac{1}{180} h^4 M$$

where

$$M = \max_{1.0 \leq x \leq 2.0} |f^{(4)}(x)| = 0.1037$$

So

$$|E| \leq \frac{1}{180}(0.1)^4(0.1037) = 5.7611 \times 10^{-8}$$

37. Evaluate the integral $\int_1^2 \ln x\, dx$ using closed Newton-Cotes formulas for $n = 3$ and $n = 4$. Also, compute the error bounds for your approximations.

Solution: Given $f(x) = \ln x,\ a = 1,\ b = 2$, then for $n = 3$ (Simpson's 3/8 rule), $h = 1/3$, we have

$$\begin{aligned} I_{SR} &= \frac{3h}{8}\Big[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\Big] \\ &= \frac{1}{8}\Big[f(1) + 3f(4/3) + 3f(5/3) + f(2)\Big] \\ &= \frac{1}{8}\Big[\ln 1 + 3\ln 4/3 + 3\ln 5/3 + \ln 2\Big] \\ &= 0.3861 \end{aligned}$$

To compute the error bound for the approximation, we have

$$|E| \leq \frac{3}{80}(1/3)^5 M$$

where

$$M = \max_{1.0 \leq x \leq 2.0} |-6/x^4| = 6.0$$

and $f^{(4)}(x) = -6/x^4$. So

$$|E| \leq \frac{3}{80}(1/3)^5(6.0) = 0.0009$$

Now for $n = 4$(Boole's rule) and $h = 1/4$, we have

$$\begin{aligned}
I_B &= \frac{2h}{45}\Big[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\Big] \\
&= \frac{1}{90}\Big[7f(1) + 32f(5/4) + 12f(6/4) + 32f(7/4) + 7f(2)\Big] \\
&= \frac{1}{90}\Big[7\ln 1 + 32\ln 5/4 + 12\ln 3/2 + 32\ln 7/4 + 7\ln 2\Big] \\
&= 0.3862
\end{aligned}$$

To compute the error bound for the approximation, we have

$$|E| \leq \frac{8}{945}(1/4)^7 M$$

where

$$M = \max_{1.0 \leq x \leq 2.0} |-120/x^6| = 120$$

and $f^{(6)}(x) = -120/x^6$. So

$$|E| \leq \frac{8}{945}(1/4)^7(120) = 0.00006$$

38. Evaluate the following integrals using closed Newton-Cotes formulas for $n = 4, 5$, and 6. Also, compute the error bounds for your approximations:

$$(\mathbf{a}) \quad \int_0^1 (x + e^{2x})\, dx \quad (\mathbf{b}) \quad \int_0^{\pi/2} \cos x\, dx \quad (\mathbf{c}) \quad \int_2^3 (x + \ln x)\, dx.$$

Solution: (a) Given $f(x) = (x + e^{2x})$, $a = 0$, $b = 1$, then for $n = 4$ ((Boole's rule)), $h = 1/4$, we have

$$\begin{aligned}
I_B &= \frac{2h}{45}\Big[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\Big] \\
&= \frac{1}{90}\Big[7f(0) + 32f(1/4) + 12f(2/4) + 32f(3/4) + 7f(1)\Big] \\
&= \frac{1}{90}\Big[7(1) + 32(1/4 + e^{1/2}) + 12(1/2 + e^1) + 32(3/4 + e^{3/2}) + 7(1 + e^2)\Big] \\
&= 3.69462
\end{aligned}$$

To compute the error bound for the approximation, we have

$$|E| \leq \frac{8}{945}(1/4)^7 M$$

where

$$M = \max_{0.0 \leq x \leq 1.0} |64e^{2x}| = 472.8996$$

and $f^{(6)}(x) = 64e^{2x}$. So

$$|E| \leq \frac{8}{945}(1/4)^7(472.8996) = 0.00024$$

For $n = 5$ ((six-point rule)), $h = 1/5$, we have

$$\begin{aligned}
I_{sp} &= \frac{5h}{288}\Big[19f(x_0) + 75f(x_1) + 50f(x_2) + 50f(x_3) + 75f(x_4) + 19f(x_5)\Big] \\
&= \frac{1}{288}\Big[19(f(0) + 75f(1/5) + 50f(2/5) + 50f(3/5) + 75f(4/5) + 19f(1)\Big] \\
&= \frac{1}{288}\Big[19(1) + 75(1/5 + e^{2/5}) + 50(2/5 + e^{4/5}) + 50(3/5 + e^{6/5}) \\
&+ 75(4/5 + e^{8/5}) + 19(1 + e^2)\Big] \\
&= 3.69458
\end{aligned}$$

To compute the error bound for the approximation, we have

$$|E| \leq \frac{275}{12096}(1/5)^7 M$$

where

$$M = \max_{0.0 \leq x \leq 1.0} |64e^{2x}| = 472.8996$$

and $f^{(6)}(x) = 64e^{2x}$. So

$$|E| \leq \frac{275}{12096}(1/5)^7(472.8996) = 0.00014$$

39. Evaluate the integral $\int_0^2 \sqrt{x+1}\, dx$ using the Weddle's rule and compute the error bound for your approximation.

Solution: Given $f(x) = \sqrt{x+1}$, $a = 0$, $b = 2$, then for $n = 6$, $h = 1/3$, we have

$$\begin{aligned}
I_W &= \frac{3h}{10}\Big[(f(x_0) + f(x_2) + f(x_4) + f(x_6)) + 5(f(x_1) + f(x_5)) + 6f(x_3)\Big] \\
&= \frac{1}{10}\Big[(f(0) + f(2/3) + f(4/3) + f(2)) \\
&+ 5(f(1/3) + f(5/3)) + 6f(1)\Big] \\
&= \frac{1}{10}\Big[(\sqrt{0+1} + \sqrt{2/3+1} + \sqrt{4/3+1} + \sqrt{2+1}) \\
&+ 5(\sqrt{1/3+1} + \sqrt{5/3+1}) + 6\sqrt{1+1}\Big] \\
&= 2.7974
\end{aligned}$$

To compute the error bound for the approximation, we have

$$|E| \leq \frac{1}{140}(1/3)^7 M$$

where

$$M = \max_{0.0 \leq x \leq 2.0} |-945/64(x+1)^{11/2}| = 14.765625$$

and $f^{(6)}(x) = -945/64(x+1)^{11/2}$. So

$$|E| \leq \frac{1}{140}(1/3)^7(14.765625) = 0.00005$$

40. Evaluate the following integrals using closed Newton-Cotes formulas for $n = 1, 2, 3, 4, 5,$ and 6. Also, compute the error bounds for your approximations:

(**a**) $\int_0^1 (x^2 + \sqrt{x+1})\ dx$ (**b**) $\int_0^{\pi/2} (\sin x - x)\ dx$ (**c**) $\int_1^2 (\sqrt{x+1} + \ln x)\ dx$.

Solution: **(a)** Given $f(x) = x^2 + \sqrt{x+1},\ a = 0,\ b = 1$, then for $n = 3$ (Simpson's 3/8 rule), $h = 1/3$, we have

$$\begin{aligned} I_{SR} &= \frac{3h}{8}\Big[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\Big] \\ &= \frac{1}{8}\Big[f(0) + 3f(1/3) + 3f(2/3) + f(1)\Big] \\ &= \frac{1}{8}\Big[(1 + 3((1/3)^2 + \sqrt{1/3+1}) + 3((2/3)^2 + \sqrt{2/3+1}) + (1+\sqrt{2})\Big] \\ &= 1.55223 \end{aligned}$$

To compute the error bound for the approximation, we have

$$|E| \leq \frac{3}{80}(1/3)^5 M$$

where

$$M = \max_{0.0 \leq x \leq 1.0} |-15/16(x+1)^{7/2}| = 0.9375$$

and $f^{(4)}(x) = -15/16(x+1)^{7/2}$. So

$$|E| \leq \frac{3}{80}(1/3)^5(0.9375) = 0.00015$$

Now for $n = 4$(Boole's rule) and $h = 1/4$, we have

$$\begin{aligned} I_B &= \frac{2h}{45}\Big[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\Big] \\ &= \frac{1}{90}\Big[7f(0) + 32f(1/4) + 12f(2/4) + 32f(3/4) + 7f(1)\Big] \\ &= \frac{1}{90}\Big[7(1) + 32((1/4)^2 + \sqrt{1/4+1}) + 12((1/2)^2 + \sqrt{1/2+1}) + 32((3/4)^2 \\ &+ \sqrt{3/4+1}) + 7(1+\sqrt{2})\Big] \\ &= 1.39134 \end{aligned}$$

To compute the error bound for the approximation, we have

$$|E| \leq \frac{8}{945}(1/4)^7 M$$

where

$$M = \max_{0.0 \leq x \leq 1.0} |-945/64(x+1)^{11/2}| = 14.7656$$

and $f^{(6)}(x) = -945/64(x+1)^{11/2}$. So

$$|E| \leq \frac{8}{945}(1/4)^7(14.7656) = 0.000008$$

41. Evaluate the integral $\int_0^1 \sqrt{x+4}\, dx$ using open Newton-Cotes formulas for $n = 0, 1$, and $n = 2$.

Solution: Given $f(x) = \sqrt{x+4},\ a = 0,\ b = 1$, then for $n = 0,\ h = 0.5$, we have

$$I_{M0} = 2hf(x_0) = 2(0.5)[\sqrt{0.5+4}] = 2.1213$$

For $n = 1,\ h = 1/3$, we have

$$\begin{aligned} I_{M1} &= \frac{3h}{2}\Big[f(x_0) + f(x_1)\Big] = \frac{1}{2}\Big[\sqrt{1/3+4} + \sqrt{2/3+4}\Big] \\ &= 2.1210 \end{aligned}$$

For $n = 2,\ h = 1/4$, we have

$$\begin{aligned} I_{M2} &= \frac{4h}{3}[2f(x_0) - f(x_1) + 2f(x_2)] \\ &= \frac{1}{3}[2f(0.25) - f(0.5) + 2f(0.75)] \\ &= \frac{1}{3}[2\sqrt{0.25+4} - \sqrt{0.5+4} + 2\sqrt{0.75+4}] = 2.1202 \end{aligned}$$

42. Evaluate the following integrals using open Newton-Cotes formulas for $n = 0, 1, 2, 3, 4$, and 5. Also, compute the error bounds for your approximations:

$$\textbf{(a)} \quad \int_1^2 (x + e^x)\, dx \quad \textbf{(b)} \quad \int_0^\pi (x + \sin x)\, dx \quad \textbf{(c)} \quad \int_2^3 \ln(x^2+4)\, dx.$$

Solution: Given $f(x) = (x + e^x),\ a = 1,\ b = 2$, then for $n = 0,\ h = 0.5$, we have

$$I_{M0} = 2hf(x_0) = 2(0.5)[1.5 + e^{1.5}] = 5.9817$$

To compute the error bound for the approximation, we have

$$|E| \leq \frac{1}{3}(0.5)^3 M$$

where

$$M = \max_{1.0 \leq x \leq 2.0} |e^x| = 7.3891$$

and $f''(x) = e^x$. So

$$|E| \leq \frac{1}{3}(0.5)^3(7.3891) = 0.3079$$

For $n = 1,\ h = 1/3$, we have

$$\begin{aligned} I_{M1} &= \frac{3h}{2}\Big[f(x_0) + f(x_1)\Big] \\ &= \frac{1}{2}\Big[(4/3 + e^{4/3}) + (5/3 + e^{5/3})\Big] \\ &= 6.0441 \end{aligned}$$

To compute the error bound for the approximation, we have

$$|E| \leq \frac{3}{4}(1/3)^3 M$$

where

$$M = \max_{1.0 \leq x \leq 2.0} |e^x| = 7.3891$$

and $f''(x) = e^x$. So

$$|E| \leq \frac{3}{4}(1/3)^3(7.3891) = 0.2053$$

For $n = 2,\ h = 1/4$, we have

$$\begin{aligned} I_{M2} &= \frac{4h}{3}[2f(x_0) - f(x_1) + 2f(x_2)] \\ &= \frac{1}{3}[2f(1.25) - f(1.5) + 2f(1.75)] \\ &= \frac{1}{3}[2(1.25 + e^{1.25}) - (1.5 + e^{1.5}) + 2(1.75 + e^{1.75}] = 6.1694 \end{aligned}$$

To compute the error bound for the approximation, we have

$$|E| \leq \frac{14}{45}(1/4)^5 M$$

where

$$M = \max_{1.0 \leq x \leq 2.0} |e^x| = 7.3891$$

and $f^{(4)}(x) = e^x$. So

$$|E| \leq \frac{14}{45}(1/4)^3(7.3891) = 0.0023$$

For $n = 3,\ h = 1/5$, we have

$$\begin{aligned} I_{M3} &= \frac{5h}{24}[11f(x_0) + f(x_1) + f(x_2) + 11f(x_3)] \\ &= \frac{1}{24}[11f(1.2) + f(1.4) + f(1.6) + 11f(1.8)] \\ &= \frac{1}{24}[11(1.2 + e^{1.2}) + (1.4 + e^{1.4}) + (1.6 + e^{1.6} + (1.8 + e^{1.8})] \\ &= 6.1698 \end{aligned}$$

To compute the error bound for the approximation, we have

$$|E| \leq \frac{95}{144}(1/5)^5 M$$

where

$$M = \max_{1.0 \leq x \leq 2.0} |e^x| = 7.3891$$

and $f^{(4)}(x) = e^x$. So

$$|E| \leq \frac{95}{144}(1/5)^3(7.3891) = 0.0016$$

For $n = 4,\ h = 1/6$, we have

$$\begin{aligned} I_{M4} &= (6h/20)[f(x_0) + f(x_2) + f(x_4)] \\ &= (1/20)[11(f(7/6) + f(11/6)) - 14(f(8/6) + f(10/6)) + 26f(9/6)] \\ &= (1/20)[11[(7/6 + e^{7/6}) + (11/6 + e^{11/6})] - 14[(8/6 + e^{8/6}) + (10/6 + e^{10/6})] \\ &+ \ 26(9/6 + e^{9/6})] = 6.1707 \end{aligned}$$

To compute the error bound for the approximation, we have

$$|E| \leq \frac{41}{144}(1/6)^7 M$$

where

$$M = \max_{1.0 \leq x \leq 2.0} |e^x| = 7.3891$$

and $f^{(6)}(x) = e^x$. So

$$|E| \leq \frac{41}{144}(1/6)^7(7.3891) = 0.000008$$

43. Evaluate the integral $\int_1^2 \ln(x+1)\ dx$ using the four-point open Newton-Cotes formula and compute the error bound for your approximation.

Solution: Given $f(x) = \ln(x+1),\ a = 1,\ b = 2,\ n = 4$, then for h = 1/6, we have

$$\begin{aligned}
I_{M4} &= (6h/20)[f(x_0) + f(x_2) + f(x_4)] \\
&= (1/20)[11(f(7/6) + f(11/6)) - 14(f(8/6) + f(10/6)) + 26f(9/6)] \\
&= (1/20)[11(\ln(7/6+1) + \ln(11/6+1)) - 14(\ln(8/6+1) + \ln(10/6+1)) \\
&+ 26\ln(9/6+1)] = 0.90954304
\end{aligned}$$

To compute the error bound for the approximation, we have

$$|E| \leq \frac{41}{140}(1/6)^7 M$$

where

$$M = \max_{1.0 \leq x \leq 2.0} |-120/(x+1)^6| = 1.8750$$

and $f^{(6)}(x) = -120/(x+1)^6$. So

$$|E| \leq \frac{41}{140}(1/6)^7(1.8750) = 0.000002$$

Chapter 6 Ordinary Differential Equations

1. Find the general solution of the differential equation

$$y' = \frac{x}{y}$$

Solution: Since the given differential equation is separable, therefore, we can write in the form

$$\frac{dy}{dx} = \frac{x}{y} \quad \text{or} \quad ydy = xdx$$

from which it follows that

$$\begin{aligned} \int ydy &= \int xdx \\ \frac{y^2}{2} &= \frac{x^2}{2} + c_1 \\ y^2 &= x^2 + c \end{aligned}$$

2. Put the following differential equations into a form for numerical solution by the Euler's method.
(a) $y + 2yy' - y' = 0$
(b) $\ln y' = x^2 - y^2$
(c) $y' - x^2 y' = y$

Solution:
(a) $y' = f(x,y) = \dfrac{y}{1-2y}$

(b) $y' = f(x,y) = e^{x^2-y^2}$

(c) $y' = f(x,y) = \dfrac{y}{1-x^2}$

3. Solve the following initial-value problems using the Euler's method.
(a) $y' = y + x^2, \quad x = 0(0.2)1, \quad y(0) = 1.$
(b) $y' = (x-1)(x+y+1), \quad x = 0(0.2)1, \quad y(0) = 1.$
(c) $y' = y + \sin(x), \quad x = 0.2(0.01)0.25, \quad y(0.2) = 1.5.$

Solution: **(a)** Since $f(x,y) = y + x^2$, and $x_0 = 0,\ y_0 = 1$, then

$$y_{i+1} = y_i + hf(x_i, y_i), \quad \text{for} \quad i = 0, 1, \ldots, 4$$

For $h = 0.2$ and taking $i = 0$, we have

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h(y_0 + x_0^2) = 1 + (0.2)[1 + 0]$$

which gives

$$y_1 = 1.2000$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4,$ are as follows

x_i	y_i
0.2000	1.2000
0.4000	1.4480
0.6000	1.7696
0.8000	2.1955
1.0000	2.7626

(b) Since $f(x, y) = (x-1)(x+y+1)$, and $x_0 = 0,\ y_0 = 1$, then

$$y_{i+1} = y_i + hf(x_i, y_i), \quad \text{for} \quad i = 0, 1, \ldots, 4$$

For $h = 0.2$ and taking $i = 0$, we have

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h[(x_0 - 1)(x_0 + y_0 + 1)] = 1 + (0.2)[(0-1)(0+1+1)]$$

which gives

$$y_1 = 0.6000$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4,$ are as follows

x_i	y_i
0.2000	0.6000
0.4000	0.3120
0.6000	0.1066
0.8000	−0.0300
1.0000	−0.1008

(c) Since $f(x, y) = y + \sin(x)$, and $x_0 = 0.2,\ y_0 = 1.5$, then

$$y_{i+1} = y_i + hf(x_i, y_i), \quad \text{for} \quad i = 0, 1, \ldots, 4$$

For $h = 0.01$ and taking $i = 0$, we have

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h[y_0 + \sin(x_0)] = 1.5 + (0.01)[1.5 + 0.1987]$$

which gives

$$y_1 = 1.5170$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4,$ are as follows

x_i	y_i
0.2100	1.5170
0.2200	1.5342
0.2300	1.5518
0.2400	1.5696
0.2500	1.5876

4. Solve the following initial-value problems and compare the numerical solutions obtained with the Euler's method using the values of $h = 0.1$ and $h = 0.2$. Compare the results to the actual values.

(a) $y' = 1 + x^2, \quad 0 \leq x \leq 1, \quad y(0) = 0, \quad y(x) = \tan x.$

(b) $y' = 2(y + 1), \quad 0 \leq x \leq 1, \quad y(0) = 0, \quad y(x) = e^{2x} - 1.$

(c) $y' = 2(y - 1)^2, \; 1 \leq x \leq 2, \quad y(1) = 0.5, \; y(x) = (2x - 1)/2x.$

Solution: **(a)** Since $f(x, y) = 1 + x^2$, and $x_0 = 0, \; y_0 = 0$, then

$$y_{i+1} = y_i + hf(x_i, y_i), \quad \text{for} \quad i = 0, 1, \ldots, 9$$

For $h = 0.1$ and taking $i = 0$, we have

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h(1 + x_0^2) = 0 + (0.1)[1 + 0]$$

which gives

$$y_1 = 0.1000$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 9$, are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
0.1000	0.1000	0.1003	0.0003
0.2000	0.2010	0.2027	0.0017
0.3000	0.3050	0.3093	0.0043
0.4000	0.4140	0.4228	0.0088
0.5000	0.5300	0.5463	0.0163
0.6000	0.6550	0.6841	0.0291
0.7000	0.7910	0.8423	0.0513
0.8000	0.9400	1.0296	0.0896
0.9000	1.1040	1.2602	0.1562
1.0000	1.2850	1.5574	0.2724

For $h = 0.2$ and taking $i = 0$, we have

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h(1 + x_0^2) = 0 + (0.2)[1 + 0]$$

which gives

$$y_1 = 0.2000$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4$, are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
0.2000	0.2000	0.2027	0.0027
0.4000	0.4080	0.4228	0.0148
0.6000	0.6400	0.6841	0.0441
0.8000	0.9120	1.0296	0.1176
1.0000	1.2400	1.5574	0.3174

(b) Since $f(x, y) = 2(y + 1)$, and $x_0 = 0, \; y_0 = 0$, then

$$y_{i+1} = y_i + hf(x_i, y_i), \quad \text{for} \quad i = 0, 1, \ldots, 9$$

For $h = 0.1$ and taking $i = 0$, we have

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h(2y_0 + 2) = 0 + (0.1)[0 + 2]$$

which gives

$$y_1 = 0.2000$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 9$, are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
0.1000	0.2000	0.2214	0.0214
0.2000	0.4400	0.4918	0.0518
0.3000	0.7280	0.8221	0.0941
0.4000	1.0736	1.2255	0.1519
0.5000	1.4883	1.7183	0.2300
0.6000	1.9860	2.3201	0.3341
0.7000	2.5832	3.0552	0.4720
0.8000	3.2998	3.9530	0.6532
0.9000	4.1598	5.0496	0.8899
1.0000	5.1917	6.3891	1.1973

For $h = 0.2$ and taking $i = 0$, we have

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h(2y_0 + 2) = 0 + (0.2)[0 + 2]$$

which gives

$$y_1 = 0.4000$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4$, are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
0.2000	0.4000	0.4918	0.0918
0.4000	0.9600	1.2255	0.2655
0.6000	1.7440	2.3201	0.5761
0.8000	2.8416	3.9530	1.1114
1.0000	4.3782	6.3891	2.0108

(c) Since $f(x, y) = 2(y - 1)^2$, and $x_0 = 1,\ y_0 = 0.5$, then

$$y_{i+1} = y_i + hf(x_i, y_i), \quad \text{for} \quad i = 0, 1, \ldots, 9$$

For $h = 0.1$ and taking $i = 0$, we have

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + 2h[(y_0 - 2)^2] = 0.5 + (0.2)[(0.5 - 1)^2]$$

which gives

$$y_1 = 0.2000$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 9,$ are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
1.1000	0.5500	0.5455	−0.0045
1.2000	0.5905	0.5833	−0.0072
1.3000	0.6240	0.6154	−0.0087
1.4000	0.6523	0.6429	−0.0095
1.5000	0.6765	0.6667	−0.0098
1.6000	0.6974	0.6875	−0.0099
1.7000	0.7157	0.7059	−0.0098
1.8000	0.7319	0.7222	−0.0097
1.9000	0.7463	0.7368	−0.0094
2.0000	0.7591	0.7500	−0.0091

For $h = 0.2$ and taking $i = 0$, we have

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + 2h[(y_0 - 2)^2] = 0.5 + (0.4)[(0.5 - 1)^2]$$

which gives

$$y_1 = 0.6000$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4,$ are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
1.2000	0.6000	0.5833	−0.0167
1.4000	0.6640	0.6429	−0.0211
1.6000	0.7092	0.6875	−0.0217
1.8000	0.7430	0.7222	−0.0208
2.0000	0.7694	0.7500	−0.0194

5. Solve the following initial-value problems using the Taylor's method of order two.

(a) $y' = 2x^2 - y, \quad x = 0(0.2)1, \quad y(0) = -1.$

(b) $y' = 3x^2 y, \quad x = 0(0.2)1, \quad y(0) = 1.$

(c) $y' = x/y - x, \quad x = 0(0.2)1, \; y(0) = 2.$

Solution: (a) Since $f(x, y) = 2x^2 - y, \; f'(x, y) = 4x - 2x^2 + y$, and $x_0 = 0, \; y_0 = -1$, then using second-order Taylor's method

$$y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2} f'(x_i, y_i), \quad \text{for} \quad i = 0, 1, 2, 3, 4$$

for $h = 0.2$ and taking $i = 0$, we have

$$y_1 = y_0 + hf(x_0, y_0) + \frac{h^2}{2} f'(x_0, y_0) = y_0 + 2h[2x_0^2 - y_0] + \frac{h^2}{2}(4x_0 - 2x_0^2 + y_0)$$

which gives

$$y_1 = -1 + (0.2)[0 + 1] + 0.02[0 - 0 - 1] = -0.8200$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4$, are as follows

x_i	y_i
0.2000	−0.8200
0.4000	−0.6420
0.6000	−0.4368
0.8000	−0.1806
1.0000	0.1463

(b) Since $f(x,y) = 3x^2y,\ f'(x,y) = 6xy + 9x^4y$, and $x_0 = 0,\ y_0 = 1$, then using

$$y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2}f'(x_i, y_i), \quad \text{for} \quad i = 0, 1, 2, 3, 4$$

for $h = 0.2$ and taking $i = 0$, we have

$$y_1 = y_0 + hf(x_0, y_0) + \frac{h^2}{2}f'(x_0, y_0) = y_0 + 2h[3x_0^2y_0] + \frac{h^2}{2}[6x_0y_0 + 9x_0^4y_0]$$

which gives

$$y_1 = 1 + (0.2)[0] + 0.02[0 + 0] = 1.0000$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4$, are as follows

x_i	y_i
0.2000	1.0000
0.4000	1.0483
0.6000	1.2041
0.8000	1.5789
1.0000	2.4532

(c) Since $f(x,y) = x/y - x,\ f'(x,y) = 1/y - x^2/y^3 + x^2/y^2 - 1$, and $x_0 = 0,\ y_0 = 2$, then using

$$y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2}f'(x_i, y_i), \quad \text{for} \quad i = 0, 1, 2, 3, 4$$

for $h = 0.2$ and taking $i = 0$, we have

$$y_1 = y_0 + hf(x_0, y_0) + \frac{h^2}{2}f'(x_0, y_0) = y_0 + 2h[x_0/y_0 - x_0] + \frac{h^2}{2}[1/y_0 - x_0^2/y_0^3 + x_0^2/y_0^2 - 1]$$

which gives

$$y_1 = 2 + (0.2)[0] + 0.02[1/2 - 0 + 0 - 1] = 1.9900$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4$, are as follows

x_i	y_i
0.2000	1.9900
0.4000	1.9603
0.6000	1.9117
0.8000	1.8458
1.0000	1.7651

6. Solve the initial-value problems by using the Taylor's method of order three of the Problem 3.

Solution: **(a)** Since $f(x,y) = y+x^2,\ f'(x,y) = y+x^2+2x,\ f''(x,y) = y+x^2+2x+2,$ and $x_0 = 0,\ y_0 = 1$, then using third-order Taylor's method

$$y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2} f'(x_i, y_i) + \frac{h^3}{6} f''(x_i, y_i), \quad \text{for} \quad i = 0, 1, 2, 3, 4$$

for $h = 0.2$ and taking $i = 0$, we have

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) + \frac{h^2}{2} f'(x_0, y_0) + \frac{h^3}{6} f''(x_0, y_0) \\ &= y_0 + h[y_0 + x_0^2] + \frac{h^2}{2}[y_0 + x_0^2 + 2x_0] + \frac{h^3}{6}[y_0 + x_0^2 + 2x_0 + 2] \end{aligned}$$

which gives

$$y_1 = 1 + (0.2)[1 + 0] + 0.02[1 + 0 + 0] + 0.0013[1 + 0 + 0 + 2] = 1.2240$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4$, are as follows

x_i	y_i
0.2000	1.2240
0.4000	1.5150
0.6000	1.9054
0.8000	2.4351
1.0000	3.1525

(b) Since

$$\begin{aligned} &f(x,y) = (x-1)(x+y+1) \\ &f'(x,y) = 2x + y + (x-1)^2(x+y+1) \\ &f''(x,y) = 2 + 3(x-1)(x+y+1) + (x-1)^2(1 + (x-1)(x+y+1)) \end{aligned}$$

and $x_0 = 0,\ y_0 = 1$, then using

$$y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2} f'(x_i, y_i) + \frac{h^3}{6} f''(x_i, y_i), \quad \text{for} \quad i = 0, 1, 2, 3, 4$$

for $h = 0.2$ and taking $i = 0$, we have

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) + \frac{h^2}{2} f'(x_0, y_0) + \frac{h^3}{6} f''(x_0, y_0) \\ &= y_0 + h[(x_0 - 1)(x_0 + y_0 + 1)] + \frac{h^2}{2}[2x_0 + y_0 + (x_0 - 1)^2(x_0 + y_0 + 1)] \\ &+ \frac{h^3}{6}[2 + 3(x_0 - 1)(x_0 + y_0 + 1) + (x_0 - 1)^2(1 + (x_0 - 1)(x_0 + y_0 + 1))] \end{aligned}$$

which gives

$$y_1 = 1 + (0.2)(-2) + 0.02(3) + 0.00133(-5) = 0.6533$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4,$ are as follows

x_i	y_i
0.2000	0.6533
0.4000	0.3979
0.6000	0.2174
0.8000	0.1060
1.0000	0.0666

(c) Since $f(x, y) = y + sin(x),\ f'(x, y) = y + sin(x) + cos(x),\ f''(x, y) = y + cos(x)$, and $x_0 = 0.2,\ y_0 = 1.5$, then using

$$y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2} f'(x_i, y_i) + \frac{h^3}{6} f''(x_i, y_i), \quad \text{for} \quad i = 0, 1, 2, 3, 4$$

for $h = 0.01$ and taking $i = 0$, we have

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) + \frac{h^2}{2} f'(x_0, y_0) + \frac{h^3}{6} f''(x_0, y_0) \\ &= y_0 + h[y_0 + sin(x_0)] + \frac{h^2}{2}[y_0 + sin(x_0) + cos(x_0)] + \frac{h^3}{6}[y_0 + cos(x_0)] \end{aligned}$$

which gives

$$\begin{aligned} y_1 &= 1.5 + (0.01)[1.5 + 0.1987] + 0.00005[1.5 + 0.1987 + 0.9801] \\ &+ 0.0000002[1.5 + 0.9801] = 1.5171 \end{aligned}$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4,$ are as follows

x_i	y_i
0.2100	1.5171
0.2200	1.5345
0.2300	1.5522
0.2400	1.5701
0.2500	1.5883

7. Solve the following initial-value problems using the Modified Euler's method.

(a) $y' = y^2x^2, \quad x = 1(0.2)2, \quad y(1) = -1.$
(b) $y' = x - y/2x, \quad x = 1(0.02)1.10, \quad y(1) = 0.25.$
(c) $y' = 1/y^2 - yx, \quad x = 1(0.2)2, \quad y(1) = 1.$

Solution: (a) Since $f(x, y) = y^2x^2$, and $x_0 = 1,\ y_0 = -1,\ h = 0.2$, then for $i = 0$, we have

$$\begin{aligned} k_1 &= f(x_0, y_0) = (y_0^2 x_0^2) = 1.0000 \\ k_2 &= f(x_1, y_0 + hk_1) = ((y_0 + hk_1)^2 x_1^2) = (-0.8)^2(1.2)^2 = 0.9216 \end{aligned}$$

Now using these values in the Modified formula, we have

$$y_1 = y_0 + \frac{h}{2}[k_1 + k_2] = -1 + 0.1(1 + 0.9216) = -0.8078$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4,$ are as follows

x_i	y_i
1.2000	−0.8078
1.4000	−0.6385
1.6000	−0.5000
1.8000	−0.3911
2.0000	−0.3075

(b) Since $f(x,y) = x - y/2x$, and $x_0 = 1,\ y_0 = 0.25,\ h = 0.2$, then for $i = 0$, we have

$$\begin{aligned} k_1 &= f(x_0, y_0) = (x_0 - y_0/2x_0) = 0.8750 \\ k_2 &= f(x_1, y_0 + hk_1) = (x_1 - (y_0 + hk_1)/2x_1) = 1.02 - 0.1311 = 0.8889 \end{aligned}$$

Now using these values in the Modified formula, we have

$$y_1 = y_0 + \frac{h}{2}[k_1 + k_2] = 0.25 + 0.01(0.8750 + 0.8889) = 0.2676$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4,$ are as follows

x_i	y_i
1.0200	0.2676
1.0400	0.2856
1.0600	0.3037
1.0800	0.3222
1.1000	0.3410

(c) Since $f(x,y) = 1/y^2 - yx$, and $x_0 = 1,\ y_0 = 1,\ h = 0.2$, then for $i = 0$, we have

$$\begin{aligned} k_1 &= f(x_0, y_0) = (1/y_0^2 - y_0 x_0) = 0 \\ k_2 &= f(x_1, y_0 + hk_1) = (1/(y_0 + hk_1)^2 - (y_0 + hk_1)x_1) = 1 - 1.2 = -0.2 \end{aligned}$$

Now using these values in the Modified formula, we have

$$y_1 = y_0 + \frac{h}{2}[k_1 + k_2] = 1 + 0.1(0 - 0.2) = 0.9800$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4,$ are as follows

x_i	y_i
1.2000	0.9800
1.4000	0.9432
1.6000	0.9013
1.8000	0.8611
2.0000	0.8252

8. Solve the following initial-value problems and compare the numerical solutions obtained with the Modified Euler's method using the values of $h = 0.05$ and $h = 0.1$ and compare the results with the actual values.

(a) $y' = x + \dfrac{3y}{x}, \quad 1 \leq x \leq 2, \quad y(1) = 0, \quad y(x) = x^3 - x^2.$

(b) $y' = \sqrt{y}, \quad 0 \leq x \leq 1, \quad y(0) = 1, \quad y(x) = (x+2)^2/4.$

(c) $y' = 4 - 3y, \quad 0 \leq x \leq 1, \quad y(0) = 5, \quad y(x) = 4/3 + 11/3e^{-3x}.$

Solution: **(a)** Since $f(x, y) = x + 3y/x$, and $x_0 = 1, \; y_0 = 0, \; h = 0.05$, then for $i = 0$, we have

$$\begin{array}{rcl} k_1 & = & f(x_0, y_0) = (x_0 + (3y_0)/x_0) = 1 \\ k_2 & = & f(x_1, y_0 + hk_1) = (x_1 + 3(y_0 + hk_1)/x_1) = 1.05 + 0.4129 = 1.1929 \end{array}$$

Now using these values in the Modified formula, we have

$$y_1 = y_0 + \frac{h}{2}\left[k_1 + k_2\right] = 0 + 0.025(1 + 1.1929) = 0.0548$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 19$, and the possible errors, are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
1.0500	0.0548	0.0551	0.0003
1.1000	0.1203	0.1210	0.0007
1.1500	0.1973	0.1984	0.0011
1.2000	0.2864	0.2880	0.0016
1.2500	0.3885	0.3906	0.0021
1.3000	0.5043	0.5070	0.0027
1.3500	0.6345	0.6379	0.0034
1.4000	0.7799	0.7840	0.0041
1.4500	0.9412	0.9461	0.0049
1.5000	1.1192	1.1250	0.0058
1.5500	1.3146	1.3214	0.0068
1.6000	1.5282	1.5360	0.0078
1.6500	1.7607	1.7696	0.0089
1.7000	2.0128	2.0230	0.0102
1.7500	2.2854	2.2969	0.0115
1.8000	2.5792	2.5920	0.0128
1.8500	2.8948	2.9091	0.0143
1.9000	3.2331	3.2490	0.0159
1.9500	3.5948	3.6124	0.0176
2.0000	3.9806	4.0000	0.0194

Similarly, for $h = 0.1$, we get

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
1.1000	0.1186	0.1210	0.0024
1.2000	0.2824	0.2880	0.0056
1.3000	0.4973	0.5070	0.0097
1.4000	0.7692	0.7840	0.0148
1.5000	1.1040	1.1250	0.0210
1.6000	1.5077	1.5360	0.0283
1.7000	1.9862	2.0230	0.0368
1.8000	2.5453	2.5920	0.0467
1.9000	3.1910	3.2490	0.0580
2.0000	3.9293	4.0000	0.0707

(b) Since $f(x, y) = \sqrt{y}$, and $x_0 = 0,\ y_0 = 1,\ h = 0.05$, then for $i = 0$, we have

$$\begin{aligned} k_1 &= f(x_0, y_0) = \sqrt{y_0} = 1 \\ k_2 &= f(x_1, y_0 + hk_1) = \sqrt{y_0 + hk_1} = \sqrt{1.05} = 1.0247 \end{aligned}$$

Now using these values in the Modified formula, we have

$$y_1 = y_0 + \frac{h}{2}[k_1 + k_2] = 1 + 0.025(1 + 1.0247) = 1.0506$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 19$, and the possible errors, are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
0.0500	1.0506	1.0506	0.0000
0.1000	1.1025	1.1025	0.0000
0.1500	1.1556	1.1556	0.0000
0.2000	1.2100	1.2100	0.0000
0.2500	1.2656	1.2656	0.0000
0.3000	1.3225	1.3225	0.0000
0.3500	1.3806	1.3806	0.0001
0.4000	1.4399	1.4400	0.0001
0.4500	1.5006	1.5006	0.0001
0.5000	1.5624	1.5625	0.0001
0.5500	1.6255	1.6256	0.0001
0.6000	1.6899	1.6900	0.0001
0.6500	1.7555	1.7556	0.0001
0.7000	1.8224	1.8225	0.0001
0.7500	1.8905	1.8906	0.0001
0.8000	1.9599	1.9600	0.0001
0.8500	2.0305	2.0306	0.0001
0.9000	2.1024	2.1025	0.0001
0.9500	2.1755	2.1756	0.0001
1.0000	2.2498	2.2500	0.0002

Similarly, for $h = 0.1$, we get

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
0.1000	1.1024	1.1025	0.0001
0.2000	1.2099	1.2100	0.0001
0.3000	1.3223	1.3225	0.0002
0.4000	1.4398	1.4400	0.0002
0.5000	1.5622	1.5625	0.0003
0.6000	1.6896	1.6900	0.0004
0.7000	1.8221	1.8225	0.0004
0.8000	1.9595	1.9600	0.0005
0.9000	2.1020	2.1025	0.0005
1.0000	2.2494	2.2500	0.0006

(c) Since $f(x, y) = 4 - 3y$, and $x_0 = 0,\ y_0 = 5,\ h = 0.05$, then for $i = 0$, we have

$$\begin{aligned} k_1 &= f(x_0, y_0) = 4 - 3y_0 = -11 \\ k_2 &= f(x_1, y_0 + hk_1) = 4 - 3(y_0 + hk_1) = 4 - 3(4.45) = -9.3500 \end{aligned}$$

Now using these values in the Modified formula, we have

$$y_1 = y_0 + \frac{h}{2}[k_1 + k_2] = 5 + 0.025(-11 - 9.3500) = 4.4913$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 19$, and the possible errors, are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
0.0500	4.4913	4.4893	−0.0020
0.1000	4.0531	4.0497	−0.0034
0.1500	3.6757	3.6713	−0.0044
0.2000	3.3507	3.3456	−0.0051
0.2500	3.0708	3.0653	−0.0055
0.3000	2.8297	2.8241	−0.0056
0.3500	2.6221	2.6164	−0.0057
0.4000	2.4433	2.4377	−0.0056
0.4500	2.2893	2.2839	−0.0054
0.5000	2.1566	2.1515	−0.0052
0.5500	2.0424	2.0375	−0.0049
0.6000	1.9440	1.9394	−0.0046
0.6500	1.8593	1.8550	−0.0043
0.7000	1.7863	1.7823	−0.0040
0.7500	1.7235	1.7198	−0.0037
0.8000	1.6693	1.6660	−0.0034
0.8500	1.6227	1.6196	−0.0031
0.9000	1.5826	1.5798	−0.0028
0.9500	1.5480	1.5454	−0.0026
1.0000	1.5182	1.5159	−0.0023

Similarly, for $h = 0.1$, we get

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
0.1000	4.0650	4.0497	−0.0153
0.2000	3.3684	3.3456	−0.0228
0.3000	2.8495	2.8241	−0.0254
0.4000	2.4629	2.4377	−0.0251
0.5000	2.1748	2.1515	−0.0234
0.6000	1.9602	1.9394	−0.0208
0.7000	1.8004	1.7823	−0.0180
0.8000	1.6813	1.6660	−0.0153
0.9000	1.5926	1.5798	−0.0128
1.0000	1.5265	1.5159	−0.0106

9. Solve the following initial-value problems using the Heun's method and the Midpoint method.
(a) $y' = (x+1)y, \quad x = 0.5(0.2)1.5, \; y(0.5) = 1.$
(b) $y' = -xy^2, \quad x = 0(0.2)1, \quad y(0) = 2.$
(c) $y' = x^2 + y^2, \quad x = 1(0.2)2, \; y(1) = -1.$

Solution: (a) Given $f(x, y) = (x+1)y$, and $x_0 = 0.5, \; y_0 = 1, \; h = 0.2:$
Heun's Method: First we find the values of k_1 and k_2 by taking $i = 0$, we have

$$\begin{aligned} k_1 &= f(x_0, y_0) = (x_0 + 1)y_0 = 1.5000 \\ k_2 &= f(x_0 + \frac{2h}{3}, y_0 + \frac{2h}{3}k_1) = [(0.6333 + 1)(1.2)] = 1.9600 \end{aligned}$$

Now using these values in the Heun's formula, we get

$$y_1 = y_0 + \frac{h}{4}[k_1 + 3k_2] = 1 + 0.05(1.5000 + 3(1.9600)) = 1.3690$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4$, are as follows

x_i	y_i
0.7000	1.3690
0.9000	1.9472
1.1000	2.8765
1.3000	4.4120
1.5000	7.0236

Midpoint Method: First we find the values of k_1 and k_2 by taking $i = 0$, we have

$$\begin{aligned} k_1 &= f(x_0, y_0) = (x_0 + 1)y_0 = 1.5000 \\ k_2 &= f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1) = (1.7)(1.15) = 1.8400 \end{aligned}$$

Now using these values in the Midpoint formula, we get

$$y_1 = y_0 + hk_2 = 1 + (0.2)(1.8400) = 1.3680$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4,$ are as follows

x_i	y_i
0.7000	1.3680
0.9000	1.9442
1.1000	2.8696
1.3000	4.3974
1.5000	6.9937

(b) Given $f(x,y) = -xy^2$, and $x_0 = 0,\ y_0 = 2,\ h = 0.2$:
Heun's Method: First we find the values of k_1 and k_2 by taking $i = 0$, we have

$$\begin{aligned} k_1 &= f(x_0, y_0) = -x_0 y_0^2 = 0 \\ k_2 &= f(x_0 + \frac{2h}{3}, y_0 + \frac{2h}{3} k_1) = [-(0.1333)(4)] = -0.5333 \end{aligned}$$

Now using these values in the Heun's formula, we get

$$y_1 = y_0 + \frac{h}{4}\left[k_1 + 3k_2\right] = 2 + 0.05(0 + 3(-0.5333)) = 1.9200$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4,$ are as follows

x_i	y_i
0.2000	1.9200
0.4000	1.7172
0.6000	1.4636
0.8000	1.2156
1.0000	0.9998

Midpoint Method: First we find the values of k_1 and k_2 by taking $i = 0$, we have

$$\begin{aligned} k_1 &= f(x_0, y_0) = -x_0 y_0^2 = 0 \\ k_2 &= f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} k_1) = -(0.1)(4) = -0.4000 \end{aligned}$$

Now using these values in the Midpoint formula, we get

$$y_1 = y_0 + hk_2 = 2 + (0.2)(-0.4000) = 1.9200$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4,$ are as follows

x_i	y_i
0.2000	1.9200
0.4000	1.7155
0.6000	1.4602
0.8000	1.2117
1.0000	0.9962

(c) Given $f(x,y) = x^2 + y^2$, and $x_0 = 1,\ y_0 = -1,\ h = 0.2:$
Heun's Method: First we find the values of k_1 and k_2 by taking $i = 0$, we have

$$k_1 = f(x_0, y_0) = x_0^2 + y_0^2 = 2$$

$$k_2 = f(x_0 + \frac{2h}{3}, y_0 + \frac{2h}{3}k_1) = [(1.1333)^2 + (-0.7334)^2] = 1.8222$$

Now using these values in the Heun's formula, we get

$$y_1 = y_0 + \frac{h}{4}[k_1 + 3k_2] = -1 + 0.05(2 + 3(1.8222)) = -0.6267$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4$, are as follows

x_i	y_i
1.2000	−0.6267
1.4000	−0.2464
1.6000	0.2073
1.8000	0.8343
2.0000	1.8688

Midpoint Method: First we find the values of k_1 and k_2 by taking $i = 0$, we have

$$k_1 = f(x_0, y_0) = x_0^2 + y_0^2 = 2$$

$$k_2 = f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1) = (1.1)^2 + (-0.8)^2 = 1.8500$$

Now using these values in the Midpoint formula, we get

$$y_1 = y_0 + hk_2 = -1 + (0.2)(1.8500) = -0.6300$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4$, are as follows

x_i	y_i
1.2000	−0.6300
1.4000	−0.2522
1.6000	0.1983
1.8000	0.8183
2.0000	1.8328

10. Solve the initial-value problems in the Problem 8 and compare the numerical solutions obtained with the Heun's method and the Midpoint method.

Solution: (a) Given $f(x,y) = x + 3y/x$, and $x_0 = 1,\ y_0 = 0,\ h = 0.05:$
Heun's Method: First we find the values of k_1 and k_2 by taking $i = 0$, we have

$$k_1 = f(x_0, y_0) = x_0 + 3y_0/x_0 = 1$$

$$k_2 = f(x_0 + \frac{2h}{3}, y_0 + \frac{2h}{3}k_1) = [1.0333 + 3(0.0333)/1.0333] = 1.1301$$

Now using these values in the Heun's formula, we get

$$y_1 = y_0 + \frac{h}{4}\left[k_1 + 3k_2\right] = 0 + 0.0125(1 + 3(1.1301)) = 0.0549$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 19$, are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
1.0500	0.0549	0.0551	0.0002
1.1000	0.1205	0.1210	0.0005
1.1500	0.1975	0.1984	0.0009
1.2000	0.2868	0.2880	0.0012
1.2500	0.3889	0.3906	0.0017
1.3000	0.5048	0.5070	0.0022
1.3500	0.6352	0.6379	0.0027
1.4000	0.7807	0.7840	0.0033
1.4500	0.9422	0.9461	0.0039
1.5000	1.1204	1.1250	0.0046
1.5500	1.3160	1.3214	0.0054
1.6000	1.5298	1.5360	0.0062
1.6500	1.7625	1.7696	0.0071
1.7000	2.0149	2.0230	0.0081
1.7500	2.2878	2.2969	0.0091
1.8000	2.5818	2.5920	0.0102
1.8500	2.8977	2.9091	0.0114
1.9000	3.2364	3.2490	0.0126
1.9500	3.5984	3.6124	0.0140
2.0000	3.9846	4.0000	0.0154

Similarly, for $h = 0.1$, we have

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
1.1000	0.1191	0.1210	0.0019
1.2000	0.2835	0.2880	0.0045
1.3000	0.4991	0.5070	0.0079
1.4000	0.7720	0.7840	0.0120
1.5000	1.1080	1.1250	0.0170
1.6000	1.5132	1.5360	0.0228
1.7000	1.9933	2.0230	0.0297
1.8000	2.5544	2.5920	0.0376
1.9000	3.2024	3.2490	0.0466
2.0000	3.9432	4.0000	0.0568

Midpoint Method: First we find the values of k_1 and k_2 by taking $i = 0$, we have

$$k_1 = f(x_0, y_0) = (x_0 + 3y_0/x_0) = 1$$

$$k_2 = f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1) = 1.025 + 3(0.025)/1.025 = 1.0982$$

Now using these values in the Midpoint formula, we get

$$y_1 = y_0 + hk_2 = 0 + (0.05)(1.0982) = 0.0549$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 19$, are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
1.0500	0.0549	0.0551	0.0002
1.1000	0.1205	0.1210	0.0005
1.1500	0.1976	0.1984	0.0008
1.2000	0.2869	0.2880	0.0011
1.2500	0.3892	0.3906	0.0015
1.3000	0.5051	0.5070	0.0019
1.3500	0.6355	0.6379	0.0024
1.4000	0.7811	0.7840	0.0029
1.4500	0.9427	0.9461	0.0034
1.5000	1.1210	1.1250	0.0040
1.5500	1.3167	1.3214	0.0047
1.6000	1.5306	1.5360	0.0054
1.6500	1.7634	1.7696	0.0062
1.7000	2.0160	2.0230	0.0070
1.7500	2.2890	2.2969	0.0079
1.8000	2.5831	2.5920	0.0089
1.8500	2.8992	2.9091	0.0099
1.9000	3.2380	3.2490	0.0110
1.9500	3.6002	3.6124	0.0121
2.0000	3.9866	4.0000	0.0134

Similarly, for $h = 0.1$, we have

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
1.1000	0.1193	0.1210	0.0017
1.2000	0.2840	0.2880	0.0040
1.3000	0.5001	0.5070	0.0069
1.4000	0.7735	0.7840	0.0105
1.5000	1.1101	1.1250	0.0149
1.6000	1.5160	1.5360	0.0200
1.7000	1.9970	2.0230	0.0260
1.8000	2.5591	2.5920	0.0329
1.9000	3.2083	3.2490	0.0407
2.0000	3.9505	4.0000	0.0495

(b) Given $f(x, y) = \sqrt{y}$, and $x_0 = 0,\ y_0 = 1,\ h = 0.05$:
Heun's Method: First we find the values of k_1 and k_2 by taking $i = 0$, we have

$$\begin{aligned} k_1 &= f(x_0, y_0) = \sqrt{y_0} = 1 \\ k_2 &= f(x_0 + \frac{2h}{3}, y_0 + \frac{2h}{3}k_1) = \sqrt{1.0333} = 1.0165 \end{aligned}$$

Now using these values in the Heun's formula, we get

$$y_1 = y_0 + \frac{h}{4}[k_1 + 3k_2] = 1 + 0.0125(1 + 3(1.0165)) = 1.0506$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 19$, are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
0.0500	1.0506	1.0506	0.0000
0.1000	1.1025	1.1025	0.0000
0.1500	1.1556	1.1556	0.0000
0.2000	1.2100	1.2100	0.0000
0.2500	1.2656	1.2656	0.0000
0.3000	1.3225	1.3225	0.0000
0.3500	1.3806	1.3806	0.0000
0.4000	1.4400	1.4400	0.0000
0.4500	1.5006	1.5006	0.0000
0.5000	1.5624	1.5625	0.0001
0.5500	1.6256	1.6256	0.0001
0.6000	1.6899	1.6900	0.0001
0.6500	1.7556	1.7556	0.0001
0.7000	1.8224	1.8225	0.0001
0.7500	1.8905	1.8906	0.0001
0.8000	1.9599	1.9600	0.0001
0.8500	2.0305	2.0306	0.0001
0.9000	2.1024	2.1025	0.0001
0.9500	2.1755	2.1756	0.0001
1.0000	2.2499	2.2500	0.0001

Similarly, for $h = 0.1$, we have

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
0.1000	1.1025	1.1025	0.0000
0.2000	1.2099	1.2100	0.0001
0.3000	1.3224	1.3225	0.0001
0.4000	1.4398	1.4400	0.0002
0.5000	1.5623	1.5625	0.0002
0.6000	1.6898	1.6900	0.0002
0.7000	1.8222	1.8225	0.0003
0.8000	1.9597	1.9600	0.0003
0.9000	2.1021	2.1025	0.0004
1.0000	2.2496	2.2500	0.0004

Midpoint Method: First we find the values of k_1 and k_2 by taking $i = 0$, we have

$$k_1 = f(x_0, y_0) = \sqrt{y_0} = 1$$

$$k_2 = f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1) = \sqrt{1.025} = 1.0124$$

Now using these values in the Midpoint formula, we get

$$y_1 = y_0 + hk_2 = 1 + (0.05)(1.0124) = 1.0506$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 19$, are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
0.0500	1.0506	1.0506	0.0000
0.1000	1.1025	1.1025	0.0000
0.1500	1.1556	1.1556	0.0000
0.2000	1.2100	1.2100	0.0000
0.2500	1.2656	1.2656	0.0000
0.3000	1.3225	1.3225	0.0000
0.3500	1.3806	1.3806	0.0000
0.4000	1.4400	1.4400	0.0000
0.4500	1.5006	1.5006	0.0000
0.5000	1.5625	1.5625	0.0000
0.5500	1.6256	1.6256	0.0000
0.6000	1.6900	1.6900	0.0000
0.6500	1.7556	1.7556	0.0001
0.7000	1.8224	1.8225	0.0001
0.7500	1.8906	1.8906	0.0001
0.8000	1.9599	1.9600	0.0001
0.8500	2.0306	2.0306	0.0001
0.9000	2.1024	2.1025	0.0001
0.9500	2.1756	2.1756	0.0001
1.0000	2.2499	2.2500	0.0001

Similarly, for $h = 0.1$, we have

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
0.1000	1.1025	1.1025	0.0000
0.2000	1.2099	1.2100	0.0001
0.3000	1.3224	1.3225	0.0001
0.4000	1.4399	1.4400	0.0001
0.5000	1.5623	1.5625	0.0002
0.6000	1.6898	1.6900	0.0002
0.7000	1.8223	1.8225	0.0002
0.8000	1.9598	1.9600	0.0002
0.9000	2.1022	2.1025	0.0003
1.0000	2.2497	2.2500	0.0003

(c) Given $f(x, y) = 4 - 3y$, and $x_0 = 0,\ y_0 = 5,\ h = 0.05$:
Heun's Method: The approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 19$,

are as follows

(**c**) $h = 0.05$	x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
	0.0500	4.4913	4.4893	−0.0020
	0.1000	4.0531	4.0497	−0.0034
	0.1500	3.6757	3.6713	−0.0044
	0.2000	3.3507	3.3456	−0.0051
	0.2500	3.0708	3.0653	−0.0055
	0.3000	2.8297	2.8241	−0.0056
	0.3500	2.6221	2.6164	−0.0057
	0.4000	2.4433	2.4377	−0.0056
	0.4500	2.2893	2.2839	−0.0054
	0.5000	2.1566	2.1515	−0.0052
	0.5500	2.0424	2.0375	−0.0049
	0.6000	1.9440	1.9394	−0.0046
	0.6500	1.8593	1.8550	−0.0043
	0.7000	1.7863	1.7823	−0.0040
	0.7500	1.7235	1.7198	−0.0037
	0.8000	1.6693	1.6660	−0.0034
	0.8500	1.6227	1.6196	−0.0031
	0.9000	1.5826	1.5798	−0.0028
	0.9500	1.5480	1.5454	−0.0026
	1.0000	1.5182	1.5159	−0.0023

Similarly, for $h = 0.1$, the approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 9$, are as follows

$h = 0.1$	x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
	0.1000	4.0650	4.0497	−0.0153
	0.2000	3.3684	3.3456	−0.0228
	0.3000	2.8495	2.8241	−0.0254
	0.4000	2.4629	2.4377	−0.0251
	0.5000	2.1748	2.1515	−0.0234
	0.6000	1.9602	1.9394	−0.0208
	0.7000	1.8004	1.7823	−0.0180
	0.8000	1.6813	1.6660	−0.0153
	0.9000	1.5926	1.5798	−0.0128
	1.0000	1.5265	1.5159	−0.0106

Midpoint Method: The approximations by taking $x_i = x_{i-1}+h, i = 1, 2, \ldots, 19$, are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
0.0500	4.4913	4.4893	−0.0020
0.1000	4.0531	4.0497	−0.0034
0.1500	3.6757	3.6713	−0.0044
0.2000	3.3507	3.3456	−0.0051
0.2500	3.0708	3.0653	−0.0055
0.3000	2.8297	2.8241	−0.0056
0.3500	2.6221	2.6164	−0.0057
0.4000	2.4433	2.4377	−0.0056
0.4500	2.2893	2.2839	−0.0054
0.5000	2.1566	2.1515	−0.0052
0.5500	2.0424	2.0375	−0.0049
0.6000	1.9440	1.9394	−0.0046
0.6500	1.8593	1.8550	−0.0043
0.7000	1.7863	1.7823	−0.0040
0.7500	1.7235	1.7198	−0.0037
0.8000	1.6693	1.6660	−0.0034
0.8500	1.6227	1.6196	−0.0031
0.9000	1.5826	1.5798	−0.0028
0.9500	1.5480	1.5454	−0.0026
1.0000	1.5182	1.5159	−0.0023

Similarly, for $h = 0.1$, the approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 9$, are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
0.1000	4.0650	4.0497	−0.0153
0.2000	3.3684	3.3456	−0.0228
0.3000	2.8495	2.8241	−0.0254
0.4000	2.4629	2.4377	−0.0251
0.5000	2.1748	2.1515	−0.0234
0.6000	1.9602	1.9394	−0.0208
0.7000	1.8004	1.7823	−0.0180
0.8000	1.6813	1.6660	−0.0153
0.9000	1.5926	1.5798	−0.0128
1.0000	1.5265	1.5159	−0.0106

11. Solve the following initial-value problems using the fourth-order Runge-Kutta formula using $h = 0.2$

(a) $y' = 1 + \dfrac{y}{x}, \quad 1 \leq x \leq 2 \quad y((1) = 1.$

(b) $y' = y \tan x, \quad 0 \leq x \leq 1, \quad y(0) = 2.$

(c) $y' = (1 - x)y^2 - y, \quad 1 \leq x \leq 2 \quad y(0) = 1.$

Solution: **(a)** We set $f(x,y) = 1 + y/x$ and $x_0 = 1,\ y_0 = 1$, and $h = 0.2$, then for $i = 0$, the calculated values of k_1, k_2, k_3, and k_4 are as follows

$$\begin{aligned}
k_1 &= f(x_0, y_0) = (1 + y_0/x_0) = 2.0000 \\
k_2 &= f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1) = [1 + 1.2/1.1] = 2.0909 \\
k_3 &= f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_2) = [1 + 1.2091/1.1] = 2.0992 \\
k_4 &= f(x_0 + h, y_0 + hk_3) = [1 + 1.4198/1.2] = 2.1832
\end{aligned}$$

Hence the first approximate solution is

$$y_1 = y_0 + \frac{h}{6}\Big[k_1 + 2k_2 + 2k_3 + k_4\Big] = [1 + (0.2/6)(12.5634)] = 1.4188$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4,$ are as follows

x_i	y_i
1.2000	1.4188
1.4000	1.8710
1.6000	2.3520
1.8000	2.8580
2.0000	3.3863

(b) We set $f(x,y) = y \tan x$ and $x_0 = 0,\ y_0 = 2$, and $h = 0.2$, then for $i = 0$, the calculated values of k_1, k_2, k_3, and k_4 are as follows

$$k_1 = 0, \quad k_2 = 0.2007, \quad k_3 = 0.2027, \quad k_4 = 0.4136$$

Hence the first approximate solution is

$$y_1 = y_0 + \frac{h}{6}\Big[k_1 + 2k_2 + 2k_3 + k_4\Big] = [2 + (0.2/6)(1.2204)] = 2.0407$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4,$ are as follows

x_i	y_i
0.2000	2.0407
0.4000	2.1714
0.6000	2.4233
0.8000	2.8707
1.0000	3.7017

(c) We set $f(x,y) = (1-x)y^2 - y$ and $x_0 = 1,\ y_0 = 1$, and $h = 0.2$, then for $i = 0$, the calculated values of k_1, k_2, k_3, and k_4 are as follows

$$k_1 = -1, \quad k_2 = -0.9810, \quad k_3 = -0.9832, \quad k_4 = -0.9324$$

Hence the first approximate solution is

$$y_1 = y_0 + \frac{h}{6}\Big[k_1 + 2k_2 + 2k_3 + k_4\Big] = [1 + (0.2/6)(-5.8608)] = 0.8046$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4,$ are as follows

x_i	y_i
1.2000	0.8046
1.4000	0.6315
1.6000	0.4892
1.8000	0.3772
2.0000	0.2910

12. Solve the following initial-value problems and compare the numerical solutions obtained with the fourth-order Runge-Kutta formula and the fourth-order Taylor's method by using the values of $h = 0.1$ and $h = 0.2$, over the interval $[a, b]$.

(a) $y' = 4 - 3y, \quad [0, 1], \quad y(0) = 5, \quad y(x) = 4/3 + 11/3e^{-3x}.$

(b) $y' = (2 - x)y, \quad [2, 3], \quad y(2) = 1, \quad y(x) = e^{-1/2(x-2)^2}.$

(c) $y' = \dfrac{1}{1 + x^2} - 2y^2, \quad [0, 1], \; y(0) = 0, \quad y(x) = \dfrac{x}{1 + x^2}.$

Solution: (a) We set $f(x, y) = 4 - 3y$ and $x_0 = 1, \; y_0 = 5$, and $h = 0.1$:
Fourth-order Runge-Kutta Formula: For $i = 0$, the calculated values of $k_1, k_2, k_3,$ and k_4 are as follows

$$\begin{aligned}
k_1 &= f(x_0, y_0) = (4 - 3y_0) = -11.0000 \\
k_2 &= f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1) = [4 - 3(4.45)] = -9.3500 \\
k_3 &= f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_2) = [4 - 3(4.5325)] = -9.5975 \\
k_4 &= f(x_0 + h, y_0 + hk_3) = [4 - 3(4.04025)] = -8.1208
\end{aligned}$$

Hence the first approximate solution is

$$y_1 = y_0 + \frac{h}{6}\Big[k_1 + 2k_2 + 2k_3 + k_4\Big] = [5 + (0.2/6)(-57.0158)] = 4.0497$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 9,$ are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
0.1000	4.0497	4.0497	−0.0001
0.2000	3.3457	3.3456	−0.0001
0.3000	2.8242	2.8241	−0.0001
0.4000	2.4378	2.4377	−0.0001
0.5000	2.1516	2.1515	−0.0001
0.6000	1.9395	1.9394	−0.0001
0.7000	1.7824	1.7823	−0.0001
0.8000	1.6660	1.6660	−0.0001
0.9000	1.5798	1.5798	−0.0001
1.0000	1.5159	1.5159	−0.0000

Similarly, for $h = 0.2$, the approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4$, are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
0.2000	3.3478	3.3456	−0.0022
0.4000	2.4401	2.4377	−0.0024
0.6000	1.9414	1.9394	−0.0020
0.8000	1.6674	1.6660	−0.0014
1.0000	1.5169	1.5159	−0.0010

Fourth-order Taylor's Method: Since $f(x, y) = 4 - 3y,\ f'(x, y) = -12 + 9y,\ f''(x, y) = 36 - 27y,\ f'''(x, y) = -108 + 81y$, and $x_0 = 0,\ y_0 = 5,\ h = 0.1$, then using fourth-order Taylor's method

$$y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2} f'(x_i, y_i) + \frac{h^3}{6} f''(x_i, y_i) + \frac{h^4}{24} f'''(x_i, y_i), \quad \text{for} \quad i = 0, 1, 2, 3, 4$$

for $h = 0.1$ and taking $i = 0$, we have

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) + \frac{h^2}{2} f'(x_0, y_0) + \frac{h^3}{6} f''(x_0, y_0) + \frac{h^4}{24} f'''(x_0, y_0) \\ &= y_0 + h[4 - 3y_0] + \frac{h^2}{2}[-12 + 9y_0] + \frac{h^3}{6}[36 - 27y_0] + \frac{h^4}{24}[-108 + 81y_0] \end{aligned}$$

which gives

$$\begin{aligned} y_1 &= 5 + (0.1)[4 - 15] + 0.005[-12 + 45] + 0.0002[36 - 135] \\ &+ 0.000004[-108 + 405] = 4.0497 \end{aligned}$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 9$, are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
0.1000	4.0497	4.0497	−0.0001
0.2000	3.3457	3.3456	−0.0001
0.3000	2.8242	2.8241	−0.0001
0.4000	2.4378	2.4377	−0.0001
0.5000	2.1516	2.1515	−0.0001
0.6000	1.9395	1.9394	−0.0001
0.7000	1.7824	1.7823	−0.0001
0.8000	1.6660	1.6660	−0.0001
0.9000	1.5798	1.5798	−0.0001
1.0000	1.5159	1.5159	−0.0000

Similarly, for $h = 0.2$, the approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4$, are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
0.2000	3.3478	3.3456	−0.0022
0.4000	2.4401	2.4377	−0.0024
0.6000	1.9414	1.9394	−0.0020
0.8000	1.6674	1.6660	−0.0014
1.0000	1.5169	1.5159	−0.0010

(b) We set $f(x, y) = (2 - x)y$ and $x_0 = 2,\ y_0 = 1$, and $h = 0.1$:
Fourth-order Runge-Kutta Formula: For $i = 0$, the calculated values of k_1, k_2, k_3, and k_4 are as follows

$$k_1 = 0, \quad k_2 = -0.0500, \quad k_3 = -0.0499, \quad k_4 = -0.0995$$

Hence the first approximate solution is

$$y_1 = y_0 + \frac{h}{6}\Big[k_1 + 2k_2 + 2k_3 + k_4\Big] = [1 + (0.2/6)(-0.2993)] = 0.9950$$

This and other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 9$, are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
2.1000	0.9950	0.9950	0.0000
2.2000	0.9802	0.9802	0.0000
2.3000	0.9560	0.9560	0.0000
2.4000	0.9231	0.9231	0.0000
2.5000	0.8825	0.8825	0.0000
2.6000	0.8353	0.8353	0.0000
2.7000	0.7827	0.7827	−0.0000
2.8000	0.7261	0.7261	−0.0000
2.9000	0.6670	0.6670	−0.0000
3.0000	0.6065	0.6065	−0.0000

Similarly, for $h = 0.2$, the approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4$, are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
2.2000	0.9802	0.9802	0.0000
2.4000	0.9231	0.9231	0.0000
2.6000	0.8353	0.8353	0.0000
2.8000	0.7261	0.7261	0.0000
3.0000	0.6065	0.6065	−0.0000

Fourth-order Taylor's Method: Since $f(x,y) = (2-x)y,\ f'(x,y) = -y + (2-x)^2 y,\ f''(x,y) = -3(2-x)y + (2-x)^3 y,\ f'''(x,y) = 3y - 6(2-x)^2 y + (2-x)^4 y$, and $x_0 = 2,\ y_0 = 1,\ h = 0.1$, then using fourth-order Taylor's method, the approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 9$, are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
2.1000	0.9950	0.9950	−0.0000
2.2000	0.9802	0.9802	−0.0000
2.3000	0.9560	0.9560	−0.0000
2.4000	0.9231	0.9231	−0.0000
2.5000	0.8825	0.8825	−0.0000
2.6000	0.8353	0.8353	−0.0000
2.7000	0.7827	0.7827	−0.0000
2.8000	0.7262	0.7261	−0.0000
2.9000	0.6670	0.6670	−0.0000
3.0000	0.6065	0.6065	−0.0000

Similarly, for $h = 0.2$, the approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4$, are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
2.2000	0.9802	0.9802	−0.0000
2.4000	0.9231	0.9231	−0.0000
2.6000	0.8353	0.8353	−0.0000
2.8000	0.7262	0.7261	−0.0000
3.0000	0.6066	0.6065	−0.0000

(c) We set $f(x,y) = \dfrac{1}{1+x^2} - 2y^2$ and $x_0 = 0,\ y_0 = 0$, and $h = 0.1$:

Fourth-order Runge-Kutta Formula: The approximations by taking $x_i =$

$x_{i-1} + h, i = 1, 2, \ldots, 9$, are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
0.1000	0.0990	0.0990	0.0000
0.2000	0.1923	0.1923	0.0000
0.3000	0.2752	0.2752	0.0000
0.4000	0.3448	0.3448	0.0000
0.5000	0.4000	0.4000	0.0000
0.6000	0.4412	0.4412	0.0000
0.7000	0.4698	0.4698	0.0000
0.8000	0.4878	0.4878	0.0000
0.9000	0.4972	0.4972	0.0000
1.0000	0.5000	0.5000	0.0000

Similarly, for $h = 0.2$, the approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 4$, are as follows

x_i	y_i	$y(x_i)$	$y(x_i) - y_i$
0.2000	0.1923	0.1923	0.0000
0.4000	0.3448	0.3448	0.0000
0.6000	0.4411	0.4412	0.0001
0.8000	0.4877	0.4878	0.0001
1.0000	0.4999	0.5000	0.0001