Solutions of Exercises Sheet #4

Solution 1:

a)

Uniform	$U(a, b)$
Parameters:	a = minimum, b = maximum, $-\infty < a < b < \infty$
PDF:	$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$
CDF:	$F(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b, \\ 1 & \text{if } b > x \end{cases}$
Inverse CDF:	$F^{-1}(p) = a + p(b-a)$ if $0 < p < 1$
Expected value:	$E[X] = \frac{a+b}{2}$
Variance:	$V[X] = \frac{(b-a)^2}{12}$
Arena™ generation:	UNIF(Min,Max[,Stream])
Spreadsheet generation:	= a + RAND()*(b-a)

X = 12 + 0.3734*(22-12) = 15.734

b)

Erlang	Erlang(r, β)
Parameters:	$r > 0$, integer, $\beta > 0$ (scale)
CDF:	$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{(-x/\beta)} \sum_{j=0}^{r-1} \frac{(x/\beta)^j}{j} & \text{if } x \geq 0 \end{cases}$
Inverse CDF:	No closed form
Expected value:	E$[X] = r\beta$
Variance:	Var$[X] = r\beta^2$
Arena™ generation:	ERLA(E$[X]$, r[,Stream])
Spreadsheet generation:	= GAMMA.INV(RAND(), r, β)

- Erlang Variable = Σ *iid* Exponential variables

Exponential	EXPO$(1/\lambda)$
Parameters:	$\lambda > 0$
PDF:	$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \lambda e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$
CDF:	$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$
Inverse CDF:	$F^{-1}(p) = (-1/\lambda) \ln\ (1 - p)$ if $0 < p < 1$
Expected value:	$E[X] = 1/\lambda$
Variance:	$Var[X] = 1/\lambda^2$
Arena™ generation:	EXPO(mean[,Stream])
Spreadsheet generation:	$= (-1/\lambda)$LN(1-RAND())

Use convolution to generate 2 exponential random variables

$X_1 = -3\ln(1-0.9559) = 9.364$

$X_2 = -3\ln(1-0.5814) = 2.612$

$X = X_1 + X_2 = 11.976$

c)

X	40	50	60	70	80
P(X=x)	0.44	0.22	0.16	0.12	0.06
F(x)	0.44	0.66	0.82	0.94	1

so,

$$F^{-1}(u) = \begin{cases} 40 & ,0 \leq u \leq .44 \\ 50 & ,.44 < u \leq .66 \\ 60 & ,.66 < u \leq .82 \\ 70 & ,.82 < u \leq .94 \\ 80 & ,.94 < u \leq 1 \end{cases}$$

H10 =IF(F10>0.94;80;IF(F10>0.82;70;IF(F10>0.66;60;IF(F10>0.44;50;40))))

	A	B	C	D	E	F	G	H	I
1	x	p(x)		F(x)					
2	40	0.44	0	0.44					
3	50	0.22	0.44	0.66					
4	60	0.16	0.66	0.82					
5	70	0.12	0.82	0.94					
6	80	0.06	0.94	1					
7									
8									
9									
10					U1=	0.9559	X1=	80	
11					U2=	0.5814	X2=	50	
12					U3=	0.6534	X3=	50	
13					U4=	0.5548	X4=	50	
14									

$U_1 = 0.9559 \rightarrow X = 80$

$U_2 = 0.5814 \rightarrow X = 50$

$U_3 = 0.6534 \rightarrow X = 50$

$U_4 = 0.5548 \rightarrow X = 50$

Solution 2:

Poisson	Pois(λ)
Parameters:	$\lambda > 0$
PMF:	$P\{X = x\} = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, \ldots$
Expected value:	$E[X] = \lambda$
Variance:	$Var[X] = \lambda$
Arena™ generation:	POIS(λ[,Stream])

Exponential	EXPO($1/\lambda$)
Parameters:	$\lambda > 0$
PDF:	$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \lambda e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$
CDF:	$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$
Inverse CDF:	$F^{-1}(p) = (-1/\lambda)\ln(1-p) \quad \text{if } 0 < p < 1$
Expected value:	$E[X] = 1/\lambda$
Variance:	$Var[X] = 1/\lambda^2$
Arena™ generation:	EXPO(mean[,Stream])
Spreadsheet generation:	= ($-1/\lambda$)LN(1-RAND())

Customers arrive at an ATM via a Poisson process with mean 7 per hour ($\lambda = 7$).

$T_i = F^{-1}(U_i) = -1/\lambda * \ln(1-U_i)$

the arrival time for the first six customers can be calculated.

Inverse CDF of Poisson dis.

$AT(i) = \sum_{j=1}^{i} T_j$

Arrival Times of First Six Customers (in hours)

Customer # i	**U_i**	**T_i=Inter-Arrival Time**	**AT(i)=Arrival time**
1	0.943	0.4092	0.4092
2	0.498	0.0985	0.4092+0.0985=0.5077
3	0.102	0.0154	0.5077+0.0154=0.5231
4	0.398	0.0725	0.5231+0.0725=0.5956
5	0.528	0.1073	0.5956+0.1073=0.7029
6	0.057	0.0084	0.7029+0.0084=0.7113

Solution 3:

To generate the demand for the first four days using the sequence of (0,1) random numbers, we first need to find the inverse CDF for the discrete distribution.

CDF of the Discrete Distribution

x_i	0	1	2
$f(x_i)$	0.3	0.2	0.5
$F(x_i)$	0.3	0.5	1.0

The above CDF can also be written as:

$$F^{-1}(x) = \begin{cases} 0 & \text{if } 0 \le x \le 0.3 \\ 1 & \text{if } 0.3 < x \le 0.5 \\ 2 & \text{if } 0.5 < x \le 1.0 \end{cases}$$

Using the above function and the random numbers, the demand for the first four days is as follows:

Demand for the First Four Days

	Day 1	Day 2	Day 3	Day 4
U_i	0.943	0.498	0.102	0.398
Demand	2	1	0	1

Solution 4:

shifted exponential distribution

$$f_T(t) = \begin{cases} \lambda e^{-\lambda(t-\delta)} & t > \delta, \\ 0 & otherwise \end{cases}$$

with $E(T) = \delta + \frac{1}{\lambda}$ and $Var(T) = \frac{1}{\lambda^2}$

Then, the cumulative distribution function and the inverse function are

$$F_T(t) = 1 - e^{-\lambda(t-\delta)}, t > \delta$$

$$T = F^{-1}(U) = \frac{-1}{\lambda}\ln(1-U) + \delta, 0 < U < 1$$

So (where $\lambda = 45$ and $\delta = 15$),

$T_1 = 15 + (-(1/45)\ln(1-.943)) = 15.064$

$T_2 = 15 + (-(1/45)\ln(1-0.398)) = 15.011$

Solution 5:

Weibull	WEIB(β, α)
Parameters:	$\beta > 0$ (scale), $\alpha > 0$ (shape)
CDF:	$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{(-x/\beta)^\alpha} & \text{if } x \geq 0 \end{cases}$
Inverse CDF:	$F^{-1}(p) = \beta[-\ln(1-p)]^{1/\alpha}$ if $0 < p < 1$
Expected value:	$E[X] = \left(\frac{\beta}{\alpha}\right)\Gamma\left(\frac{1}{\alpha}\right)$
Variance:	$\mathrm{Var}[X] = \left(\frac{\beta^2}{\alpha}\right)\{2\Gamma\left(\frac{2}{\alpha}\right) - \left(\frac{1}{\alpha}\right)(\Gamma\left(\frac{1}{\alpha}\right))^2\}$
Arena™ generation:	WEIB(scale, shape[,Stream])
Spreadsheet generation:	$= (\beta)(-LN(1 - \mathrm{RAND}()) \wedge (1/\alpha)$

So,

$U_1 = 0.943 \rightarrow$ X_1 = 3[-ln(1-0.943)]^(1/2) = 5.0776

$U_2 = 0.398 \rightarrow$ X_2 = 3[-ln(1-0.398)]^(1/2) = 2.1372

Solution 6:

Weibull	WEIB(β, α)
Parameters:	$\beta > 0$ (scale), $\alpha > 0$ (shape)
CDF:	$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{(-x/\beta)^\alpha} & \text{if } x \geq 0 \end{cases}$
Inverse CDF:	$F^{-1}(p) = \beta[-\ln(1-p)]^{1/\alpha}$ if $0 < p < 1$
Expected value:	$E[X] = \left(\frac{\beta}{\alpha}\right)\Gamma\left(\frac{1}{\alpha}\right)$
Variance:	$\mathrm{Var}[X] = \left(\frac{\beta^2}{\alpha}\right)\{2\Gamma\left(\frac{2}{\alpha}\right) - \left(\frac{1}{\alpha}\right)(\Gamma\left(\frac{1}{\alpha}\right))^2\}$
Arena™ generation:	WEIB(scale, shape[,Stream])
Spreadsheet generation:	$= (\beta)(-LN(1 - \mathrm{RAND}()) \wedge (1/\alpha)$

Simulation from Truncated Dist.

- The new pdf g(x) with $a \leq X \leq b$.

$$g(x) = \frac{f(x)}{F(b) - F(a)}; \quad a \leq x \leq b \Rightarrow$$

1: Generate $u \sim U(0,1)$
2: $W = F(a) + (F(b) - F(a))u$
3: $X = F^{-1}(W)$

Notice that the range is truncated, we have ($\alpha = 2$, $\beta = 3$, a=1.5, b=4.5):

F(1.5) = 1-exp(-(1.5/3)^2) = 0.22119

F(4.5) = 1-exp(-(4.5/3)^2) = 0.8946

W = 0.22119 + (0.8946 – 0.22119)*0.943 = 0.8562169

X = 3[-ln(1-0.8562169)]^(1/2) = 4.1779

Solution 7:

Uniform	$U(a, b)$
Parameters:	a = minimum, b = maximum, $-\infty < a < b < \infty$
PDF:	$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$
CDF:	$F(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \le x \le b, \\ 1 & \text{if } b > x \end{cases}$
Inverse CDF:	$F^{-1}(p) = a + p(b - a)$ if $0 < p < 1$
Expected value:	$E[X] = \frac{a+b}{2}$
Variance:	$V[X] = \frac{(b-a)^2}{12}$
Arena™ generation:	UNIF(Min,Max[,Stream])
Spreadsheet generation:	= a + RAND()*(b-a)

Equally likely means uniformly distributed: U(a=0.02, b = 0.05), using inverse transform:

X = a + (b-a)*U

For $U_1 = 0.943$ and $U_2 = 0.398$

$X_1 = 0.02 + (0.05-0.02)*0.943 = 0.04829$

$X_2 = 0.02 + (0.05-0.02)*0.398 = 0.03194$

Solution 8:

	A	B	C	D	E	F
1	customer	U	Inter-Arrival Time	Arrival time	U	server
2	1	0.943	0.2864704	0.2864704	0.498	1
3	2	0.102	0.01075852	0.29722892	0.398	1
4	3	0.528	0.07507763	0.37230655	0.057	1
5	4	0.372	0.04652151	0.41882806	0.272	1
6	5	0.409	0.05259393	0.47142199	0.943	2
7	6	0.899	0.22926348	0.70068547	0.398	1
8	7	0.204	0.02281561	0.72350107	0.294	1
9	8	0.4	0.05108256	0.77458364	0.794	2
10	9	0.156	0.01696028	0.79154392	0.997	2

	A	B	C	D	E	F
1	customer	U	Inter-Arrival Time	Arrival time	U	server
2	1	0.943	=-(1/10)*LN(1-B2)	=C2	0.498	=IF(E2<=0.6,1,2)
3	2	0.102	=-(1/10)*LN(1-B3)	=D2+C3	0.398	=IF(E3<=0.6,1,2)
4	3	0.528	=-(1/10)*LN(1-B4)	=D3+C4	0.057	=IF(E4<=0.6,1,2)
5	4	0.372	=-(1/10)*LN(1-B5)	=D4+C5	0.272	=IF(E5<=0.6,1,2)
6	5	0.409	=-(1/10)*LN(1-B6)	=D5+C6	0.943	=IF(E6<=0.6,1,2)
7	6	0.899	=-(1/10)*LN(1-B7)	=D6+C7	0.398	=IF(E7<=0.6,1,2)
8	7	0.204	=-(1/10)*LN(1-B8)	=D7+C8	0.294	=IF(E8<=0.6,1,2)
9	8	0.4	=-(1/10)*LN(1-B9)	=D8+C9	0.794	=IF(E9<=0.6,1,2)
10	9	0.156	=-(1/10)*LN(1-B10)	=D9+C10	0.997	=IF(E10<=0.6,1,2)
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Solution 9:

a)

The triangular(a, c, b) distribution has probability density function

$$f(x) = \begin{cases} \frac{2(x-a)}{(b-a)(c-a)} & a < x < c \\ \frac{2(b-x)}{(b-a)(b-c)} & c \le x < b \end{cases}$$

and cumulative distribution function

$$F(x) = \begin{cases} \frac{(x-a)^2}{(b-a)(c-a)} & a < x < c \\ 1 - \frac{(x-b)^2}{(b-a)(b-c)} & c \le x < b. \end{cases}$$

Equating the cumulative distribution function to u, where $0 < u < 1$ yields an inverse cumulative distribution function

$$F^{-1}(u) = \begin{cases} a + \sqrt{(b-a)(c-a)u} & 0 < u < \frac{c-a}{b-a} \\ b - \sqrt{(b-a)(b-c)(1-u)} & \frac{c-a}{b-a} \le u < 1. \end{cases}$$

b)

$U_1 = 0.943$ and $U_2 = 0.398$

(c-a)/(b-a) = 0.375

For $U_1 = 0.943$, since 0.943 > 0.375, we have X = b-SQRT((b-a)(b-c)*(1-U_1)) = 8.8304

For $U_2 = 0.398$, since 0.398 > 0.375, we have X = b-SQRT((b-a)(b-c)*(1-U_2)) = 6.1989

Solution 10:

a) For $x < -1, F(x) = 0$

For $-1 \leq x \leq 1, F(x) = \frac{1}{2}(x^3 + 1)$

For $x > 1, F(x) = 1$

$$\therefore F^{-1}(u) = \sqrt[3]{2u - 1}$$

b) $F^{-1}(0.943) = 0.9604$

$F^{-1}(0.398) = -0.5886765$

Solution 11:

a) For $x < 2, F(x) = 0$

For $2 \leq x \leq 4, F(x) = \frac{x^2}{4} - x + 1$

For $x > 4, F(x) = 1$

Solve the following equation for x:

$$\therefore x^2 - 4x + 4(1 - u) = 0$$

Using the quadratic equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Yields

$$x = \frac{4 \pm \sqrt{16 - 4 * 4(1 - u)}}{2}$$

$$x = \frac{4 \pm 4\sqrt{u}}{2} = 2 \pm 2\sqrt{u}$$

Since the final number must be $2 \leq x \leq 4$, we have

$$x = 2 + 2\sqrt{u} = 2(1 + \sqrt{u})$$

b) $F^{-1}(0.943) = 3.94216$

$F^{-1}(0.398) = 3.2617$

Solution 12:

a) For $x < 0, F(x) = 0$

For $0 \leq x \leq 5, F(x) = \frac{x^2}{25}$

For $x > 5, F(x) = 1$

$$\therefore F^{-1}(u) = 5\sqrt[2]{u}$$

b) $F^{-1}(0.943) = 4.8554$

$F^{-1}(0.398) = 3.1544$

Solution 13:

a) For $x \leq 1, F(x) = 0$

For $x > 1, F(x) = 1 - \frac{1}{b^2}$

$$\therefore F^{-1}(u) = \sqrt[2]{\frac{1}{1-u}}$$

b) $F^{-1}(0.943) = 4.1885$

$F^{-1}(0.398) = 1.2888$