Some Continuous Probability Distributions: Part II

Gamma Distribution

Chi Square Distribution

T Student Distribution

The Gamma Distribution and The Gamma Function:

Gamma Function:

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x}\, dx$$

Properties of the Gamma function:

–For any $\alpha > 1$, $\Gamma(\alpha) = (\alpha-1)\cdot\Gamma(\alpha-1)$

–For any positive integer, n, $\Gamma(n) = (n-1)!$,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y}\, dy$$

$$u = y^{\alpha} \Rightarrow du = \alpha y^{\alpha-1} dy$$

$$dv = e^{-y} dy \Rightarrow v = -e^{-y}$$

$$\Gamma(\alpha+1) = \int_0^{\infty} y^{\alpha} e^{-y}\, dy \quad \text{Integrating by Parts:}$$

$$\Rightarrow \Gamma(\alpha+1) = \int_0^{\infty} y^{\alpha} e^{-y}\, dy = uv - \int v\, du = -\, y^{\alpha} e^{-y}\Big|_0^{\infty} + \int_0^{\infty} \alpha y^{\alpha-1} e^{-y}\, dy =$$

$$= -0 - (-0) + \alpha \int_0^{\infty} y^{\alpha-1} e^{-y}\, dy = \alpha\Gamma(\alpha) \quad \text{(Recursive Property)}$$

Note that if α is a positive integer number, then $\Gamma(\alpha) = (\alpha-1)!$

Example: $\Gamma(10)=\int_0^\infty y^9 e^{-y}dy = 9!$

$Prove\ \ that:$ $$\int_0^\infty x^b e^{-ay}dy = \Gamma(b+1)/a^{b+1}$$

$let\ \ y = ax\ \ then\ dy = adx,\ \ \ dx = dy/a$

$then,$

$$\int_0^\infty x^b e^{-ay}dy = \Gamma(b+1)/a^{b+1} =$$

$$\int_0^\infty (y/a)^b e^{ay}dy/a = (1/a^{b+1})\int_0^\infty y^b e^{-y}dy = \Gamma(b+1)/(a^{b+1})$$

$Example:$

$$\int_0^\infty x^{10}e^{-4x}dx = \Gamma(11)/4^{11}$$

Realation with Beta Function:

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt = \Gamma(x)\Gamma(y)/\Gamma(x+y)$$

Gamma, Chi and Exponential Distributions

A continuous rv X has a Gamma Distribution if the pdf of X is

$$f(x;\alpha,\beta)=\begin{cases}\dfrac{1}{\beta^{\alpha}\Gamma(\alpha)}x^{\alpha-1}e^{\frac{-x}{\beta}} & ,\ \mathrm{x}\ge 0\\ 0 & ,\mathrm{otherwise}\end{cases}$$

Where α > 0 and β > 0.

The standard Gamma distribution has β=1

$$\mu=\alpha\beta,\qquad \sigma^2=\alpha\beta^2,\qquad M_x(t)=\left(\frac{1}{1-\beta t}\right)^{\alpha}$$

In Gamma pdf

$Exp(x;\ \beta)$ — $if\ \ \alpha=1$

$if\ \ \alpha=v/2,\ \ \beta=2$ — $Chi-Square\ Dis.$ *with v degree of freedom*

$$f(x;\beta)=\begin{cases}\dfrac{1}{\beta}\ e^{-x/\beta} & ;\ x>0\\ 0\ ;\ elsewhere\end{cases}$$

$$\mu=\alpha\beta,$$

$$\sigma^2=\alpha\beta^2,$$

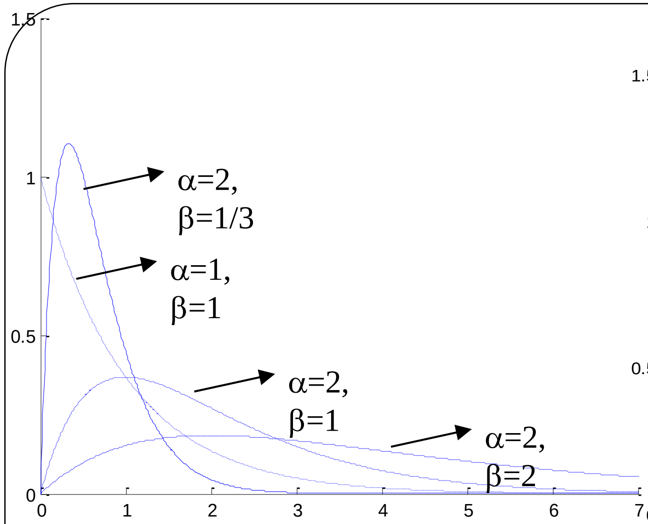
$$M_x(t)=\left(\frac{1}{1-\beta t}\right)^{\alpha}$$

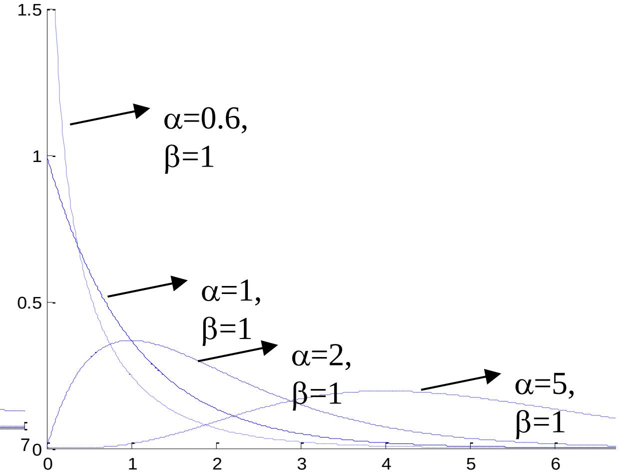
$$f(x;v)=\begin{cases}\dfrac{1}{2^{v/2}\Gamma(v/\alpha)}x^{(v/2)-1}e^{\frac{-x}{2}} & ,\ \mathrm{x}\ge 0\\ 0 & ,\mathrm{otherwise}\end{cases}$$

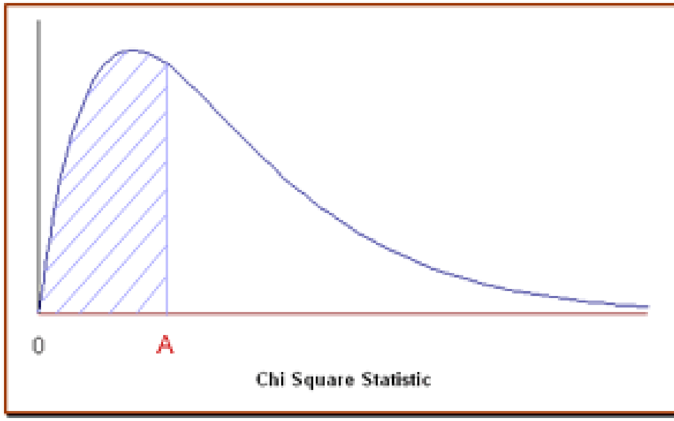
$$\mu=v,$$

$$\sigma^2=2v,$$

$$M_x(t)=\left(\frac{1}{1-2t}\right)^{v/2},\ t\ne 1/2$$







Why do we need gamma distribution?

Any normal distribution is bell-shaped and symmetric. There are many practical situations that do not fit to symmetrical distribution.

The **Gamma family** pdfs can yield a wide variety of skewed distributions.

β is called the scale parameter because values other than 1 either stretch or compress the pdf in the x-direction.

The (CDF) of the standard gamma r.v. X: (Optional subject)

$$F(x;\alpha)=\int_0^x \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)}dy\,,\quad \mathbf{x>0}$$

is called the **incomplete gamma function**. There are extensive tabulations available for F(x;α). We will refer to page 742 on Devore, for α = 1,2,…,10 and x = 1,2,..,15.

- Let X have a gamma distribution with parameters α and β. For any x>0, the cdf of X is given by:

$$p(X\le x)=F(x;\alpha,\beta)=F(\frac{x}{\beta};\alpha)$$

where F(•;α) is the incomplete gamma function.

Example:

X = the survival time in weeks. α=8, β=15.

Expected survival time E(X)=(8)(15)=120 weeks.

V(X) = $(8)(15)^2$ = 1800, σ_x= 42.43

P(60≤X≤120) = P(X≤120) - P(X≤60)

= F(120/15;8) – F(60/15;8)

= F(8;8) – F(4;8) = 0.547 – 0.051 = 0.496

P(X≥30) = 1 – P(X<30) = 1 – P(X≤30)

= 1 – F(30/15;8) = 0.999

Exercises

6.45 The length of time for one individual to be served at a cafeteria is a random variable having an exponential distribution with a mean of 4 minutes. What is the probability that a person is served in less than 3 minutes on at least 4 of the next 6 days?

Solution

6.45 $P(X < 3) = \frac{1}{4}\int_0^3 e^{-x/4}\,dx = -e^{-x/4}\Big|_0^3 = 1 - e^{-3/4} = 0.5276.$

Let Y be the number of days a person is served in less than 3 minutes. Then

$$P(Y \geq 4) = \sum_{x=4}^{6} b(y; 6, 1 - e^{-3/4}) = \binom{6}{4}(0.5276)^4(0.4724)^2 + \binom{6}{5}(0.5276)^5(0.4724)$$
$$+ \binom{6}{6}(0.5276)^6 = 0.3968.$$

6.39 If a random variable X has the gamma distribution with $\alpha = 2$ and $\theta = 1$, find $P(1.8 < X < 2.4)$.

Solution

6.39 $P(1.8 < X < 2.4) = \int_{1.8}^{2.4} xe^{-x}\,dx = [-xe^{-x} - e^{-x}]\big|_{1.8}^{2.4} = 2.8e^{-1.8} - 3.4e^{-2.4} = 0.1545.$

6.42 Suppose that the time, in hours, taken to repair a heat pump is a random variable X having a gamma distribution with parameters $\alpha = 2$ and $\beta = 1/2$. What is the probability that the next service call will require

(a) at most 1 hour to repair the heat pump?

(b) at least 2 hours to repair the heat pump?

Solution

6.42 (a) $P(X < 1) = 4\int_0^1 xe^{-2x}\,dx = [-2xe^{-2x} - e^{-2x}]\big|_0^1 = 1 - 3e^{-2} = 0.5940.$

(b) $P(X > 2) = 4\int_0^\infty xe^{-2x}\,dx = [-2xe^{-2x} - e^{-2x}]\big|_2^\infty = 5e^{-4} = 0.0916.$

6.44 In a certain city, the daily consumption of electric power, in millions of kilowatt-hours, is a random variable X having a gamma distribution with mean $\mu = 6$ and variance $\sigma^2 = 12$.

(a) Find the values of α and β.

(b) Find the probability that on any given day the daily power consumption will exceed 12 million kilowatt-hours.

Solution

6.44 (a) $\mu = \alpha\beta = 6$ and $\sigma^2 = \alpha\beta^2 = 12$. Substituting $\alpha = 6/\beta$ into the variance formula we find $6\beta = 12$ or $\beta = 2$ and then $\alpha = 3$.

(b) $P(X > 12) = \frac{1}{16}\int_{12}^{\infty} x^2 e^{-x/2}\, dx$. Integrating by parts twice gives

$$P(X > 12) = \frac{1}{16}\left[-2x^2e^{-x/2} - 8xe^{-x/2} - 16e^{-x/2}\right]\Big|_{12}^{\infty} = 25e^{-6} = 0.0620.$$

The T Distribution

Student's t-distribution has the following probability density function

$$f(t) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})} \left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}},$$

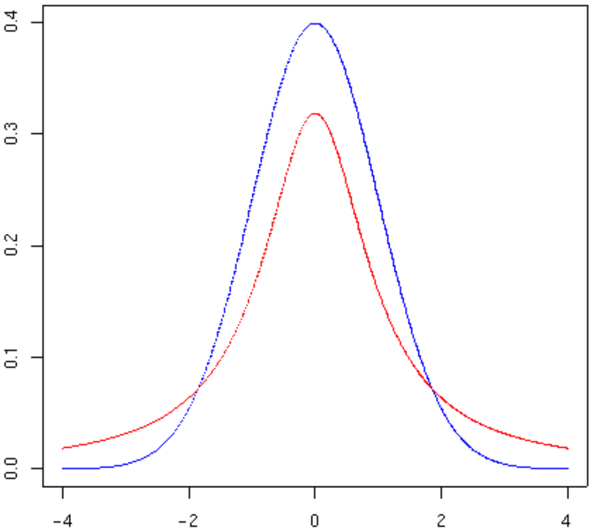
where ν is the number of degrees of freedom and Γ is the Gamma function. This may also be written as

$$f(t) = \frac{1}{\sqrt{\nu}\,B\left(\frac{1}{2},\frac{\nu}{2}\right)} \left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}},$$

where B is the Beta function

Some properties

1- It has mean of zero.
2- It is symmetric about the mean.
3- It ranges from -∞ to ∞.
4- compared to the normal distribution, the t distribution is less peaked in the center and has higher tails.
5- It depends on the degrees of freedom (n-1).
6- t distribution approaches the standard norma distribution as (n-1) approaches ∞.



Density of the *t*-distribution for v=1

Special cases

There are some special cases from t distribution

$$f(t) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})}\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

Certain values of ν give an especially simple form.

- $\nu = 1$

  Distribution function:

  $$F(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(x).$$

  Density function:

  $$f(x) = \frac{1}{\pi(1+x^2)}.$$

  See Cauchy distribution

- $\nu = 2$

  Distribution function:

  $$F(x) = \frac{1}{2}\left[1 + \frac{x}{\sqrt{2+x^2}}\right].$$

  Density function:

  $$f(x) = \frac{1}{(2+x^2)^{\frac{3}{2}}}.$$

- $\nu = 3$

  Density function:

  $$f(x) = \frac{6\sqrt{3}}{\pi\,(3+x^2)^2}.$$

- $\nu = \infty$

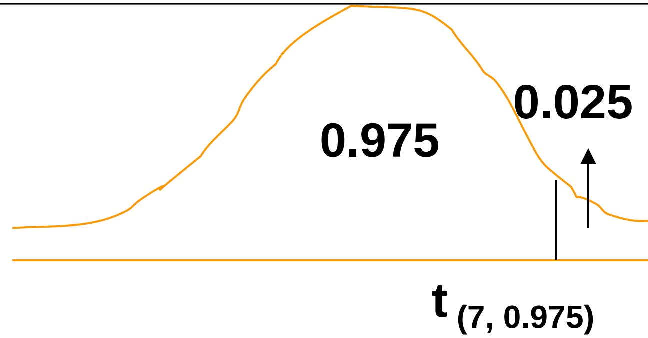
  Density function:

  $$f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.$$

  See Normal distribution

Examples

t (7, 0.975) = 2.3646



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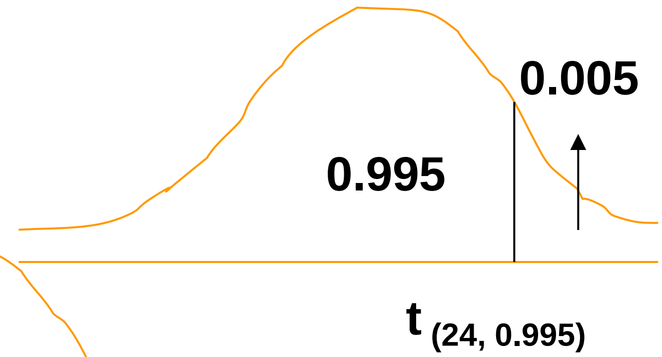
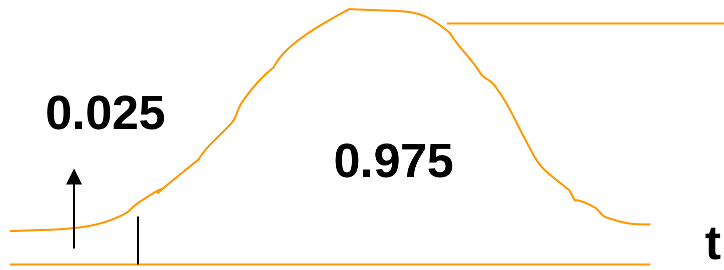
t (24, 0.995) = 2.7696



If P ($T_{(18)}$ > t) = 0.975,
then t = -2.1009



If P ($T_{(22)}$ < t) = 0.99,
then t = 2.508

Also,

$t_{0.95,10}$ = 1.8125

$t_{0.975,18}$ = 2.1009

$t_{0.01,20}$ = - 2.528

$t_{0.10,29}$ = - 1.311