Problem 1 Let $S = \{(1,2,2), (-1,0,2)\} \subseteq \mathbb{R}^3$.

(i) Prove that S is Linear indep ?

(ii) Find a basis B such that $S \subseteq B$?

(iii) Find The arthonormal basis of $\mathbb{R}^3$?

Solution (i) suppose that

$$\lambda_1(1,2,2) + \lambda_2(-1,0,2) = (0,0,0)$$

Then

$$\left.\begin{array}{l} \lambda_1 - \lambda_2 = 0 \\ 2\lambda_2 = 0 \\ 2\lambda_2 + 2\lambda_2 = 0 \end{array}\right\} \Rightarrow \lambda_2 = \lambda_1 = 0$$

Hence S is Linear indep.

(ii) we will extend S to B by adding a vector from the normal basis :

$$B = \{(1,2,2), (-1,0,2), \underline{(0,1,0)}\}$$

Notice that

$$\begin{vmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 2 & 2 & 0 \end{vmatrix} = -1 \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} = -4 \neq 0$$

So, B is Linear indep. As $|B| = 3 = \text{Dim}(\mathbb{R}^3)$,

B is basis.

(iii) we will use Gram-Smidth to find arthonormal basis by using $B = \{\overset{v_1}{(1,2,2)}, \overset{v_2}{(-1,0,2)}, \overset{v_3}{(0,1,0)}\}$

$$e_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{1+4+4}}(1,2,2) = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$e_2 = \frac{v_2 - \langle v_2, e_1\rangle e_1}{\|v_2 - \langle v_2, e_1\rangle e_1\|} \quad (*)$$

$$\text{So} \quad v_2 - \langle v_2, e_1\rangle e_1 = (-1,0,2) - \left(\frac{-1}{3} + \frac{4}{3}\right)\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$
$$= (-1,0,2) - \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \left(\frac{-4}{3}, \frac{-2}{3}, \frac{4}{3}\right)$$

$$\|v_2 - \langle v_2, e_1\rangle e_1\| = \sqrt{\frac{16}{9} + \frac{4}{9} + \frac{16}{9}} = \frac{6}{3} = 2$$

$$\text{So, } e_2 = \left(\frac{4}{6}, \frac{-2}{6}, \frac{4}{6}\right) = \left(\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3}\right)$$

$$e_3 = \frac{v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2}{\|v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2\|} \quad (\#)$$

Complete

The arthonormal basis is $\{e_1, e_2, e_3\}$ ∎

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Problem 2

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 5 & 3 \end{bmatrix}$

(i) find $rank(A)$ and nullity (A^t) ?

(ii) If $B = \{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$ and ${}_S P_B = A$ where S is the standered basis of $\mathbb{R}^3$. find $[v_2]_S$.

Solution

(i) we will write A on the row-echolon form:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 5 & 3 \end{bmatrix} \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 2 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$B_{\text{Column-space}} = \{(1,0,0), (1,1,0)\}$

$\Rightarrow$ Dim (column-space) $= 2 =$ Rank(A).

Therefore

$Rank(A^t) = 2$.

Now, since $Rank(A^t) + nullity(A^t) = \underbrace{n}_{\text{Column of } A^t} = 3$

nullity $(A^t) = 1$.

(ii) $[v_2]_S = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$ Since ${}_S P_B = \begin{bmatrix} [v_1]_S & [v_2]_S & [v_3]_S \end{bmatrix}$.

Problem 3

Let $|A| = 2$ where $size(A) = 4\times 4$. Find

$$\Big|\, |A^2 \cdot 3A^{-1}|\, I \Big| \quad ?$$

Solution

$A^2 \cdot (3A^{-1}) = 3\, A^2 A^{-1} = 3A$

So, $|A^2 \cdot 3A^{-1}| = |3A| = 3^4 |A| = 2(3^4)$

Hence

$$\Big|\, |A^2 \cdot 3A^{-1}|\, I \Big| = \big| 2(3^4)\, I \big|$$
$$= \left[2(3^4)\right]^4 |I| = 2^4 (3^{16}) \quad \blacksquare$$

Problem 4

Let $A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 5 & 6 & 8 \\ 0 & 1 & 7 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 3 & 5 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 7 & 1 \end{bmatrix}$

(i) find $(AB^t)^{-1}$ if it is possible ?

(ii) find solutions of $(AB^t)X = 0$

(iii) Are A and B equivalent ?

Hints $\quad AB^t$ of size 3×3.

(i) To find $(AB^t)^{-1}$, use the rule $C^{-1} = \frac{1}{|C|}\, adj(C)$.

where $C = AB^t$

(ii) The system $(AB^t)X = CX = 0$ is Homogenous and squar.

So, Examine $|C|$!!

(iii) A and B are Equivalent iff $rank(A) = rank(B)$