

Surfaces in the space \mathbb{R}^3

We have been exploring vectors and vector operations in three-dimensional space, and we have developed equations to describe lines, planes, and spheres. In this section, we use our knowledge of planes and spheres, which are examples of three-dimensional figures called *surfaces*, to explore a variety of other surfaces that can be graphed in a three-dimensional coordinate system.

Identifying Cylinders

The first surface we'll examine is the cylinder. Although most people immediately think of a hollow pipe or a soda straw when they hear the word cylinder, here we use the broad mathematical meaning of the term. As we have seen, cylindrical surfaces don't have to be circular. A rectangular heating duct is a cylinder, as is a rolled-up yoga mat, the cross-section of which is a spiral shape.

In the two-dimensional coordinate plane, the equation $x^2 + y^2 = 9$ describes a circle centered at the origin with radius 3. In three-dimensional space, this same equation represents a surface. Imagine copies of a circle stacked on top of each other centered on the z-axis, forming a hollow tube. We can then construct a cylinder from the set of lines parallel to the z-axis passing through the circle $x^2 + y^2 = 9$ in the xy-plane, as shown in the figure. In this way, any curve in one of the coordinate planes can be extended to become a surface.

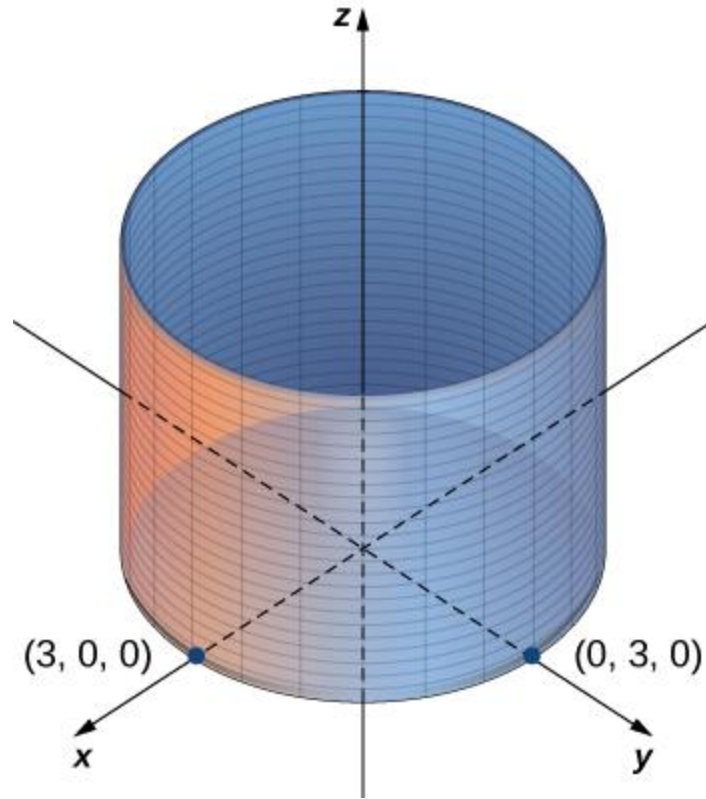


Figure : In three-dimensional space, the graph of equation $x^2 + y^2 = 9$ is a cylinder with radius 3 centered on the z-axis. It continues indefinitely in the positive and negative directions.

DEFINITION: CYLINDERS AND RULINGS

A set of lines parallel to a given line passing through a given curve is known as a cylindrical surface, or *cylinder*. The parallel lines are called *rulings*.

From this definition, we can see that we still have a cylinder in three-dimensional space, even if the curve is not a circle. Any curve can form a cylinder, and the rulings that compose the cylinder may be parallel to any given line.

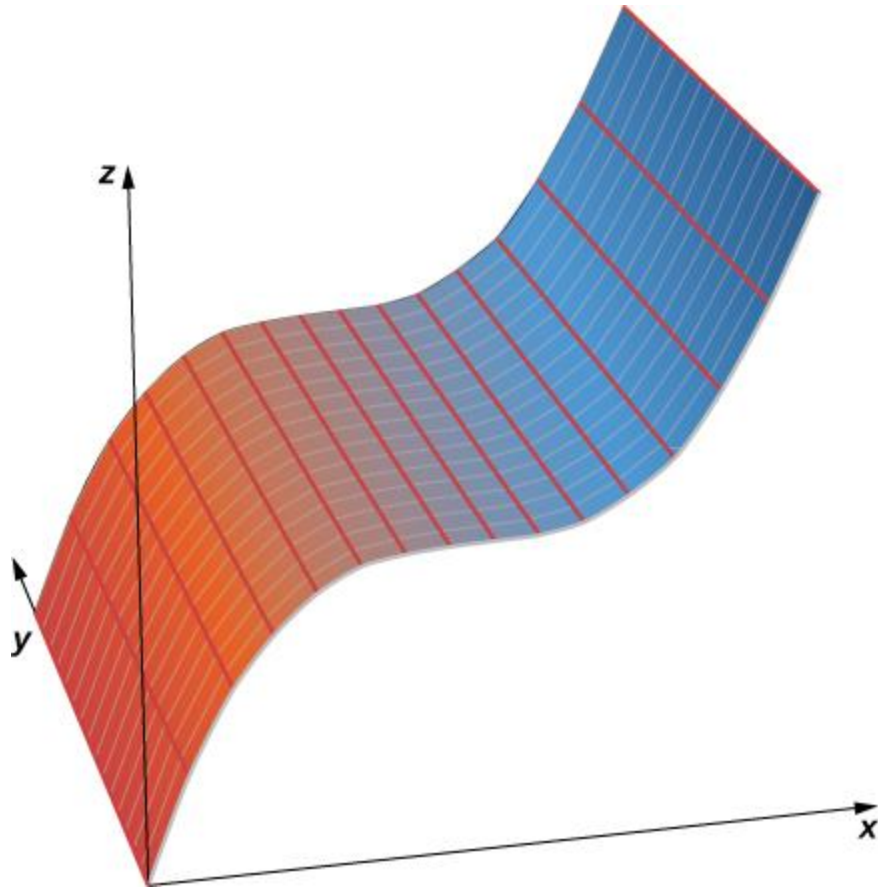


Figure: In three-dimensional space, the graph of equation $z = x^3$ is a cylinder, or a cylindrical surface with rulings parallel to the y-axis.

EXAMPLE : GRAPHING CYLINDRICAL SURFACES

Sketch the graphs of the following cylindrical surfaces.

- $x^2 + z^2 = 25$
- $z = 2x^2 - y$
- $y = \sin x$

Solution

a. The variable y can take on any value without limit. Therefore, the lines ruling this surface are parallel to the y-axis. The intersection of this surface with the xz -plane forms a circle centered at the origin with radius 5.

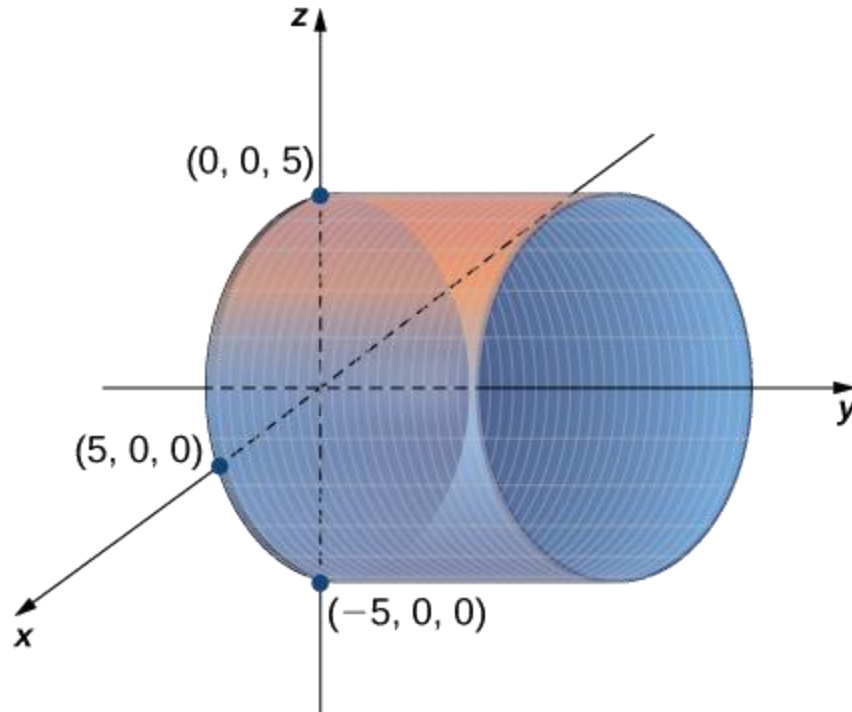
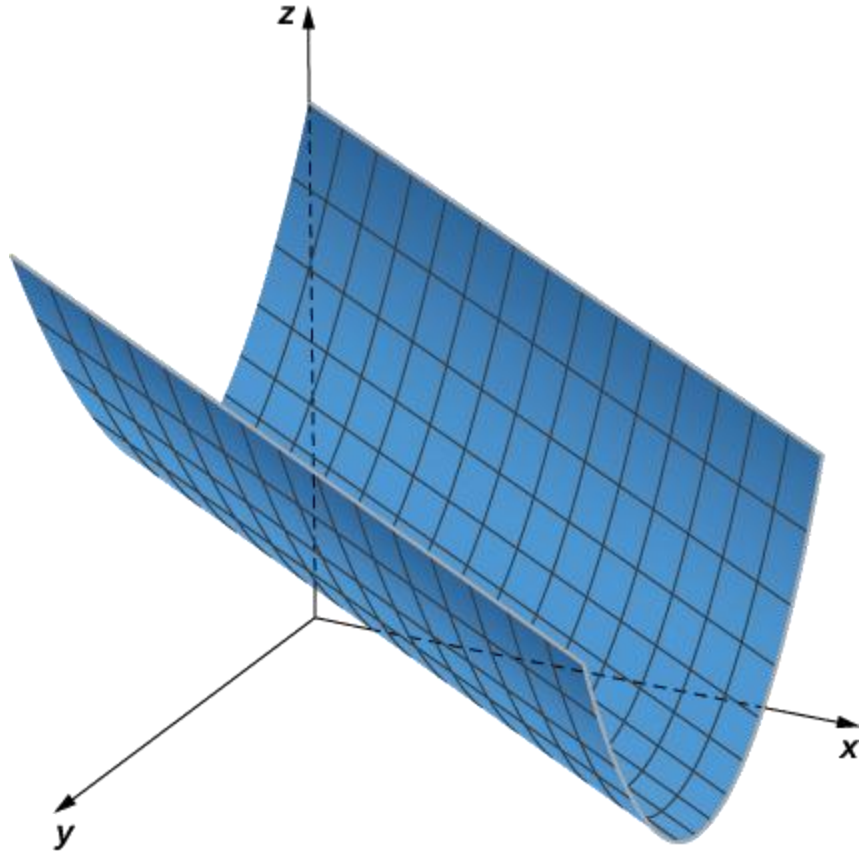


Figure : The graph of equation $x^2 + z^2 = 25$ is a cylinder with radius 5 centered on the y-axis.

b. In this case, the equation contains all three variables —x,y, and z— so none of the variables can vary arbitrarily. The easiest way to visualize this surface is to use a computer graphing utility.



Figure

c. In this equation, the variable z can take on any value without limit. Therefore, the lines composing this surface are parallel to the z -axis. The intersection of this surface with the yz -plane outlines curve $y = \sin x$.

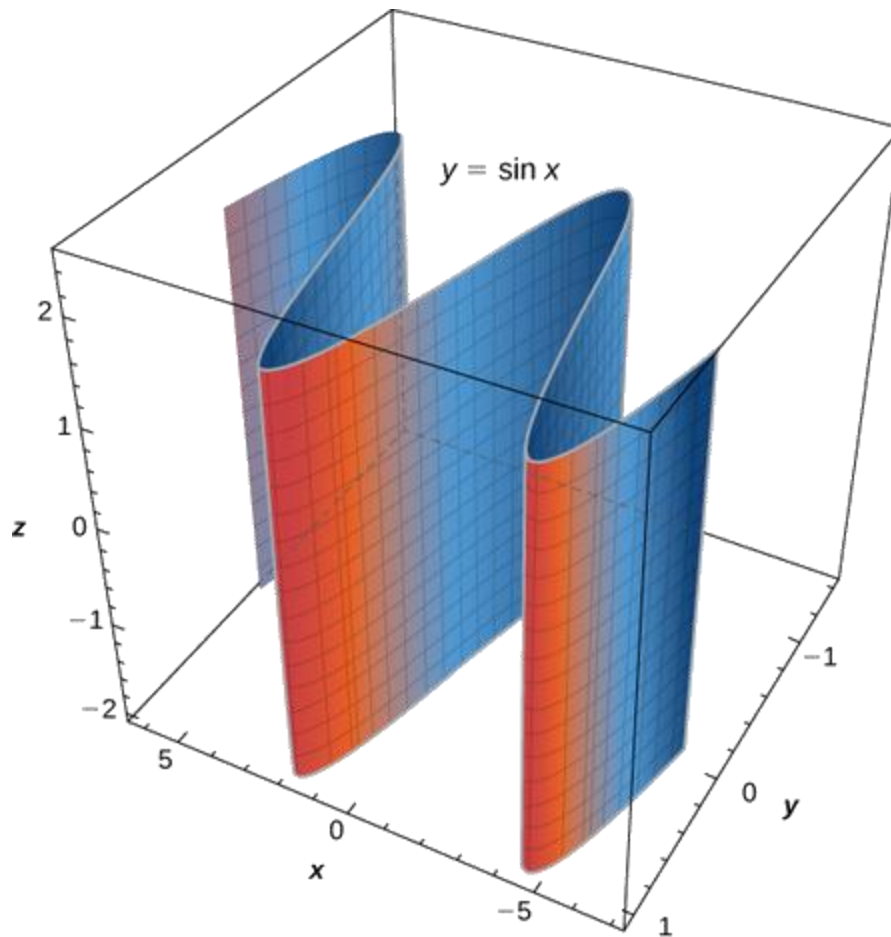


Figure : The graph of equation $y = \sin x$ is formed by a set of lines parallel to the z -axis passing through curve $y = \sin x$ in the xy -plane.

EXERCISE

Sketch or use a graphing tool to view the graph of the cylindrical surface defined by equation $z = y^2$.

Answer When sketching surfaces, we have seen that it is useful to sketch the intersection of the surface with a plane parallel to one of the coordinate planes. These curves are called traces. We can see them in the plot of the cylinder.

DEFINITION: TRACES

The *traces* of a surface are the cross-sections created when the surface intersects a plane parallel to one of the coordinate planes. Traces are useful in sketching cylindrical surfaces. For a cylinder in three dimensions, though, only one set of traces is useful. Notice, in Figure, that the trace of the graph of

$z = \sin x$ in the xz -plane is useful in constructing the graph. The trace in the xy -plane, though, is just a series of parallel lines, and the trace in the yz -plane is simply one line.

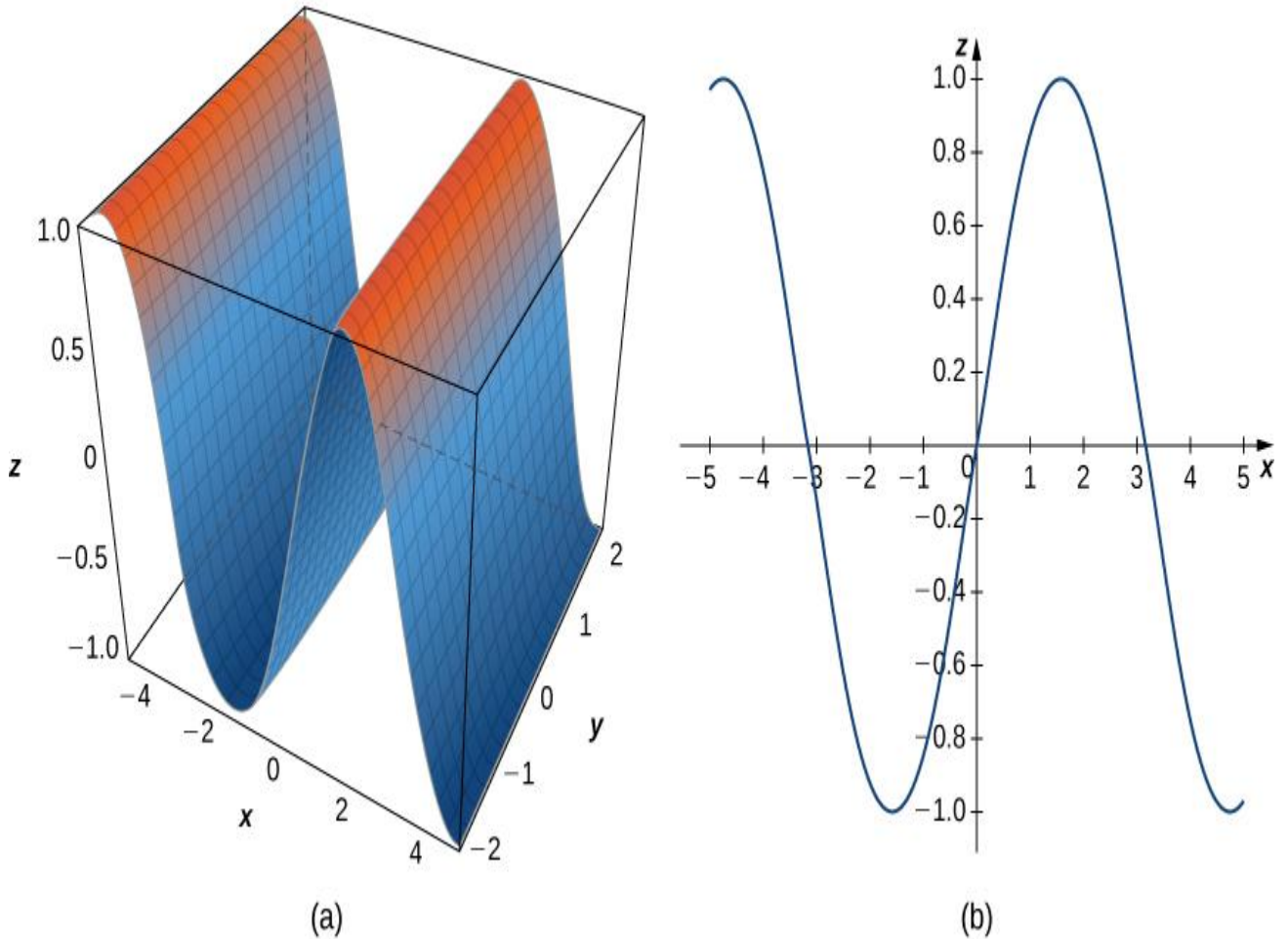


Figure : (a) This is one view of the graph of equation $z = \sin x$. (b) To find the trace of the graph in the xz -plane, set $y = 0$. The trace is simply a two-dimensional sine wave.

Cylindrical surfaces are formed by a set of parallel lines. Not all surfaces in three dimensions are constructed so simply, however. We now explore more complex surfaces, and traces are an important tool in this investigation.

Quadric Surfaces

We have learned about surfaces in three dimensions described by first-order equations; these are planes. Some other common types of surfaces can be described by second-order equations. We can view these surfaces as three-dimensional extensions of the conic sections we discussed earlier: the ellipse, the parabola, and the hyperbola. We call these graphs quadric surfaces

DEFINITION: QUADRIC SURFACES AND CONIC SECTIONS

Quadric surfaces are the graphs of equations that can be expressed in the form

$$A x^2 + B y^2 + C z^2 + D xy + E xz + F yz + G x + H y + J z + K = 0$$

When a quadric surface intersects a coordinate plane, the trace is a **conic section**.

An *ellipsoid* is a surface described by an equation of the form $a^2x^2 + b^2y^2 + c^2z^2 = 1$.

Set $x=0$ to see the trace of the ellipsoid in the yz -plane. To see the traces in the y - and xz -planes, set $z=0$ and $y=0$, respectively. Notice that, if $a=b$, the trace in the xy -plane is a circle. Similarly, if $a=c$, the trace in the xz -plane is a circle and, if $b=c$, then the trace in the yz -plane is a circle. A sphere, then, is an ellipsoid with $a=b=c$.

EXAMPLE : SKETCHING AN ELLIPSOID

Sketch the ellipsoid

$$2^2x^2 + 3^2y^2 + 5^2z^2 = 1.$$

Solution

Start by sketching the traces. To find the trace in the xy -plane, set $z=0$: $2^2x^2 + 3^2y^2 = 1$. To find the other traces, first set $y=0$ and then set $x=0$.

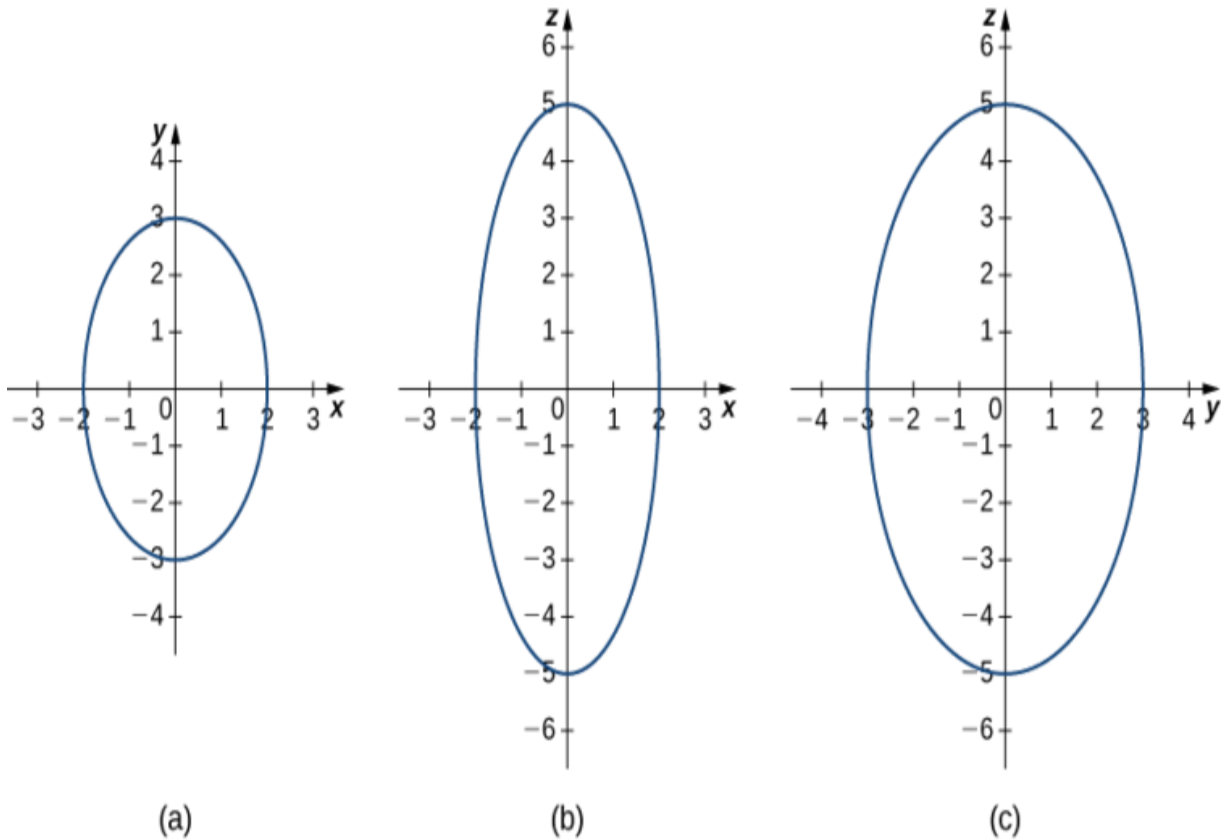


Figure : (a) This graph represents the trace of equation $2^2x^2 + 3^2y^2 + 5^2z^2 = 1$ in the xy -plane, when we set $z=0$.

(b) When we set $y=0$, we get the trace of the ellipsoid in the xz -plane, which is an ellipse.

(c) When we set $x=0$, we get the trace of the ellipsoid in the yz -plane, which is also an ellipse.

Now that we know what traces of this solid look like, we can sketch the surface in three dimensions.

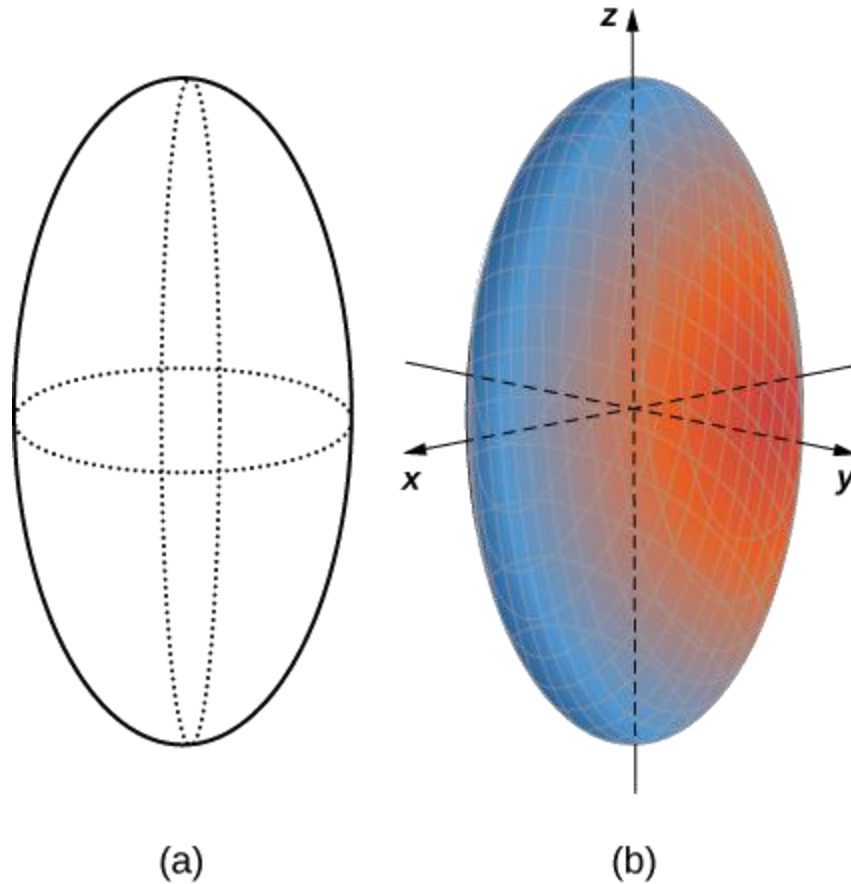


Figure (a) The traces provide a framework for the surface.

(b) The center of this ellipsoid is the origin.

The trace of an ellipsoid is an ellipse in each of the coordinate planes. However, this does not have to be the case for all quadric surfaces. Many quadric surfaces have traces that are different kinds of conic sections, and this is usually indicated by the name of the surface. For example, if a surface can be described by an equation of the form $a^2x^2 + b^2y^2 = cz$.

then we call that surface an **elliptic paraboloid**. The trace in the xy -plane is an ellipse, but the traces in the xz -plane and yz -plane are parabolas.

Other elliptic paraboloids can have other orientations simply by interchanging the variables to give us a different variable in the linear term of the equation $a^2x^2 + c^2z^2 = by$ or $b^2y^2 + c^2z^2 = ax$.

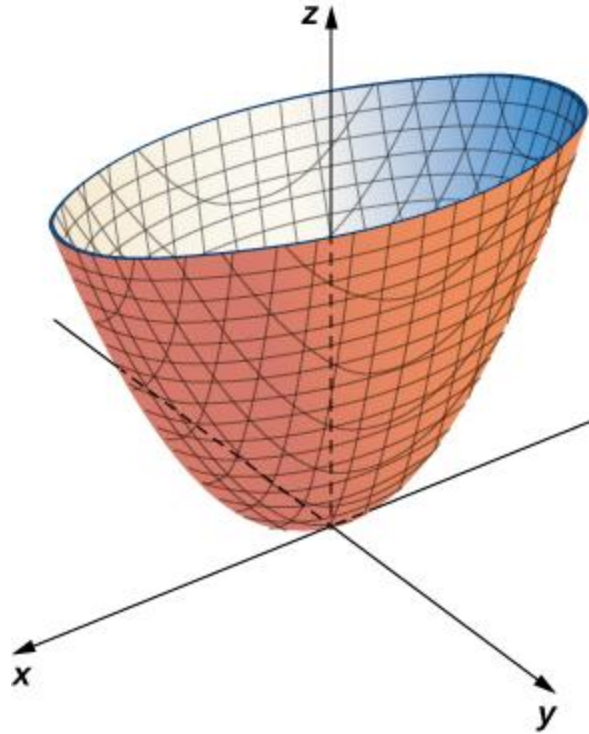


Figure: This quadric surface is called an elliptic paraboloid.

EXAMPLE: IDENTIFYING TRACES OF QUADRIC SURFACES

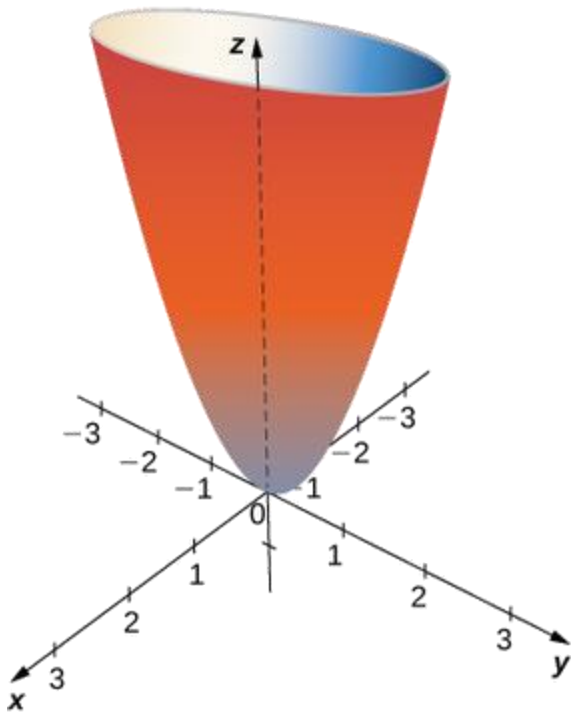
Describe the traces of the elliptic paraboloid $x^2 + 2^2 y^2 = \frac{z}{5}$.

Solution

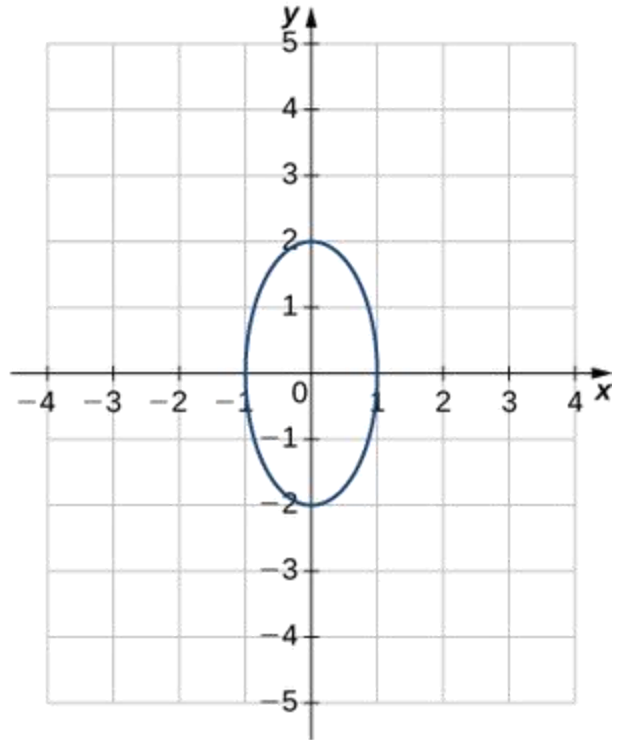
To find the trace in the xy -plane, set $z=0$: $x^2 + 2^2 y^2 = 0$. The trace in the plane $z=0$ is simply one point, the origin. Since a single point does not tell us what the shape is, we can move up the z -axis to an arbitrary plane to find the shape of other traces of the figure.

The trace in plane $z=5$ is the graph of equation: $x^2 + 2^2 y^2 = 1$, which is an ellipse. In the xz -plane, the equation becomes $z = 5x^2$. The trace is a parabola in this plane and in any plane with the equation $y=b$.

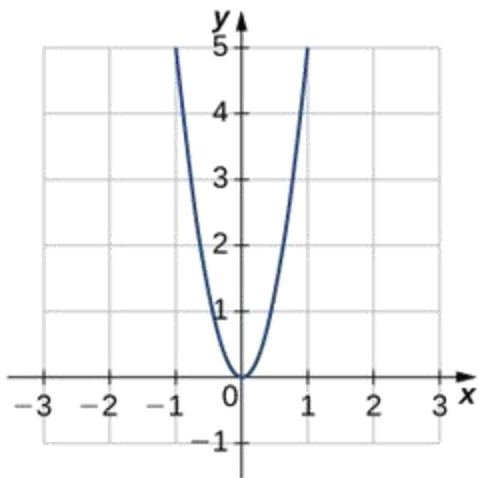
In planes parallel to the yz -plane, the traces are also parabolas, as we can see in Figure below.



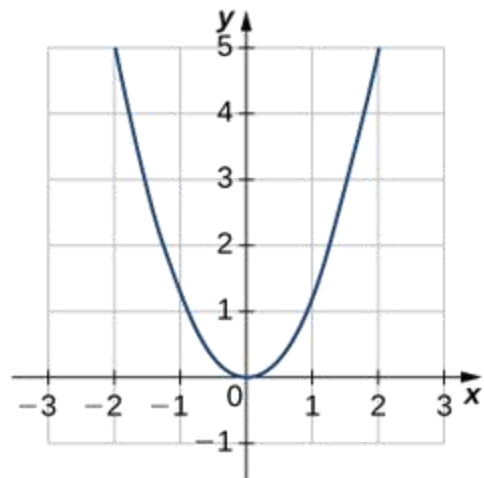
(a)



(b)



(c)



(d)

Figure : (a) The paraboloid $x^2 + 2^2 y^2 = 5z$.

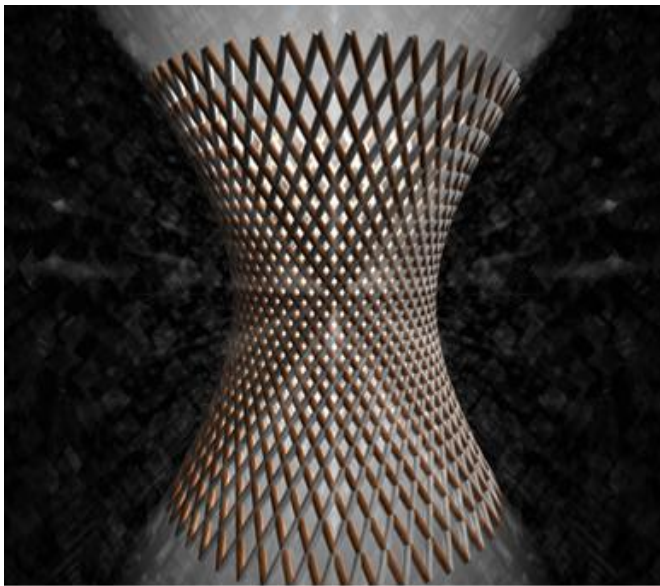
(b) The trace in plane $z=5$.

(c) The trace in the xz -plane. (d) The trace in the yz -plane.

EXERCISE:

A **hyperboloid of one sheet** is any surface that can be described with an equation of the form $a^2x^2 + b^2y^2 - c^2z^2 = 1$. Describe the traces of the hyperboloid of one sheet given by equation $3^2x^2 + 2^2y^2 - 5^2z^2 = 1$

Hyperboloids of one sheet have some fascinating properties. For example, they can be constructed using straight lines, such as in the sculpture in Figure (a). In fact, cooling towers for nuclear power plants are often constructed in the shape of a hyperboloid. The builders are able to use straight steel beams in the construction, which makes the towers very strong while using relatively little material (Figure b).



(a)



(b)

Figure : (a) A sculpture in the shape of a hyperboloid can be constructed of straight lines.

(b) Cooling towers for nuclear power plants are often built in the shape of a hyperboloid.

EXAMPLE : FINDING THE FOCUS OF A PARABOLIC REFLECTOR

Energy hitting the surface of a parabolic reflector is concentrated at the focal point of the reflector. If the surface of a parabolic reflector is described by equation $100x^2 + 100y^2 = 4z$, where is the focal point of the reflector?

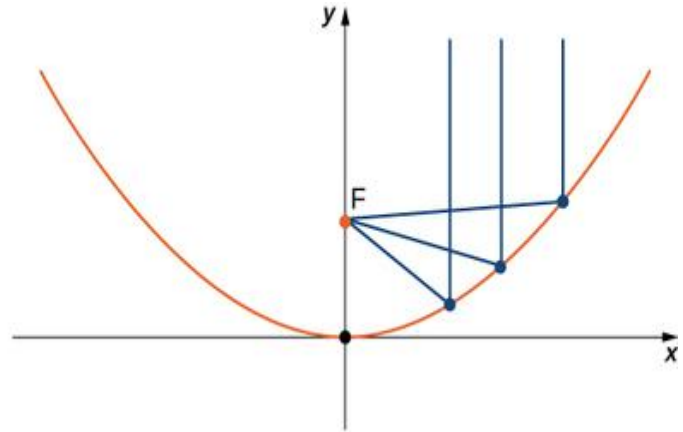


Figure : Energy reflects off of the parabolic reflector and is collected at the focal point. (credit: modification of CGP Grey, Wikimedia Commons)

Solution

Since z is the first-power variable, the axis of the reflector corresponds to the z -axis. The coefficients of x^2 and y^2 are equal, so the cross-section of the **paraboloid** perpendicular to the z -axis is a circle. We can consider a trace in the xz -plane or the yz -plane; the result is the same. Setting $y=0$, the trace is a parabola opening up along the z -axis, with standard equation $100x^2 = 4pz$, where p is the focal length of the parabola. We get p is 6.25 m, which tells us that the focus of the paraboloid is 6.25 m up the axis from the vertex. Because the vertex of this surface is the origin, the focal point is $(0,0,6.25)$.

Seventeen standard quadric surfaces can be derived from the general equation

$$A x^2 + B y^2 + C z^2 + D xy + E xz + F yz + G x + H y + J z + K = 0$$

The following figures summarize the most important ones.

Characteristics of Common Quadric Surfaces

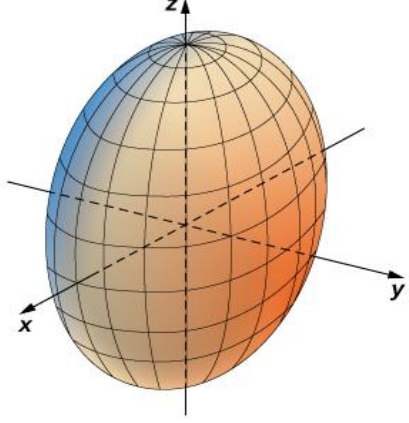
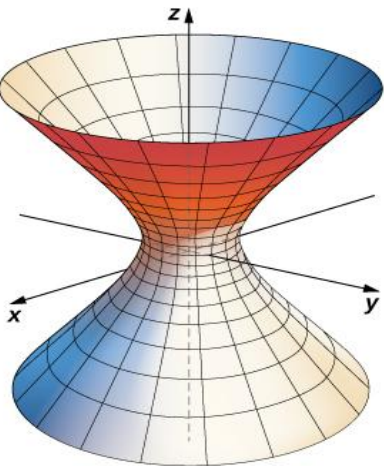
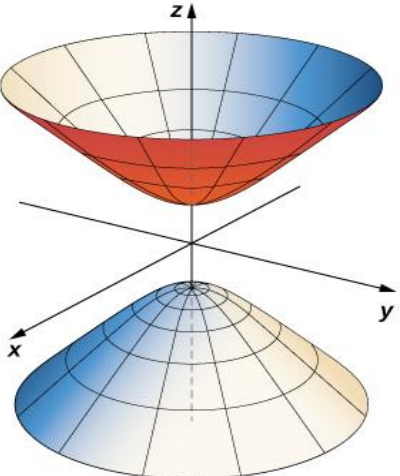
<p style="text-align: center;">Ellipsoid</p> $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ <p><i>Traces</i> In plane $z = p$: an ellipse In plane $y = q$: an ellipse In plane $x = r$: an ellipse</p> <p>If $a = b = c$, then this surface is a sphere.</p>	
<p style="text-align: center;">Hyperboloid of One Sheet</p> $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ <p><i>Traces</i> In plane $z = p$: an ellipse In plane $y = q$: a hyperbola In plane $x = r$: a hyperbola</p> <p>In the equation for this surface, two of the variables have positive coefficients and one has a negative coefficient. The axis of the surface corresponds to the variable with the negative coefficient.</p>	
<p style="text-align: center;">Hyperboloid of Two Sheets</p> $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ <p><i>Traces</i> In plane $z = p$: an ellipse or the empty set (no trace) In plane $y = q$: a hyperbola In plane $x = r$: a hyperbola</p> <p>In the equation for this surface, two of the variables have negative coefficients and one has a positive coefficient. The axis of the surface corresponds to the variable with a positive coefficient. The surface does not intersect the coordinate plane perpendicular to the axis.</p>	

Figure : Characteristics of Common Quadratic Surfaces: Ellipsoid, Hyperboloid of One Sheet, Hyperboloid of Two Sheets.

Characteristics of Common Quadric Surfaces

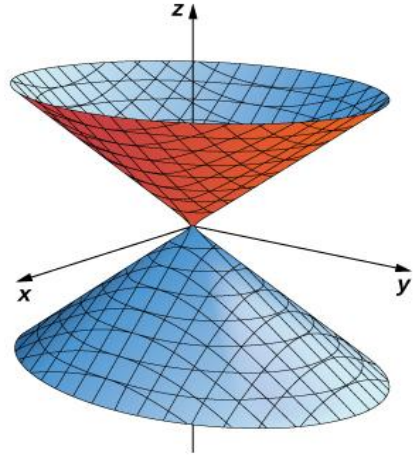
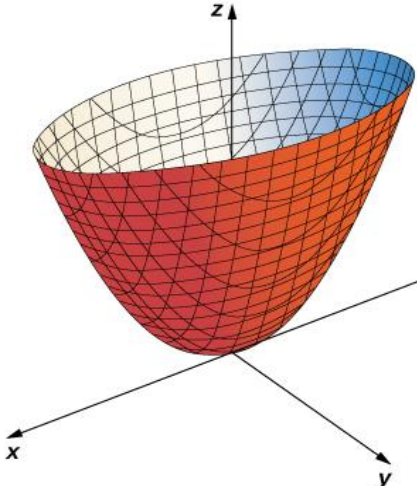
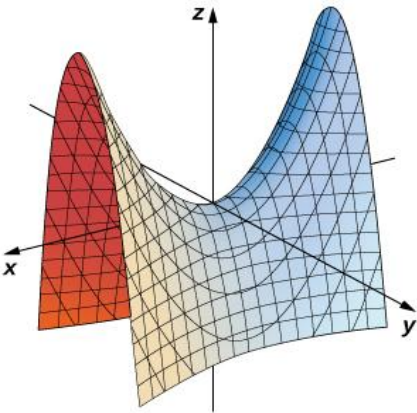
<p style="text-align: center;">Elliptic Cone</p> $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ <p><i>Traces</i> In plane $z = p$: an ellipse In plane $y = q$: a hyperbola In plane $x = r$: a hyperbola In the xz - plane: a pair of lines that intersect at the origin In the yz - plane: a pair of lines that intersect at the origin</p> <p>The axis of the surface corresponds to the variable with a negative coefficient. The traces in the coordinate planes parallel to the axis are intersecting lines.</p>	
<p style="text-align: center;">Elliptic Paraboloid</p> $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ <p><i>Traces</i> In plane $z = p$: an ellipse In plane $y = q$: a parabola In plane $x = r$: a parabola</p> <p>The axis of the surface corresponds to the linear variable.</p>	
<p style="text-align: center;">Hyperbolic Paraboloid</p> $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ <p><i>Traces</i> In plane $z = p$: a hyperbola In plane $y = q$: a parabola In plane $x = r$: a parabola</p> <p>The axis of the surface corresponds to the linear variable.</p>	

Figure: Characteristics of Common Quadratic Surfaces: **Elliptic Cone, Elliptic Paraboloid, Hyperbolic Paraboloid.**

EXAMPLE : IDENTIFYING EQUATIONS OF QUADRIC SURFACES

Identify the surfaces represented by the given equations.

a. $16x^2 + 9y^2 + 16z^2 = 144$

b. $9x^2 - 18x + 4y^2 + 16y - 36z + 25 = 0$

Solution

a. The x, y, and z terms are all squared, and are all positive, so this is probably an ellipsoid. However, let's put the equation into the standard form for an ellipsoid just to be sure. We have

$$16x^2 + 9y^2 + 16z^2 = 144.$$

Dividing through by 144 gives

$$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{9} = 1.$$

So, this is, in fact, an **ellipsoid, centered at the origin.**

b. We first notice that the z term is raised only to the first power, so this is either an elliptic paraboloid or a hyperbolic paraboloid. We also note there are x terms and y terms that are not squared, so this quadric surface is not centered at the origin. We need to complete the square to put this equation in one of the standard forms. We have

$$9x^2 - 18x + 4y^2 + 16y - 36z + 25 = 0 \Leftrightarrow$$

$$9(x^2 - 2x) + 4(y^2 + 4y) - 36z + 25 = 0 \Leftrightarrow$$

$$9((x-1)^2 - 1) + 4((y+2)^2 - 2^2) - 36z + 25 = 0 \Leftrightarrow$$

$$9(x-1)^2 + 4(y+2)^2 = 36z \Leftrightarrow$$

$$\frac{(x-1)^2}{2^2} + \frac{(y+2)^2}{3^2} = z$$

This is an **elliptic paraboloid centered at (1, -2, 0).**