Mathematical Expressions

1. $\Rightarrow$ is the symbol for implying.
2. $\Leftrightarrow$ is the symbol for “$\Rightarrow$ and $\Leftarrow$. Also, the expression “iff" means if and only if .
3. $b > a$ means b is greater than a and $a < b$ means a is less than b.
4. $b \geq a$ to denote that b is greater than or equal to a.

Example 0.1

1. $x = 2 \Rightarrow x+1 = 3$.
2. A triangle is equilateral iff its angles all measure 60" .
3. $x > 2 \Rightarrow x = \{3,4,\ldots\}$.
4. $x \geq 2 \Rightarrow x = \{2,3,4,\ldots\}$.

■

Set of Numbers & Notation

1. Natural numbers: $\mathbb{N} = \{1,2,3,\ldots\}$
2. Whole numbers: $\mathbb{W} = \{0,1,2,3,\ldots\}$
3. Integers: $\mathbb{Z} = \{1,2,3,\ldots\}$
4. Rational numbers: $\mathbb{Q} = \{\frac{a}{b} \mid a,b \in \mathbb{Z} \text{ and } b \neq 0\}$
5. Irrational numbers: $\mathbb{I} = \{x \mid x \text{ is a real number that is not rational}\}$
6. Real numbers: $\mathbb{R}$ contains all the previous sets.

Example 0.2

1. $-1 \in \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$
2. $\frac{1}{2} \in \mathbb{Q},$, and $\mathbb{R}$
3. $\sqrt{2} \in \mathbb{I},$, and $\mathbb{R}$

■

Fraction Operations

- Adding (or subtracting) two fractions:
  1. Find the least common denominator.
  2. Write both original fractions as equivalent fractions with the least common denominator.
  3. Add (or subtract) the numerators.
  4. Write the result with the denominator.

Example 0.3

$$\frac{2}{3} + \frac{3}{5}$$

The least common denominator is 15

$$\frac{2.5}{3.5} + \frac{3.3}{5.3} = \frac{10}{15} + \frac{9}{15} = \frac{19}{15}$$

■

- Multiplying two fractions:
  1. Multiply the numerator by the numerator.
  2. Multiply the denominator by the denominator.

$$\frac{a}{b} . \frac{c}{d} = \frac{ac}{bd}$$

where $b \neq 0$ and $d \neq 0$.

Example 0.4

$$\frac{2}{3} . \frac{3}{5} = \frac{6}{15} = \frac{2}{5}$$

■

- Dividing two fractions:
  1. Change the division sign to multiplication.
  2. Invert the second fraction and multiply the fractions.

■ **Example 0.5**

$$\frac{2}{3} \div \frac{3}{5} = \frac{2}{3} . \frac{5}{3} = \frac{10}{9}$$

■

Exponents

The basic rules given here will be mentioned in detail in Chapter **??**.

Assume n is a positive integer and a is a real number. The expression a^n is given by

$$a^n = a.a\ldots a\ .$$

Basic Rules:
For every $x, y > 0$ and $a, b \in \mathbb{R}$, then

1. $x^0 = 1$.
2. $x^a x^b = x^{a+b}$.
3. $\frac{x^a}{x^b} = x^{a-b}$.
4. $(x^a)^b = x^{ab}$.
5. $(xy)^a = x^a y^a$.
6. $x^{-a} = \frac{1}{x^a}$.

■ **Example 0.6**
1. $2^3 = 2.2.2 = 8$
2. $(-2)^3 = (-2).(-2).(-2) = -8$
3. $x^3.x^4 = x^7$
4. $(x^3)^4 = x^12$
5. $5^{(-2)} = \frac{1}{25}$ ■

Algebraic Expressions

Let a and b be real numbers. Then,

1. $(a+b)^2 = a^2 + 2ab + b^2$
2. $(a-b)^2 = a^2 - 2ab + b^2$
3. $(a+b)(a-b) = a^2 - b^2$
4. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
5. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
6. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
7. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

■ **Example 0.7**
1. $(x+3)^2 = x^2 + 6x + 9$
2. $(y-1)^2 = y^2 - 2y + 1$
3. $(x+2)(x-2) = x^2 - 4$
4. $(x+3)^3 = x^3 + 9x^2 + 27x + 27$
5. $(y-1)^3 = y^3 - 3y^2 + 3y - 1$
6. $x^3 + 8 = (x+2)(x^2 - 2x + 4)$
7. $y^3 - 27 = (y-3)(y^2 + 3y + 9)$

■

Solving Linear Equations

A linear equation is an equation that can be written in the form
$$ax+b=0\ .$$
x is unknown variable and $a,b \in \mathbb{R}$ where $a \neq 0$. To solve the equation, we subtract b from both sides and then divide the result by a:
$$ax+b=0 \Rightarrow ax+b-b=0-b \Rightarrow ax=-b \Rightarrow x=\frac{-b}{a}\ .$$

■ **Example 0.8** Solve the following equation $x+2=5$.

Solution:
$3x+2=5 \Rightarrow 3x=5-2 \Rightarrow 3x=3 \Rightarrow x=\frac{3}{3}=1$. ■

Solving Quadratic Equations

A quadratic equation is an equation that can be written in the form
$$ax^2+bx+c=0\ ,$$
where a, b, and c are constants and $a \neq 0$.

Students can solve the quadratic equations either by factorization method or by the quadratic formula.

• **Factorization Method**

The method is built on

1. finding the factors of c that add up to b, and
2. using the fact that if $x,y \in \mathbb{R}$, then
$$xy=0 \Rightarrow x=0 \text{ or } y=0\ .$$

■ **Example 0.9** $x^2+2x-8=0$
Note that, $2\times(-4)=-8=c$, but $2+(-4)=-2\neq b$.
Now, $-2\times 4=-8=c$ and $-2+4=2=b$.
By factoring the left side, we have
$$(x-2)(x+4)=0$$
$$\Rightarrow x-2=0 \ \text{ or } \ x+4=0$$
$$\Rightarrow x=2 \ \text{ or } \ x=-4\ .$$
■

■ **Example 0.10** $x^2+5x+6=0$
Factoring the left side yields
$$(x+2)(x+3)=0$$
$$\Rightarrow x+2=0 \ \text{ or } \ x+3=0$$
$$\Rightarrow x=-2 \ \text{ or } \ x=-3\ .$$
■

• **Quadratic Formula Solutions**
We can solve the quadratic equations by the quadratic formula:
$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}\ .$$
Remark: The expression b^2-4ac is called the discriminant of the quadratic equation.

1. If $b^2-4ac>0$, then the equation has two distinct solutions.
2. If $b^2-4ac=0$, then the equation has one distinct solution.
3. If $b^2-4ac<0$, then the equation has no real solutions.

■ **Example 0.11** Solve the following quadratic equations:

1. $x^2+2x-8=0$
2. $x^2+2x+1=0$
3. $x^2+2x+8=0$

Solution:
1. $a=1,\ b=2,\ c=-8$
$$x = \frac{-2\pm\sqrt{4+32}}{2} = \frac{-2\pm 6}{2}$$
Thus, the solution are $x=2$ and $x=-4$.
2. $a=1,\ b=2,\ c=1$
Since $b^2-4ac=2^2-4(1)(1)=0$, then there is one solution $x=-1$.
3. Since $b^2-4ac=2^2-4(1)(8)<0$, then there is no real solutions. ■

Systems of Equations

A system of equations consists of two or more equations with a same set of unknowns. The equations in the system can be linear or non-linear, but for the purpose of this book, we consider the linear ones.

Consider systems consist of two equations where a system of two equations in two unknowns x and y can be written as
$$ax+by=c$$
$$dx+ey=f\ .$$
To solve the system, we try to find values for each of the unknowns that will satisfy every equation in the system. Students can use elimination or substitution.

■ **Example 0.12** Solve the following system of equations:
$$x-3y=4$$
$$2x+y=6\ .$$
■

Solution:

• Use elimination:

By multiplying the second equation by 3

$$x - 3y = 4$$

$$6x + 3y = 18 \ .$$

By adding the two equations, we have

$$7x = 22 \Rightarrow x = \frac{22}{7}$$

Substituting the value of x into the first or second equation yields $y = -\frac{2}{7}$.

• Use substitution:

From the first equation, we can have $x = 4 + 3y$. By substituting that into the second equation , we have

$$2(4+3y) + y = 6$$

$$\Rightarrow 7y + 8 = 6$$

$$\Rightarrow y = -\frac{2}{7}$$

Thus, by substituting into $x = 4 + 3y$, we have $\frac{22}{7}$.

Pythagoras Theorem

If c denotes the length of the hypotenuse and a and b denote the lengths of the other two sides, the Pythagorean theorem can be expressed as the

$$a^2 + b^2 = c^2 \ \Rightarrow c = \sqrt{a^2 + b^2} \ .$$

If a and c are known and b is unknown, then

$$b = \sqrt{c^2 - a^2} \ .$$

Similarly, if b and c are known and a is unknown, then

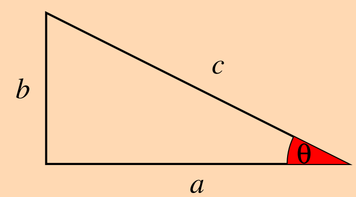
$$a = \sqrt{c^2 - b^2}$$

The trigonometric functions for a right triangle:

$$\cos\theta = \frac{a}{c} \quad \cot\theta = \frac{a}{b}$$

$$\sin\theta = \frac{b}{c} \quad sec\theta = \frac{c}{a}$$

$$\tan\theta = \frac{b}{a} \quad csc\theta = \frac{c}{b}$$



a is adjacent
b is opposite
c is hypotenuse

Trigonometric Functions

If (x,y) is a point on the unit circle, and if the ray from the origin $(0,0)$ to that point (x,y) makes an angle θ with the positive x-axis, then

$$\cos\theta = x\ ,\quad \sin\theta = y\ ,$$

- Each point (x,y) on the unit circle can be written as $(\cos\theta, \sin\theta)$.
- From the equation $x^2+y^2=1$, we have

$$\cos^2\theta + \sin^2\theta = 1\ .$$

  From this,

$$1+\tan^2\theta = sec^2\theta\ ,$$

$$\cot^2\theta + 1 = csc^2\theta\ .$$

- Exact values of trigonometric functions of most commonly used angles:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1
$\tan\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	undefined	0

- Trigonometric functions of negative angles:

$$\cos(-\theta) = \cos(\theta),\ \sin(-\theta) = -\sin(\theta),$$

$$\tan(-\theta) = -\tan(\theta)$$

- Double and half angle formulas

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

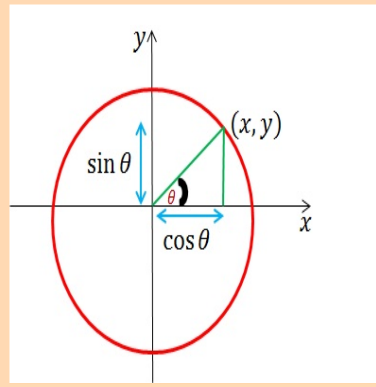
$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}},\ \cos\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$$

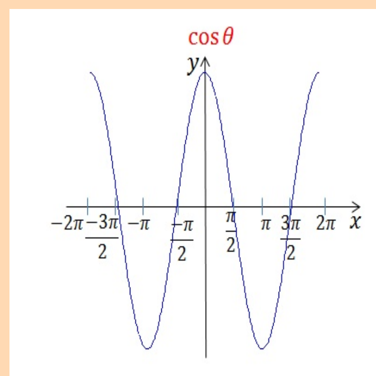
- Angle addition formulas

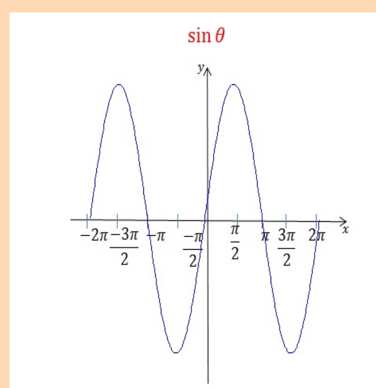
$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1\cos\theta_2 \pm \cos\theta_1\sin\theta_2$$

$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2$$

$$\tan(\theta_1 \pm \theta_2) = \frac{\tan\theta_1 \pm \tan\theta_2}{1 \mp \tan\theta_1\tan\theta_2}$$







Distance Formula

Let $P=(x_1,y_1)$ and $Q=(x_2,y_2)$ are two points in the Cartesian plane. Then the distance between P and Q is

$$PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\ .$$

Example 0.13 Find the distance between $P=(0,1)$ and $Q=(1,4)$.

$$PQ=\sqrt{(1-0)^2+(4-1)^2}=\sqrt{10}$$

■

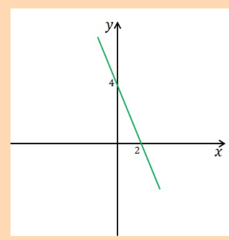
Lines

The general linear equation in two variables x and y can be written in the form:

$$ax+by+c=0\ ,$$

where a, b and c are constants with a and b not both 0.

Example 0.14 $2x-y=4$ $a=2$, $b=-1$, $c=-4$ ■



Slope

The general linear equation can be rewritten as

$$ax+by+c=0 \Rightarrow by=-ax+c \Rightarrow y=-\frac{a}{b}x+\frac{c}{b} \Rightarrow y=mx+d\ ,$$

where m is the slope.
By knowing $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ lying on a straight line, the slope is

$$m=\frac{y_2-y_1}{x_2-x_1}\ .$$

Example 0.15 Find the slope of the line $2x-5y+9=0$.
Solution:
$2x-5y+9=0 \Rightarrow -5y=-2x-9 \Rightarrow y=\frac{2}{5}x+\frac{9}{5}$.
Thus, the slope is $\frac{2}{5}$. Alternatively, take any two points lie on that line like $(-2,1)$ and $(3,3)$. Then,

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{3-1}{3-(-2)}=\frac{2}{5}\ .$$

■

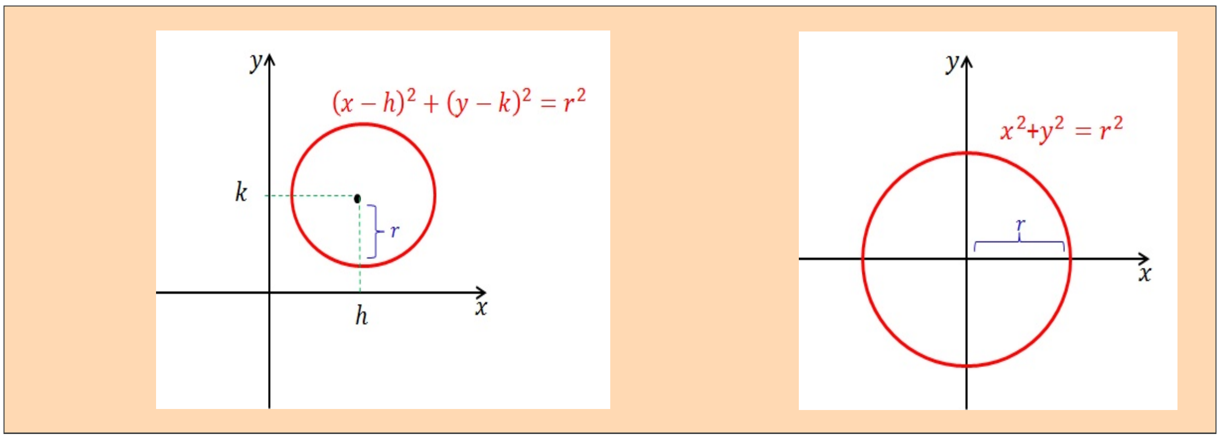
Circles

Let $C(h,k)$ be the center of a circle and r be the radius. Then, the equation of the circle is

$$(x-h)^2+(y-k)^2=r^2$$

If $h=k=0$, the center of the circle is at the origin $(0,0)$ and the equation of the circle becomes

$$x^2+y^2=r^2 .$$



➤ Conic Sections

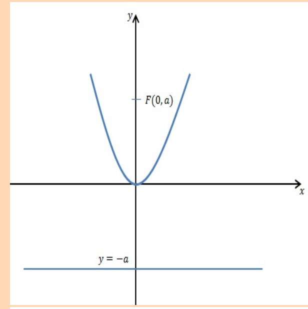
• Parabola:

A parabola is the set of all points in the plane equidistant from a fixed point F (called the focus) and a fixed line D (called the directrix) in the same plane.

(1) The vertex of the parabola is the origin $(0,0)$:

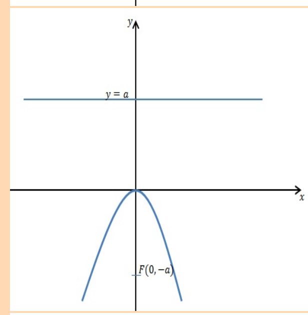
(A) $x^2 = 4ay$, where $a > 0$.

- The parabola opens upward.
- Focus: $F(0,a)$.
- Directrix equation: $y = -a$.
- Parabola axis: the y-axis.



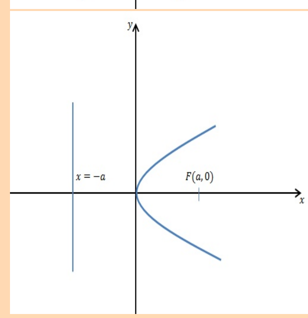
(B) $x^2 = -4ay$, where $a > 0$.

- The parabola opens downward.
- Focus: $F(0,-a)$.
- Directrix equation: $y = a$.
- Parabola axis: the y-axis.



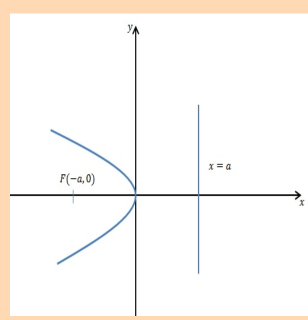
(C) $y^2 = 4ax$, where $a > 0$.

- The parabola opens to the right.
- Focus: $F(a,0)$.
- Directrix equation: $x = -a$.
- Parabola axis: the x-axis.



(D) $y^2 = -4ax$, where $a > 0$.

- The parabola opens to the left.
- Focus: $F(-a,0)$.
- Directrix equation: $x = a$.
- Parabola axis: the x-axis.



(2) The general formula of a parabola:

(A) $(x-h)^2 = 4a(y-k)$, where $a > 0$.
- Vertex: $V(h,k)$.
- The parabola opens upwards.
- Focus: $F(h,k+a)$.
- Directrix equation: $y = k-a$.
- Parabola axis: parallel to the y-axis.

(B) $(x-h)^2 = -4a(y-k)$, where $a > 0$.
- Vertex: $V(h,k)$.
- The parabola open downwards.
- Focus: $F(h,k-a)$.
- Directrix equation: $y = k+a$.
- Parabola axis: parallel to the y-axis.

(C) $(y-k)^2 = 4a(x-h)$, where $a > 0$
- Vertex: $V(h,k)$.
- The parabola opens to the right.
- Focus: $F(h+a,k)$.
- Directrix equation: $x = h-a$.
- Parabola axis: parallel to the x-axis.

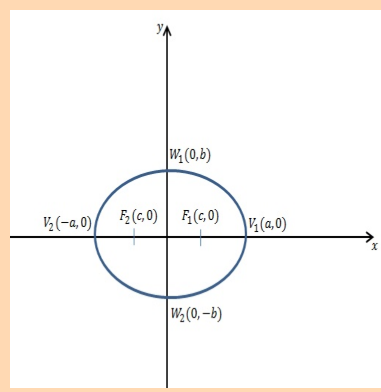
(D) $(y-k)^2 = -4a(x-h)$, where $a > 0$
- Vertex: $V(h,k)$.
- The parabola opens to the left.
- Focus: $F(h-a,k)$.
- Directrix equation: $x = h+a$.
- Parabola axis: parallel to the x-axis.

• Ellipse:

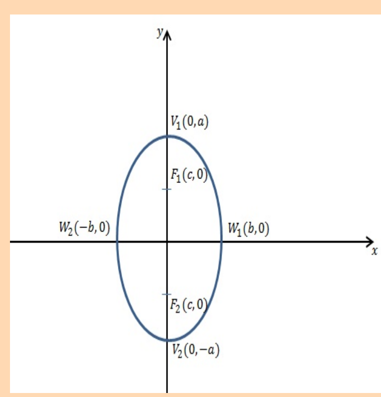
An ellipse is the set of all points in the plane for which the sum of the distances to two fixed points is constant.

(1) The center of the ellipse is the origin $(0,0)$:

(A) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b$ and $c = \sqrt{a^2 - b^2}$.
- Foci: $F_1(-c,0)$ and $F_2(c,0)$.
- Vertices: $V_1(-a,0)$ and $V_2(a,0)$.
- Major axis: the x-axis, its length is $2a$.
- Minor axis: the y-axis, its length is $2b$.
- Minor endpoints axis: $W_1(0,b)$ and $W_2(0,-b)$.



(B) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a < b$ and $c = \sqrt{b^2 - a^2}$.
- Foci: $F_1(0,c)$ and $F_2(0,-c)$.
- Vertices: $V_1(0,b)$ and $V_2(0,-b)$.
- Major axis: the y-axis, its length is $2b$.
- Minor axis: the x-axis, its length is $2a$.
- Minor endpoints axis: $W_1(-a,0)$ and $W_2(a,0)$.



(2) The general formula of the ellipse:

(A) $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ where $a > b$ and $c = \sqrt{a^2 - b^2}$.

- Center: $P(h,k)$.
- Foci: $F_1(h-c,k)$ and $F_2(h+c,k)$.
- Vertices: $V_1(h-a,k)$ and $V_2(h+a,k)$.
- Major axis: parallel to the x-axis, its length is $2a$.
- Minor axis: parallel to the y-axis, its length is $2b$.
- Minor endpoints axis: $W_1(h,k+b)$ and $W_2(h,k-b)$.

(B) $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ where $a < b$ and $c = \sqrt{b^2 - a^2}$.

- Center: $P(h,k)$.
- Foci: $F_1(h,k+c)$ and $F_2(h,k-c)$.
- Vertices: $V_1(h,k+b)$ and $V_2(h,k-b)$.
- Major axis: parallel to the y-axis, its length is $2b$.
- Minor axis: parallel to the x-axis, its length is $2a$.
- Minor endpoints axis: $W_1(h-a,k)$ and $W_2(h+a,k)$.

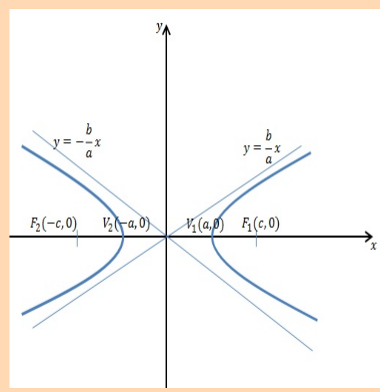
• Hyperbola:

A hyperbola is the set of all points in the plane for which the absolute difference of the distances between two fixed points is constant.

(1) The center of the hyperbola is the origin $(0,0)$:

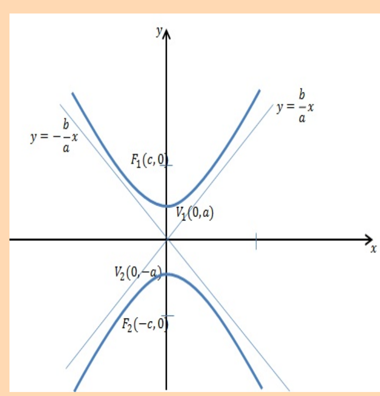
(A) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $a > b > 0$ and $c = \sqrt{a^2 + b^2}$.

- Foci: $F_1(-c,0)$ and $F_2(c,0)$.
- Vertices: $V_1(-a,0)$ and $V_2(a,0)$.
- Transverse axis: the x-axis, its length is $2a$.
- Asymptotes: $y = \pm\frac{b}{a}x$.



(B) $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ where $0 < a < b$ and $c = \sqrt{a^2 + b^2}$.

- Foci: $F_1(0,c)$ and $F_2(0,-c)$.
- Vertices: $V_1(0,b)$ and $V_2(0,-b)$.
- Transverse axis: the y-axis, its length is $2b$.
- Asymptotes: $y = \pm\frac{b}{a}x$.



(2) The general formula of the hyperbola:

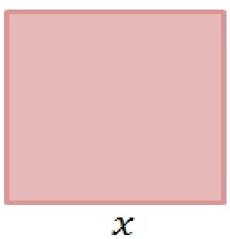
(A) $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ where $a > b > 0$ and $c = \sqrt{a^2+b^2}$.

- Center: $P(h,k)$.
- Foci: $F_1(h-c,k)$ and $F_2(h+c,k)$.
- Vertices: $V_1(h-a,k)$ and $V_2(h+a,k)$.
- Transverse axis: parallels to the x-axis, its length is $2a$.
- Asymptotes: $(y-k) = \pm\frac{b}{a}(x-h)$.

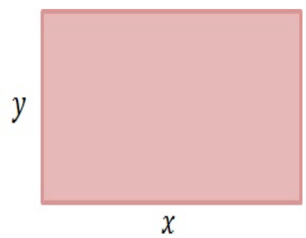
(B) $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$ where $0 < a < b$ and $c = \sqrt{a^2+b^2}$.

- Center: $P(h,k)$.
- Foci: $F_1(h,k+c)$ and $F_2(h,k-c)$.
- Vertices: $V_1(h,k+b)$ and $V_2(h,k-b)$.
- Transverse axis: parallels to the y-axis, its length is $2b$.
- Asymptotes: $(y-k) = \pm\frac{b}{a}(x-h)$.

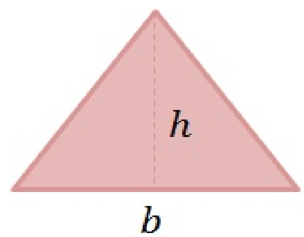
Area and Volume of Special Shapes:



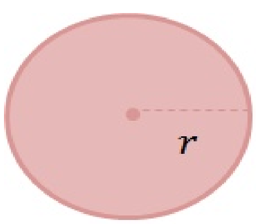
$Area = x^2$



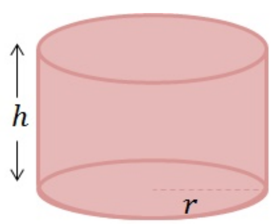
$Area = xy$



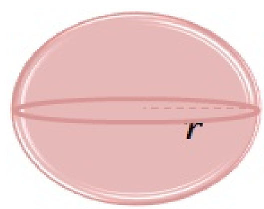
$Area = \frac{1}{2}bh$



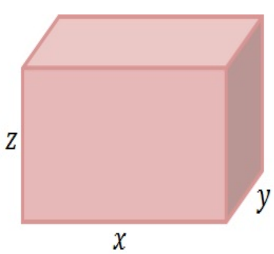
$Area = \pi r^2$



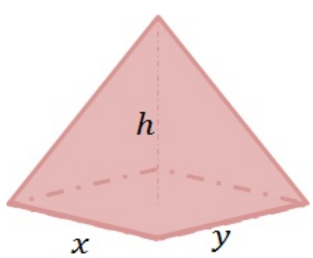
$Volume = \pi r^2 h$



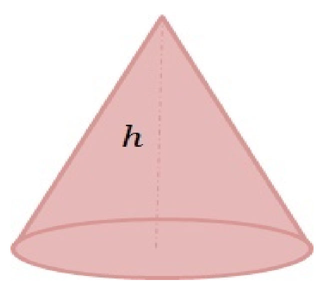
$Volume = \frac{4}{3}\pi r^3$



$Volume = xyz$



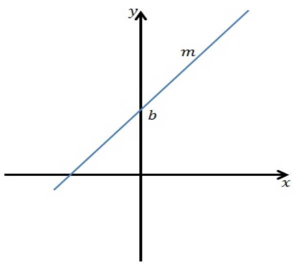
$Volume = \frac{1}{3}xyh$



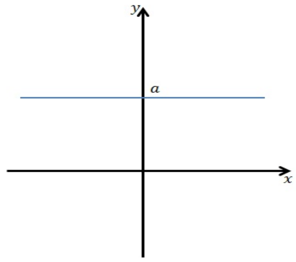
$Volume = \frac{1}{3}\pi r^2 h$

➤ **Sketch of Some functions:**

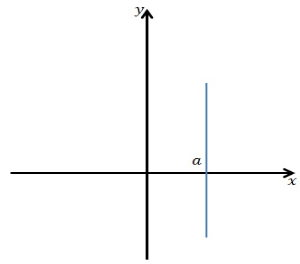
$y = mx + b$



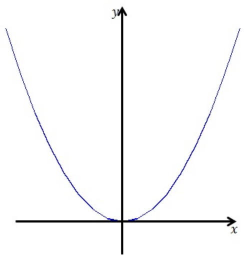
$y = a$



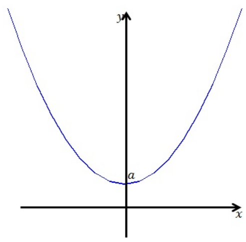
$x = a$



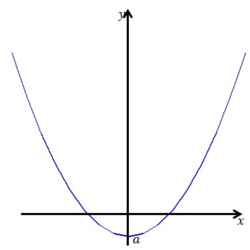
$y = x^2$



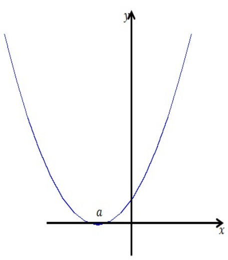
$y = x^2 + a$



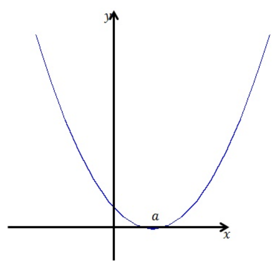
$y = x^2 - a$



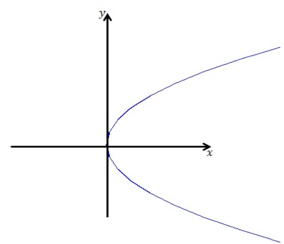
$y = (x + a)^2$



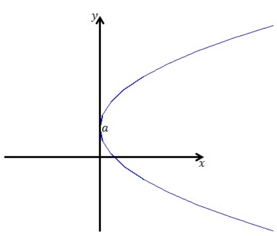
$y = (x - a)^2$



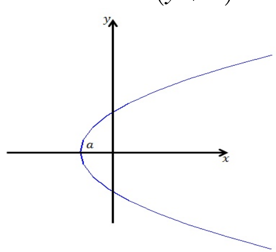
$x = y^2$



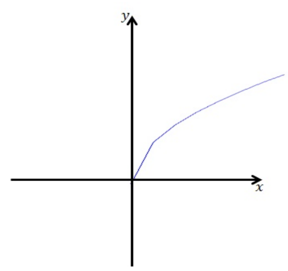
$x = y^2 + a$



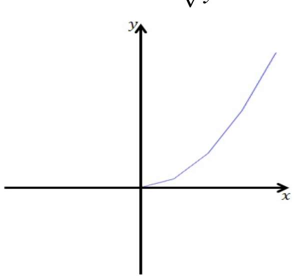
$x = (y + a)^2$



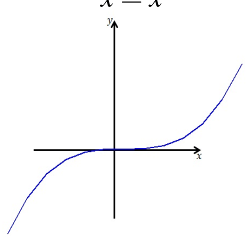
$y = \sqrt{x}$



$x = \sqrt{y}$



$x = x^3$



$y = | x |$