Chapter 1

1. (*a*) Applying formula (1.1)

$$A(t) = t^2 + 2t + 3 \quad \text{and} \quad A(0) = 3$$

so that

$$a(t) = \frac{A(t)}{k} = \frac{A(t)}{A(0)} = \frac{1}{3}\left(t^2 + 2t + 3\right).$$

(*b*) The three properties are listed on p. 2.

(1) $a(0) = \frac{1}{3}(3) = 1.$

(2) $a'(t) = \frac{1}{3}(2t + 2) > 0 \quad \text{for} \quad t \geq 0,$

so that $a(t)$ is an increasing function.

(3) $a(t)$ is a polynomial and thus is continuous.

(*c*) Applying formula (1.2)

$$\begin{aligned} I_n &= A(n) - A(n-1) = \left[n^2 + 2n + 3\right] - \left[(n-1)^2 + 2(n-1) + 3\right] \\ &= n^2 + 2n + 3 - n^2 + 2n - 1 - 2n + 2 - 3 \\ &= 2n + 1. \end{aligned}$$

2. (*a*) Appling formula (1.2)

$$\begin{aligned} I_1 + I_2 + \ldots + I_n &= \left[A(1) - A(0)\right] + \left[A(2) - A(1)\right] + \cdots + \left[A(n) - A(n-1)\right] \\ &= A(n) - A(0). \end{aligned}$$

(*b*) The LHS is the increment in the fund over the *n* periods, which is entirely attributable to the interest earned. The RHS is the sum of the interest earned during each of the *n* periods.

3. Using ratio and proportion

$$\frac{5000}{11{,}130}\left(12{,}153.96 - 11{,}575.20\right) = \$260.$$

4. We have $a(t) = at^2 + b,$ so that

$$\begin{aligned} a(0) &= \phantom{9a + {}} b = 1 \\ a(3) &= 9a + b = 1.72. \end{aligned}$$

Solving two equations in two unknowns $a = .08$ and $b = 1$. Thus,

$$a(5) = 5^2(.08) + 1 = 3$$

$$a(10) = 10^2(.08) + 1 = 9.$$

and the answer is $100\dfrac{a(10)}{a(5)} = 100\dfrac{9}{3} = 300.$

5. (*a*) From formula (1.4*b*) and $A(t) = 100 + 5t$

$$i_5 = \frac{A(5) - A(4)}{A(4)} = \frac{125 - 120}{120} = \frac{5}{120} = \frac{1}{24}.$$

(*b*) $i_{10} = \dfrac{A(10) - A(9)}{A(9)} = \dfrac{150 - 145}{145} = \dfrac{5}{145} = \dfrac{1}{29}.$

6. (*a*) $A(t) = 100(1.1)^t$ and

$$i_5 = \frac{A(5) - A(4)}{A(4)} = \frac{100\left[(1.1)^5 - (1.1)^4\right]}{100(1.1)^4} = 1.1 - 1 = .1.$$

(*b*) $i_{10} = \dfrac{A(10) - A(9)}{A(9)} = \dfrac{100\left[(1.1)^{10} - (1.1)^9\right]}{100(1.1)^9} = 1.1 - 1 = .1.$

7. From formula (1.4*b*)

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)}$$

so that

$$A(n) - A(n-1) = i_n A(n-1)$$

and

$$A(n) = (1 + i_n)A(n-1).$$

8. We have $i_5 = .05$, $i_6 = .06$, $i_7 = .07$, and using the result derived in Exercise 7

$$\begin{aligned} A(7) &= A(4)(1+i_5)(1+i_6)(1+i_7) \\ &= 1000(1.05)(1.06)(1.07) = \$1190.91. \end{aligned}$$

9. (*a*) Applying formula (1.5)

$$615 = 500(1 + 2.5i) = 500 + 1250i$$

so that

$$1250i = 115 \quad \text{and} \quad i = 115/1250 = .092, \quad \text{or} \ \ 9.2\%.$$

(*b*) Similarly,

$$630 = 500(1 + .078t) = 500 + 39t$$

so that

$$39t = 130 \quad \text{and} \quad t = 130/39 = 10/3 = 3\tfrac{1}{3} \ \text{years.}$$

10. We have

$$1110 = 1000(1 + it) = 1000 + 1000it$$
$$1000it = 110 \quad \text{and} \quad it = .11$$

so that

$$500\left[1 + \left(\frac{3}{4}\right)(i)(2t)\right] = 500[1 + 1.5it]$$
$$= 500[1 + (1.5)(.11)] = \$582.50.$$

11. Applying formula (1.6)

$$i_n = \frac{i}{1 + i(n-1)} \quad \text{and} \quad .025 = \frac{.04}{1 + .04(n-1)}$$

so that

$$.025 + .001(n-1) = .04, \quad .001n = .016, \quad \text{and} \quad n = 16.$$

12. We have

$$i_1 = .01 \quad i_2 = .02 \quad i_3 = .03 \quad i_4 = .04 \quad i_5 = .05$$

and adapting formula (1.5)

$$1000\left[1 + (i_1 + i_2 + i_3 + i_4 + i_5)\right] = 1000(1.15) = \$1150.$$

13. Applying formula (1.8)

$$600(1+i)^2 = 600 + 264 = 864$$

which gives

$$(1+i)^2 = 864/600 = 1.44, \quad 1 + i = 1.2, \quad \text{and} \quad i = .2$$

so that

$$2000(1+i)^3 = 2000(1.2)^3 = \$3456.$$

14. We have

$$\frac{(1+i)^n}{(1+j)^n} = (1+r)^n \quad \text{and} \quad 1 + r = \frac{1+i}{1+j}$$

so that

$$r = \frac{1+i}{1+j} - 1 = \frac{(1+i)-(1+j)}{1+j} = \frac{i-j}{1+j}.$$

This type of analysis will be important in Sections 4.7 and 9.4.

15. From the information given:

$$\begin{aligned} (1+i)^a &= 2 & \quad (1+i)^a &= 2 \\ 2(1+i)^b &= 3 & \quad (1+i)^b &= 3/2 \\ 3(1+i)^c &= 15 & \quad (1+i)^c &= 5 \\ 6(1+i)^n &= 10 & \quad (1+i)^n &= 5/3. \end{aligned}$$

By inspection $\frac{5}{3} = 5 \cdot \frac{2}{3} \cdot \frac{1}{2}$. Since exponents are addictive with multiplication, we have $n = c - a - b$.

16. For one unit invested the amount of interest earned in each quarter is:

Quarter:	1	2	3	4
Simple:	.03	.03	.03	.03
Compound:	$1.03-1$	$(1.03)^2-1.03$	$(1.03)^3-(1.03)^2$	$(1.03)^4-(1.03)^3$

Thus, we have

$$\frac{D(4)}{D(3)} = \frac{\left[(1.03)^4-(1.03)^3\right]-.03}{\left[(1.03)^3-(1.03)^2\right]-.03} = 1.523.$$

17. Applying formula (1.12)

$$\begin{aligned} A&: \quad 10,000\left[(1.06)^{-18}+(1.06)^{-19}\right] = 6808.57 \\ B&: \quad 10,000\left[(1.06)^{-20}+(1.06)^{-21}\right] = \underline{6059.60} \\ && \text{Difference} = \$748.97. \end{aligned}$$

18. We have

$$v^n + v^{2n} = 1$$

and multiplying by $(1+i)^{2n}$

$$(1+i)^n + 1 = (1+i)^{2n}$$

or

$$(1+i)^{2n} - (1+i)^n - 1 = 0 \quad \text{which is a quadratic.}$$

Solving the quadratic

$$(1+i)^n = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1+\sqrt{5}}{2} \quad \text{rejecting the negative root.}$$

Finally,

$$(1+i)^{2n} = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{3+\sqrt{5}}{2}.$$

19. From the given information $500(1+i)^{30} = 4000$ or $(1+i)^{30} = 8.$
The sum requested is

$$10,000\left(v^{20}+v^{40}+v^{60}\right) = 10,000\left(8^{-2/3}+8^{-4/3}+8^{-2}\right)$$
$$= 10,000\left(\frac{1}{4}+\frac{1}{16}+\frac{1}{64}\right) = \$3281.25.$$

20. (*a*) Applying formula (1.13) with $a(t) = 1+it = 1+.1t$, we have

$$d_5 = \frac{I_5}{A_5} = \frac{a(5)-a(4)}{a(5)} = \frac{1.5-1.4}{1.5} = \frac{.1}{1.5} = \frac{1}{15}.$$

(*b*) A similar approach using formula (1.18) gives

$$a^{-1}(t) = 1-dt = 1-.1t$$

and

$$d_5 = \frac{I_5}{A_5} = \frac{a(5)-a(4)}{a(5)} = \frac{(1-.5)^{-1}-(1-.4)^{-1}}{(1-.5)^{-1}}$$
$$= \frac{1/.5-1/.6}{1-1/.5} = \frac{2-5/3}{2} = \frac{6-5}{2\cdot 3} = \frac{1}{6}.$$

21. From formula (1.16) we know that $v = 1-d$, so we have

$$200+300(1-d) = 600(1-d)^2$$
$$6d^2-12d+6-2-3+3d = 0$$
$$6d^2-9d+1 = 0 \quad \text{which is a quadratic.}$$

Solving the quadratic

$$d = \frac{9 \pm \sqrt{(-9)^2-(4)(6)(1)}}{2\cdot 6} = \frac{9-\sqrt{57}}{12}$$

rejecting the root > 1, so that

$$d = .1208, \quad \text{or } 12.08\%.$$

22. Amount of interest: $iA = 336.$
Amount of discount: $dA = 300.$

Applying formula (1.14)

$$i = \frac{d}{1-d} \quad \text{and} \quad \frac{336}{A} = \frac{300/A}{1-300/A} = \frac{300}{A-300}$$

so that

$$336(A-300) = 300A$$

$$36A = 100{,}800 \quad \text{and} \quad A = \$2800.$$

23. Note that this Exercise is based on material covered in Section 1.8. The quarterly discount rate is .08/4 = .02, while 25 months is $8\frac{1}{3}$ quarters.

(*a*) The exact answer is

$$5000v^{25/3} = 5000(1-.02)^{25/3} = \$4225.27.$$

(*b*) The approximate answer is based on formula (1.20)

$$5000v^8\left(1-\tfrac{1}{3}d\right) = 5000(1-.02)^8\left[1-\left(\tfrac{1}{3}\right)(.02)\right] = \$4225.46.$$

The two answers are quite close in value.

24. We will algebraically change both the RHS and LHS using several of the basic identities contained in this Section.

$$\text{RHS} = \frac{(i-d)^2}{1-v} = \frac{(id)^2}{d} = i^2 d \quad \text{and}$$

$$\text{LHS} = \frac{d^3}{(1-d)^2} = \frac{i^3v^3}{v^2} = i^3 v = i^2 d.$$

25. Simple interest: $a(t) = 1 + it$ from formula (1.5).
Simple discount: $a^{-1}(t) = 1 - dt$ from formula (1.18).

Thus,

$$1 + it = \frac{1}{1-dt}$$

and

$$1 - dt + it - idt^2 = 1$$

$$it - dt = idt^2$$

$$i - d = idt.$$

26. (*a*) From formula (1.23*a*)

$$\left(1-\frac{d^{(4)}}{4}\right)^{-4}=\left(1+\frac{i^{(3)}}{3}\right)^{3}$$

so that

$$d^{(4)}=4\left[1-\left(1+\frac{i^{(3)}}{3}\right)^{-\frac{3}{4}}\right].$$

(*b*)

$$\left(1+\frac{i^{(6)}}{6}\right)^{6}=\left(1-\frac{d^{(2)}}{2}\right)^{-2}$$

so that

$$i^{(6)}=6\left[\left(1-\frac{d^{(2)}}{2}\right)^{-\frac{1}{3}}-1\right].$$

27. (*a*) From formula (1.24)

$$i^{(m)}-d^{(m)}=\frac{i^{(m)}d^{(m)}}{m}$$

so that

$$i^{(m)}=d^{(m)}\left(1+\frac{i^{(m)}}{m}\right)=d^{(m)}(1+i)^{1/m}.$$

(*b*) $i^{(m)}$ measures interest at the ends of *m*ths of a year, while $d^{(m)}$ is a comparable measure at the beginnings of *m*ths of a year. Accumulating $d^{(m)}$ from the beginning to the end of the *m*thly periods gives $i^{(m)}$.

28. (*a*) We have $j=\frac{i^{(4)}}{4}=\frac{.06}{4}=.015$ and $n=2\cdot 4=8$ quarters, so that the accumulated value is

$$100(1.015)^{8}=\$112.65.$$

(*b*) Here we have an unusual and uncommon situation in which the conversion frequency is less frequent than annual. We have $j=4(.06)=.24$ per 4-year period and $n=2(1/4)=1/2$ such periods, so that the accumulated value is

$$100(1-.24)^{-.5}=100(.76)^{-.5}=\$114.71.$$

29. From formula (1.24)

$$i^{(m)}-d^{(m)}=\frac{i^{(m)}d^{(m)}}{m}$$

so that

$$m = \frac{i^{(m)} d^{(m)}}{i^{(m)} - d^{(m)}} = \frac{(.1844144)(.1802608)}{.1844144 - .1802608} = 8.$$

30. We know that

$$1 + \frac{i^{(4)}}{4} = (1+i)^{\frac{1}{4}} \quad \text{and} \quad 1 + \frac{i^{(5)}}{5} = (1+i)^{\frac{1}{5}}$$

so that

$$\text{RHS} = (1+i)^{\frac{1}{4}-\frac{1}{5}} = (1+i)^{\frac{1}{20}}$$

$$\text{LHS} = (1+i)^{\frac{1}{n}} \quad \text{and} \quad n = 20.$$

31. We first need to express *v* in terms of $i^{(4)}$ and $d^{(4)}$ as follows:

$$v = 1 - d = \left(1 - \frac{d^{(4)}}{4}\right)^4 \quad \text{so that} \quad d^{(4)} = 4\left(1 - v^{.25}\right)$$

and

$$v = (1+i)^{-1} = \left(1 - \frac{i^{(4)}}{4}\right)^{-4} \quad \text{so that} \quad i^{(4)} = 4\left(v^{-.25} - 1\right).$$

Now

$$r = \frac{i^{(4)}}{d^{(4)}} = \frac{4\left(v^{-.25} - 1\right)}{4\left(1 - v^{.25}\right)} = v^{-.25} \quad \text{so that} \quad v^{.25} = r^{-1} \quad \text{and} \quad v = r^{-4}.$$

32. We know that $d < i$ from formula (1.14) and that $d^{(m)} < i^{(m)}$ from formula (1.24). We also know that $i^{(m)} = i$ and $d^{(m)} = d$ if $m = 1$. Finally, in the limit $i^{(m)} \to \delta$ and $d^{(m)} \to \delta$ as $m \to \infty$. Thus, putting it all together, we have

$$d < d^{(m)} < \delta < i^{(m)} < i.$$

33. (*a*) Using formula (1.26), we have

$$A(t) = Ka^t b^{t^2} d^{c^t}$$

$$\ln A(t) = \ln K + t \ln a + t^2 \ln b + c^t \ln d$$

and

$$\delta_t = \frac{d}{dt} \ln A(t) = \ln a + 2t \ln b + c^t \ln c \ln d.$$

(*b*) Formula (1.26) is much more convenient since it involves differentiating a sum, while formula (1.25) involves differentiating a product.

34. Fund A: $a^A(t) = 1 + .10t$ and $\delta_t^A = \dfrac{\frac{d}{dt} a^A(t)}{a^A(t)} = \dfrac{.10}{1+.10t}$.

Fund B: $a^B(t) = (1-.05t)^{-1}$ and $\delta_t^B = \dfrac{\frac{d}{dt} a^B(t)}{a^B(t)} = \dfrac{.05}{1+.05t}$.

Equating the two and solving for t, we have

$$\frac{.10}{1+.10t} = \frac{.05}{1-.05t} \quad \text{and} \quad .10 - .005t = .05 + .005t$$

so that $.01t = .05$ and $t = 5$.

35. The accumulation function is a second degree polynomial, i.e. $a(t) = at^2 + bt + c$.

$a(0)$	=	$c =$	1	from Section 1.2
$a(.5)$	=	$.25a + .5b + c =$	1.025	5% convertible semiannually
$a(1)$	=	$a + b + c =$	1.07	7% effective for the year

Solving three equations in three unknowns, we have

$$a = .04 \quad b = .03 \quad c = 1.$$

36. Let the excess be denoted by E_t. We then have

$$E_t = (1+it) - (1+i)^t$$

which we want to maximize. Using the standard approach from calculus

$$\frac{d}{dt} E_t = i - (1+i)^t \ln(1+i) = i - \delta(1+i)^t = 0$$

$$(1+i)^t = \frac{i}{\delta} \quad \text{and} \quad t\ln(1+i) = t\delta = \ln i - \ln \delta$$

so that

$$t = \frac{\ln i - \ln \delta}{\delta}.$$

37. We need to modify formula (1.39) to reflect rates of discount rather than rates of interest. Then from the definition of equivalency, we have

$$a(3) = (1+i)^3 = (1-d_1)^{-1}(1-d_2)^{-1}(1-d_3)^{-1}$$

$$= (.92)^{-1}(.93)^{-1}(.94)^{-1} = .804261^{-1}$$

and

$$i = (.804264)^{-1/3} - 1 = .0753, \quad \text{or} \quad 7.53\%.$$

38. (*a*) From formula (1.39)

$$a(n) = (1+i_1)(1+i_2)\ldots(1+i_n) \quad \text{where} \quad 1 + i_k = (1+r)^k (1+i)$$

so that

$$a(n) = [(1+r)(1+i)]\left[(1+r)^2(1+i)\right]\ldots\left[(1+r)^n(1+i)\right]$$

and using the formula for the sum of the first n positive integers in the exponent, we have

$$a(n) = (1+r)^{n(n+1)/2}(1+i)^n.$$

(b) From part (a)

$$(1+j)^n = (1+r)^{n(n+1)/2}(1+i)^n \quad \text{so that} \quad j = (1+r)^{(n+1)/2} - 1.$$

39. Adapting formula (1.42) for $t = 10$, we have

$$a(10) = e^{5(.06)}e^{5\delta} = 2, \quad \text{so that} \quad e^{5\delta} = 2e^{-.3}$$

and

$$\delta = \frac{1}{5}\ln\left(2e^{-.3}\right) = .0786, \quad \text{or} \quad 7.86\%.$$

40. Fund X: $a^X(20) = e^{\int_0^{20}(.01t+.1)dt} = e^4$ performing the integration in the exponent.

Fund Y: $a^Y(20) = (1+i)^{20} = e^4$ equating the fund balances at time $t = 20$.

The answer is

$$a^Y(1.5) = (1+i)^{1.5} = \left[(1+i)^{20}\right]^{.075} = \left(e^4\right)^{.075} = e^{.3}.$$

41. Compound discount:

$$a(3) = (1-d_1)^{-1}(1-d_2)^{-1}(1-d_3)^{-1} = (.93)^{-1}(.92)^{-1}(.91)^{-1} = 1.284363$$

using the approach taken in Exercise 37.

Simple interest: $a(3) = 1 + 3i$.

Equating the two and solving for i, we have

$$1 + 3i = 1.284363 \quad \text{and} \quad i = .0948, \quad \text{or} \quad 9.48\%.$$

42. Similar to Exercise 35 we need to solve three equations in three unknowns. We have

$$A(t) = At^2 + Bt + C$$

and using the values of $A(t)$ provided

$$\begin{aligned} A(0) &= & C &= 100 \\ A(1) &= & A + B + C &= 110 \\ A(2) &= & 4A + 2B + C &= 136 \end{aligned}$$

which has the solution $A = 8 \quad B = 2 \quad C = 100$.

(*a*) $i_2 = \dfrac{A(2)-A(1)}{A(1)} = \dfrac{136-110}{110} = \dfrac{26}{110} = .236, \text{ or } 23.6\%.$

(*b*) $\dfrac{A(1.5)-A(.5)}{A(1.5)} = \dfrac{121-103}{121} = \dfrac{18}{121} = .149, \text{ or } 14.9\%.$

(*c*) $\delta_t = \dfrac{A'(t)}{A(t)} = \dfrac{16t+2}{8t^2+2t+100}$ so that $\delta_{1.2} = \dfrac{21.2}{113.92} = .186, \text{ or } 18.6\%.$

(*d*) $\dfrac{A(.75)}{A(1.25)} = \dfrac{106}{115} = .922.$

43. The equation for the force of interest which increases linearly from 5% at time $t = 0$ to 8% at time $t = 6$ is given by

$$\delta_t = .05 + .005t \quad \text{for} \quad 0 \le t \le 6.$$

Now applying formula (1.27) the present value is

$$1,000,000a^{-1}(6) = 1,000,000e^{-\int_0^6 (.05+.005t)dt} = 1,000,000e^{-.39} = \$677,057.$$

44. The interest earned amounts are given by

$$A: \quad X\left[\left(1+\frac{i}{2}\right)^{16} - \left(1+\frac{i}{2}\right)^{15}\right] = X\left(1+\frac{i}{2}\right)^{15}\left(\frac{i}{2}\right)$$

$$B: \quad 2X \cdot \frac{i}{2}.$$

Equating two expressions and solving for *i*

$$X\left(1+\frac{i}{2}\right)^{15}\left(\frac{i}{2}\right) = 2X \cdot \frac{i}{2} \qquad \left(1+\frac{i}{2}\right)^{15} = 2 \qquad i = 2\left(2^{1/15}-1\right) = .0946, \text{ or } 9.46\%.$$

45. Following a similar approach to that taken in Exercise 44, but using rates of discount rather than rates of interest, we have

$$A: \quad X = 100\left[(1-d)^{-11} - (1-d)^{-10}\right] = 100(1-d)^{-10}\left[(1-d)^{-1} - 1\right]$$

$$B: \quad X = 50\left[(1-d)^{-17} - (1-d)^{-16}\right] = 50(1-d)^{-16}\left[(1-d)^{-1} - 1\right].$$

Equating the two expressions and solving for *d*

$$100(1-d)^{-10} = 50(1-d)^{-16} \quad (1-d)^{-6} = 2 \quad (1-d)^{-1} = 2^{1/6}.$$

Finally, we need to solve for *X*. Using *A* we have

$$X = 100 \cdot 2^{10/6}\left(2^{1/6} - 1\right) = 38.88.$$

46. For an investment of one unit at $t = 2$ the value at $t = n$ is

$$a(n) = e^{\int_2^n \delta_t dt} = e^{2\int_2^n (t-1)^{-1} dt} = e^{2\ln(t-1)]_2^n} = \frac{(n-1)^2}{(2-1)^2} = (n-1)^2 .$$

Now applying formula (1.13)

$$d_n = \frac{a(n+1) - a(n)}{a(n+1)} = \frac{n^2 - (n-1)^2}{n^2}$$

and

$$1 - dn = \left(\frac{n-1}{n}\right)^2 .$$

Finally, the equivalent $d_n^{(2)}$ is

$$d_n^{(2)} = 2\left[1 - (1-d_n)^{1/2}\right] = 2\left[1 - \frac{n-1}{n}\right] = \frac{2}{n}.$$

47. We are given $i = .20 = \frac{1}{5}$, so that

$$d = \frac{i}{1+i} = \frac{1/5}{1+1/5} = \frac{1}{6}.$$

We then have

$$\text{PV}_A = (1.20)^{-1}\left[1 + \frac{1}{2}\cdot\frac{1}{5}\right]^{-1}$$

$$\text{PV}_B = (1.20)^{-1}\left[1 - \frac{1}{2}\cdot\frac{1}{6}\right]$$

and the required ratio is

$$\frac{\text{PV}_A}{\text{PV}_B} = \frac{\left(1 + 1/10\right)^{-1}}{1 - 1/12} = \frac{10}{11}\cdot\frac{12}{11} = \frac{120}{121}.$$

48. (*a*) $i = e^\delta - 1 = \delta + \frac{\delta^2}{2!} + \frac{\delta^3}{3!} + \frac{\delta^4}{4!} + \cdots$

using the standard power series expansion for e^δ.

(*b*) $\delta = \ln(1+i) = i - \frac{i^2}{2} + \frac{i^3}{3} - \frac{i^4}{4} + \cdots$

using a Taylor series expansion.

(*c*) $d = \dfrac{i}{1+i} = i(1+i)^{-1} = i\left(1 - i^2 + i^2 - i^3 + \cdots\right) = i - i^2 + i^3 - i^4 + \cdots$

using the sum of an infinite geometric progression.

(*d*) $\delta = -\ln(1-d) = -\left(-d - \dfrac{d^2}{2} - \dfrac{d^3}{3} - \dfrac{d^4}{4} - \cdots\right)$

adapting the series expansion in part (*b*).

49. (*a*) $\dfrac{dd}{di} = \dfrac{d}{di}\left(\dfrac{i}{1+i}\right) = \dfrac{(1+i)-i}{(1+i)^2} = (1+i)^{-2}$.

(*b*) $\dfrac{d\delta}{di} = \dfrac{d}{di}\ln(1+i) = \dfrac{1}{1+i} = (1+i)^{-1}$.

(*c*) $\dfrac{d\delta}{di} = \dfrac{d}{dv}(-\ln v) = -\dfrac{1}{v} = -v^{-1}$.

(*d*) $\dfrac{dd}{d\delta} = \dfrac{d}{d\delta}\left(1-e^{-\delta}\right) = -e^{-\delta}(-1) = e^{-\delta}$.

50. (*a*) (1) $a(t) = e^{\int_0^t (a+br)dr} = e^{at+bt^2/2}$.

(2) $1+i_n = \dfrac{a(n)}{a(n-1)} = \dfrac{e^{an+.5bn^2}}{e^{a(n-1)+.5b(n-1)^2}} = e^{an+.5bn^2-an+a-.5bn^2+bn-.5b} = e^{(a-b/2)+bn}$.

(*b*) (1) $a(t) = e^{\int_0^t ab^r dr} = e^{a(b^t-1)/\ln b}$.

(2) $1+i_n = \dfrac{a(n)}{a(n-1)} = e^{\frac{a}{\ln b}\left[(b^n-1)-(b^{n-1}-1)\right]} = e^{a(b-1)b^{n-1}/\ln b}$.

Chapter 13

1. Stock / Option

(*a*) $\frac{84-80}{80}=+5\%$ $\qquad \frac{0-2}{2}=-100\%$

(*b*) $\frac{80-80}{80}=0\%$ $\qquad \frac{0-2}{2}=-100\%$

(*c*) $\frac{78-80}{80}=-2.5\%$ $\qquad \frac{2-2}{2}=0\%$

(*d*) $\frac{76-80}{80}=-5\%$ $\qquad \frac{4-2}{2}=+100\%$

(*e*) \$78, from part (*c*) above

(*f*) $TVP = P - IVP = 2-0 = \$2$

2. (*a*) $IVC = S - E = 100 - 98 = \$2$

(*b*) $TVC = C - IVC = 6-2 = \$4$

(*c*) $IVP = \$0$ since $S \geq E$

(*d*) $TVP = P - IVP = 2-0 = \$2$

3. Profit position = − Cost of \$40 call + Cost of \$45 call
+ Value of \$40 call − Value of \$45 call

(*a*) $-3+1+0-0=-\$2$

(*b*) $-3+1+0-0=-\$2$

(*c*) $-3+1+2.50-0=\$.50$

(*d*) $-3+1+5-0=\$3$

(*e*) $-3+1+10-5=\$3$

4. See answers to the Exercises on p. 623.

5. (*a*) Break-even stock prices = $E+C+P$ and $E-C-P$.

(*b*) Largest amount of loss = $C+P$

6. (*a*) The shorter-term option should sell for a lower price than the longer-term option. Thus, sell one \$5 option and buy one \$4 option. Adjust position in 6 months.

   (*b*) If $S \leq 50$ in 6 months, profit is:

   \$1 if $S = 48$ in one year.

   \$1 if $S = 50$ in one year.

   \$3 if $S = 52$ in one year.

   If $S > 50$ is 6 months, profit is:

   \$3 if $S = 48$ in one year.

   \$1 if $S = 50$ in one year.

   \$1 if $S = 52$ in one year.

7. See answers to the Exercises on p. 623.

8. *P* increases as *S* decreases, the opposite of calls.

   P increases as *E* increases, the opposite of calls.

   P increases at *t* increases, since longer-term options are more valuable than shorter-term options.

   P increases as σ increases, since all option values increase as volatility increases.

   P increases as *i* decreases, the opposite of calls. The replicating transaction for calls involves lending money, while the replicating transaction for puts involves borrowing money.

9. Figure 13.5 provides the explanation.

10. (*a*) 0 from Figure 13.5.

    (*b*) $S - Ee^{-\delta n}$ from Figure 13.5.

    (*c*) *S*, since the call is equivalent to the stock.

    (*d*) 0, since the option is far "out of the money."

    (*e*) $S - E,$ if $S \geq E$

    0, if $S < E,$ the *IVC*.

    (*f*) *S* from Figure 13.5.

11. Using put-call parity, we have

$$S + P = v^t E + C \quad \text{or} \quad C = S + P - v^t E.$$

In the limit as $S \to \infty$, $P \to 0$, so that

$$C = S + 0 - v^t E = S - v^t E.$$

12. Using put-call parity, we have

$$S + P = v^t E + C$$

$$49 + P = \left(1 + \frac{.09}{12}\right)^{-3}(50) + 1 \text{ and } P = \$.89.$$

13. Buy the call. Lend \$48.89. Sell the stock short. Sell the put. Guaranteed profit of $-1 + 48.89 + 49 + 2 = \$1.11$ at inception.

14. See Answers to the Exercises on p. 624.

15. (*a*) At $S = 45$, profit is

$$(2)(4) - 3 - 6 + 0 + 0 + 0 = -\$1$$

At $S = 50$, profit is

$$(2)(4) - 3 - 6 + 5 + 0 + 0 = +\$4$$

At $S = 55$, profit is

$$(2)(4) - 3 - 6 + 10 - (5)(2) + 0 = -\$1$$

(*b*) See Answers to the Exercises on p. 624.

16. (*a*) The percentage change in the stock value is $+10\%$ in an up move, and -10% in a down move. The risk-free rate of interest is $i = .06$. Let *p* be the probability of an up move. We have

$$p(.10) + (1 - p)(-.10) = .06$$

or $.20p = .16$ and $p = .8$.

(*b*) Using formula (13.12)

$$C = \frac{p \cdot V_U + (1 - p)V_D}{1 + i} = \frac{(.8)(10) + (.2)(0)}{1.06} = \$7.55.$$

17. (*a*) Using formula (13.8)

$$\Delta = \frac{V_U - V_D}{S_U - S_D} = \frac{10-0}{110-90} = 1/2.$$

(*b*) Bank loan = Value of stock − Value of 2 calls = $100 - 2(7.55) = 84.906$ for 2 calls.
For one call the loan would be $\frac{84.906}{2} = \$42.45.$

18.

Year 1	Year 2	Probability	Stock Value
Up	Up	$(.8)(.8) = .64$	$100(1.1)^2 = 121$
Up	Down	$(.8)(.2) = .16$	$100(1.1)(.9) = 99$
Down	Up	$(.2)(.8) = .16$	$100(.9)(1.1) = 99$
Down	Down	$(.2)(.2) = .04$	$100(.9)^2 = 81$

We then have

$$C = \frac{(.64)(121-100)}{(1.06)^2} = \$11.96.$$

19. (*a*) Using the formula (13.7)

$$k = e^{\sigma\sqrt{h}} - 1 = e^{.3\sqrt{.125}} - 1 = .11190.$$

(*b*) Up move: $90(1+k) = 100.071$

Down move: $90(1+k)^{-1} = 80.943$

Now

$$100.071p + 80.943(1-p) = 90e^{.125(.1)} = 91.132$$

and solving, we obtain $p = .5327$.

(*c*) Applying formula (13.13) with the values of k and p obtained in parts (*a*) and (*b*) above together with $n = 8$, we obtain $C = \$10.78$. This, compare with the answer of $10.93 in Example 13.7.

20. Using formula (13.12) together with the stock values obtained in Exercise 18, $p = .8$ and $i = .06$ we obtain

$$P = \frac{(.16)(100-99) + (.16)(100-99) + (.04)(100-81)}{(1.06)^2} = \$.96.$$

21. The value of a put = 0 if $S(1+k)^{n-2t} \geq E = E - S(1+k)^{n-t}$ if $S(1+k)^{n-2t} < E$ or $\max\left[0, E - S(1+k)^{n-2t}\right]$. Thus, the value of an European put becomes

$$P = \frac{1}{(1+i)^n} \sum_{t=0}^{n} \binom{n}{t} p^{n-t} (1-p)^t \max\left[0, E - S(1+k)^{n-2t}\right].$$

22. We are asked to verify that formulas (13.14) and (13.16) together satisfy formula (13.5). We have

$$S + P = S + Ee^{-\delta n}\left[1 - N(d_2)\right] - S\left[1 - N(d_1)\right]$$
$$S + P = v^n E - Ee^{-\delta n} N(d_2) + SN(d_1)$$
$$= v^n E + C \quad \text{validating the result.}$$

23. Applying formula (13.16) directly, we have

$$P = 100e^{-.1}(1 - .4333) - 90(1 - .5525) = \$11.00.$$

The result could also be obtained using put-call parity with formula (13.5).

24. Applying formulas (13.14) and (13.15) repeatedly with the appropriate inputs gives the following:

(*a*) 5.76

(*b*) 16.73

(*c*) 8.66

(*d*) 12.58

(*e*) 5.16

(*f*) 15.82

(*g*) 5.51

(*h*) 14.88

25. We modify the final equation in the solution for Example 13.8 to obtain

$$C = \left(90 - 360e^{-.1}\right)(.5525) - \left(100e^{-.1}\right)(.4333) = \$8.72.$$

26. The price of the noncallable bond is $B^{nc} = 100$ since the bond sells at par. The price of the callable bond can be obtained from formula (13.17) as

$$B^c = B^{nc} - C$$

Thus, the problem becomes one of estimating the value of the embedded option using the Black Scholes formula. This formula places a value of 2.01 on the embedded call. The answer is then $100.00 - 2.01 = \$97.99$.

27. We modify the put-call parity formula to obtain

$$S - PV \text{ dividends } + P = v^t E + C$$

$$49 - .50a_{\overline{3}|.0075} + P = (1.0075)^{-3}(50) + 1$$

and solving for P we obtain

$$P = 2.37.$$

28. The average stock price is

$$\frac{10.10 + 10.51 + 11.93 + 12.74}{4} = 11.32$$

and the option payoff is $11.32 - 9 = \$2.32$.

Chapter 12

1. $\mathrm{E}\left[a^{-1}(n)\right]=\mathrm{E}\left[\prod_{t=1}^{n}(1+i_t)^{-1}\right]$

$$=\prod_{t=1}^{n}\mathrm{E}\left[1+i_t\right]^{-1} \quad \text{from independence}$$

$$=(1+\bar{i})^{-n}.$$

2. $\mathrm{E}\left[a_{\overline{n}|}\right]=\mathrm{E}\left[\sum_{t=1}^{n}\prod_{s=1}^{t}(1+i_s)^{-1}\right]$

$$=\sum_{t=1}^{n}\prod_{s=1}^{t}\mathrm{E}\left[1+i_s\right]^{-1} \quad \text{from independence}$$

$$=\sum_{t=1}^{n}(1+\bar{i})^{-t}=a_{\overline{n}|\,\bar{i}}.$$

3. (*a*) Year 1: 8% given.

Year 2: $.5(.07+.09)=.08$, or 8%.

Year 3: $.25[.06+2(.08)+.10]=.08$, or 8%.

(*b*) Year 1: $\sigma=0$, no variance.

Year 2: $\sigma^2=.5\left[(.07-.08)^2+(.09-.08)^2\right]=.0001$

$$\sigma=\sqrt{.0001}=.01.$$

Year 3: $\sigma^2=.25\left[(.06-.08)^2+2(.08-.08)^2+(.10-.08)^2\right]$

$$=.0002$$

$$\sigma=\sqrt{.0002}=.01\sqrt{2}$$

(*c*) $1000(1.08)(1.09)(1.10)=\$1294.92.$

(*d*) $1000(1.08)(1.07)(1.06)=\$1224.94.$

(*e*) $1000(1.08)^3=\$1259.71.$

(*f*) $.25(1000)[(1.08)(1.09)(1.10)+(1.08)(1.09)(1.08)$

$$+(1.08)(1.07)(1.08)+(1.08)(1.07)(1.06)]$$

$$=.25[1294.92+1271.38+1248.05+1224.94]$$

$$=\$1259.82$$

(*g*) $\sigma^2 = .25\big[(1294.92-1259.82)^2 + (1271.38-1259.82)^2$
$+ (1248.05-1259.82)^2 + (1224.94-1259.82)^2\big]$
$= 2720.79$
$\sigma = \sqrt{2720.79} = 52.16.$

4. (*a*) $\mathrm{E}\left[(1+i_t)^{-1}\right] = \dfrac{1}{.09-.07}\displaystyle\int_{.07}^{.09} \frac{1}{1+t}dt$
$= \dfrac{1}{.09-.07}\ln(1+t)\Big]_{.07}^{.09} = .925952.$

Then set $(1+\bar{i})^{-1} = .925952$ and solve $\bar{i} = .07997$.

(*b*) We have $a^{-1}(3) = (1.07997)^{-3} = .79390.$

(c) $\mathrm{E}\left[(1+i_t)^{-2}\right] = \dfrac{1}{.09-.07}\displaystyle\int_{.07}^{.09} \frac{1}{(1+t)^2}\,dt$
$= \left[\dfrac{-1}{.09-.07}\cdot\dfrac{1}{1+t}\right]_{.07}^{.09} = .857412.$

Then set $(1+\bar{k})^{-1} = .857412$ and solve $\bar{k} = .16630$.

(*d*) Applying formula (12.10), we have

$$\mathrm{Var}\left[a^{-1}(3)\right] = (.857412)^3 - (.925952)^6 = .0000549$$

and the standard deviation is $\sqrt{.0000549} = .00735.$

5. (*b*) Applying formula (12.11), we have

$$\mathrm{E}\left[a_{\overline{3|}}\right] = a_{\overline{3|}\,\bar{i}} = a_{\overline{3|}.07997} = 2.5772.$$

(*d*) Applying formula (12.14), we have

$$\mathrm{Var}\left[a_{\overline{3|}}\right] = \frac{m_2^a + m_1^a}{m_2^a - m_1^a}a_{\overline{3|}\,\bar{k}} - \frac{2m_2^a}{m_2^a - m_1^a}a_{\overline{3|}\,\bar{i}} - \left(a_{\overline{3|}\,\bar{i}}\right)^2$$
$$= \frac{.857412+.925952}{.857412-.925952}(2.2229) - \frac{(2)(.857412)}{.857412-.925952}(2.5772) - (2.5772)^2$$
$$= .005444$$

and the standard deviation is $\sqrt{.005444} = .0735.$

6. The random variable $i_t^{(2)}/2$ will be normal with $\mu = 3\%$ and $\sigma = .25\%$.

   (*a*) Applying formula (12.1), we have

   $$\mathrm{E}[100a(4)] = 100(1.03)^4 = 112.55.$$

   Applying formula (12.3), we have

   $$\begin{aligned}\mathrm{Var}\left[100a_{\overline{4|}}\right] &= 10,000\left[\left(1+2\bar{i}+\bar{i}^2+s^2\right)^4-\left(1+\bar{i}\right)^8\right]\\ &= 10,000\left[\left\{1+(2)(.03)+(.03)^2+.0025\right\}^4-(1.03)^8\right]\\ &= 119.828\end{aligned}$$

   and the standard deviation is $\sqrt{119.828} = 10.95$.

   (*b*) Applying formula (12.5), we have

   $$\mathrm{E}\left[100\ddot{s}_{\overline{4|}}\right] = 100\ddot{s}_{\overline{4|}.03} = 430.91.$$

   Applying formula (12.8), we have

   $$\begin{aligned}m_1^s &= 1.03\\ m_2^s &= 1+2(.03)+(.03)^2+.0025 = 1.0634\end{aligned}$$

   and

   $$\begin{aligned}\mathrm{Var}\left[100\ddot{s}_4\right] &= 10,000\left[\frac{1.0634+1.03}{1.0634-1.03}(4.67549)-\frac{(2)(1.0634)}{1.0634-1.03}(4.3091)-(4.3091)^2\right]\\ &= 944.929\end{aligned}$$

   and the standard deviation is $\sqrt{944.929} = 30.74$.

7. (*a*) $\mathrm{E}\left[s_{\overline{n|}}\right] = \mathrm{E}\left[\ddot{s}_{\overline{n+1|}}-1\right] = \ddot{s}_{\overline{n+1|}\,\bar{i}}-1.$

   (*b*) $\mathrm{Var}\left[s_{\overline{n|}}\right] = \mathrm{Var}\left[\ddot{s}_{\overline{n+1|}}-1\right] = \mathrm{Var}\left[\ddot{s}_{\overline{n+1|}}\right].$

   (*c*) $\mathrm{E}\left[\ddot{a}_{\overline{n|}}\right] = \mathrm{E}\left[1+a_{\overline{n-1|}}\right] = 1+a_{\overline{n-1|}\,\bar{i}}\,.$

   (*d*) $\mathrm{Var}\left[\ddot{a}_{\overline{n|}}\right] = \mathrm{Var}\left[1+a_{\overline{n-1|}}\right] = \mathrm{Var}\left[a_{\overline{n-1|}}\right].$

8. We know that $1+i$ is lognormal with $\mu = .06$ and $\sigma^2 = .01$. From the solution to Example 12.3(1), we have $\bar{i} = .067159$ and then

   $$\begin{aligned}s^2 &= e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right) = e^{2(.06)+.01}\left(e^{.01}-1\right)\\ &= e^{.13}\left(e^{.01}-1\right) = .011445.\end{aligned}$$

We then apply formula (12.4*a*) to obtain

$$\begin{aligned}\text{Var}[a(n)] &= (1+2\bar{i}+\bar{i}^2+s^2)^n-(1+\bar{i})^{2n}\\ &= [1+2(.067159)+(.067159)^2+.011445]^5-(1.067159)^{10}\\ &= .09821\end{aligned}$$

and the standard deviation $=\sqrt{.09821}=.3134$ agreeing with the other approach.

9. (*a*) Formula (12.5) with $\bar{i}=e^{\mu+\sigma^2/2}-1$.

   (*b*) Formulas (12.6), (12.7) and (12.8) with $\bar{j}=e^{2\mu+2\sigma}$.

   (*c*) Formula (12.11) with $\bar{i}=e^{\mu-\sigma^2/2}-1$.

   (*d*) Formulas (12.12), (12.13) and (12.14) with $\bar{k}=e^{-2\mu+2\sigma^2}$.

10. (*a*) $\text{E}[1+i_t]=e^{.06+.0001/2}=1.06189$

    mean $=\text{E}[a(10)]=(1.06189)^{10}=1.823.$

    $\text{Var}[a(10)]=e^{(2)(10)(.06)+(10)(.0001)}\left(e^{(10)(.0001)}-1\right)$

    $=e^{1.201}\left(e^{.001}-1\right)=.003325$

    and $\text{s.d.}=\sqrt{.003325}=.058$.

    (*b*) Mean $=\text{E}\left[\ddot{s}_{\overline{10|}}\right]=\ddot{s}_{\overline{10|}.06189}=14.121$

    s.d. using formula (12.8) = .297.

    (*c*) $\text{E}\left[(1+i_t)^{-1}\right]=e^{-.06+.0001/2}=.941812$

    mean $=\text{E}\left[a^{-1}(10)\right]=(.941812)^{10}=.549$

    $\text{Var}\left[a^{-1}(10)\right]=e^{-1.2+.001}\left(e^{.001}-1\right)=.000302$

    and s.d. $=\sqrt{.000302}=.017$.

    (*d*) We have $(1+\bar{i})^{-1}=.941812$ or $\bar{i}=.06178$

    and $(1+\bar{k})^{-1}=e^{-.12+.0001}e^{.0001}=e^{-.1198}$

    $=.887098$ or $\bar{k}=.12727.$

    Mean $=\text{E}\left[a_{\overline{10|}}\right]=a_{\overline{10|}.06178}=7.298.$

    s.d. using formula (12.14) = .134.

11. $\mathrm{E}\left[1+i_t\right]=e^{\mu+\sigma^2/2}=1.067.$

$\mathrm{Var}\left[1+i_t\right]=e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)=.011445.$

Solving two equations in two unknowns gives

$$\mu=.06 \quad \sigma^2=.01$$

Therefore $\delta_{[t]}$ follows a normal distribution with mean $=.06$ and $\mathrm{var}=.01.$

12. $\mathrm{E}\left[1+i_t\right]=1.08=e^{\mu+\sigma^2/2}=e^{\mu+.0001/2}$ so that $\mu=.07691.$

$\mathrm{Var}\left[1+i_t\right]=e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)=(1.08)^2\left(e^{.0001}-1\right)=.00011665.$

$\mathrm{E}\left[a(3)\right]=(1.08)^3=1.25971.$

$\mathrm{Var}\left[a(3)\right]=\left[1+2(.08)+(.08)^2+.00011665\right]^3-(1.08)^6$

$=.0004762$ and $\text{s.d.}=\sqrt{.0004762}=.02182.$

The 95% confidence interval is

$1.25971\pm 1.96(.02182)$ or $(1.21693, 1.30247).$

13. $\mathrm{E}\left[s_{\overline{3}|}\right]=\mathrm{E}\left[\ddot{s}_{\overline{4}|}-1\right]=\ddot{s}_{\overline{4}|.08}-1=s_{\overline{3}|.08}=3.246=\text{mean}.$

$\mathrm{Var}\left[s_{\overline{3}|}\right]=\mathrm{Var}\left[\ddot{s}_{\overline{4}|}-1\right]=\mathrm{Var}\left[\ddot{s}_{\overline{4}|}\right].$

$\mathrm{Var}=65.62$ using formula (12.8).

14. $\mathrm{E}\left[\ln\left(1+i_t\right)\right]=\dfrac{.07+.09}{2}=.08=\mu.$

$\mathrm{Var}\left[\ln\left(1+i_t\right)\right]=\dfrac{(.09-.07)^2}{2}=\dfrac{.0001}{3}=\sigma^2.$

$\mathrm{E}\left[\ln a^{-1}(30)\right]=-30\mu=-30(.08)=-2.4.$

$\mathrm{Var}\left[\ln a^{-1}(30)\right]=30\sigma^2=30\left(\dfrac{.0001}{3}\right)=.001.$

The 95th percentile of $\ln a^{-1}(30)$ is

$$-2.4+1.645\sqrt{.001}=-2.34798.$$

Thus, $100,000e^{-2.34798}=\$9556.20.$

15. Continuing Example 12.7:

$$\delta_{[6]} = .08 + .6(.091 - .08) + .2(.095 - .08) = .0896$$
$$\delta_{[7]} = .08 + .6(.0896 - .08) + .2(.091 - .08) = .0880$$
$$\delta_{[8]} = .08 + .6(.0880 - .08) + .2(.0896 - .08) = .0867.$$

16. (*a*) Formula (12.33)

$$\text{Var}\left[\delta_{[t]}\right] = \frac{1-k_2}{1+k_2} \cdot \frac{\sigma^2}{(1-k_2)^2 k_1^2}$$
$$= \frac{\sigma^2}{1-k_1} \quad \text{if } k_2 = 0$$

which is formula (12.30) with $k_1 = k$.

(*b*) Formula (12.34)

$$\text{Cov}\left[\delta_{[s]}, \delta_{[t]}\right] = \text{Var}\left[\delta_{[t]}\right]\left[\tau g_1^{t-s} + (1-\tau)\, g_2^{t-s}\right].$$

We set $k_2 = 0,$ so that

$$\tau = 1 \qquad g_1 = k_1 \qquad g_2 = 0$$

from formula (12.35). We also substitute the result from part (*a*).

Thus, $\text{Cov}\left[\delta_{[s]}, \delta_{[t]}\right] = \dfrac{\sigma^2}{1-k_1^2} k_1^{t-s}$

which is formula (12.31) with $k_1 = k$.

17. Use formula (12.33) with $k_1 = .6$ and $k_2 = .2$. Find the empirical estimate for $\text{Var}\left[\delta_{[t]}\right]$ based upon the sample data for $\delta_{[t]}$ given in Example 12.6. This will result in one equation in one unknown that can be solved for σ^2.

18. (*a*) Applying formula (12.33)

$$\text{Var}\left[\delta_{[t]}\right] = \frac{1-k_2}{1+k_2} \cdot \frac{\sigma^2}{(1-k_2)^2 - k_1^2}$$
$$= \frac{1-.2}{1+.2} \cdot \frac{.0002}{(1-.2)^2 - (.6)^2} = .0004762.$$

(*b*) Applying formulas (12.34), (12.35) and (12.36) with $k_1 = .6$ and $k_2 = .2$ and with $t - s = 2$ gives the answer .0001300.

19. (*a*) Applying formula (12.29) twice, we have

$$.096 = \delta + k(.100 - \delta)$$
$$.100 = \delta + k(.105 - \delta).$$

Solving these two equations in two unknowns, we have

$$k = .08 \quad \text{and} \quad \delta = .08\,.$$

Therefore

$$\delta^E_{[4]} = .08 + .8(.095 - .08) = .092.$$

(*b*) Applying formula (12.31), we have

$$\text{Cov}\left[\delta_{[s]}, \delta_{[t]}\right] = \frac{\sigma^2}{1-k_2} k^{t-s} = (.0001)(.8)^{6-3} = .0000512.$$

20. There are 9 paths each with probability 1/9:

$$.06/.02/.02-.04k \qquad .06/.06/.02 \qquad .06/.10/.02+.04k$$
$$.06/.02/.06-.04k \qquad .06/.06/.06 \qquad .06/.10/.06+.04k$$
$$.06/.02/.10-.04k \qquad .06/.06/.10 \qquad .06/.10/.10+.04k$$

(*a*) $\text{E}[a(2)] = \frac{1}{9}[(1.02)(1.02-.04k) + (1.02)(1.06-.04k) + 7 \text{ more terms}]$

$$= \frac{1}{3}\left[(1.02)(1.06-.04k) + (1.06)^2 + (1.10)(1.06-.04k)\right]$$
$$= (1.06)^2 + \frac{1}{3}(.0032)k.$$

(*b*) $\text{E}\left[a(2)^2\right] = \frac{1}{9}\left[(1.02)^2(1.02-.04k)^2 + (1.02)^2(1.06-.04k)^2 + 7 \text{ more terms}\right]$

$$= \frac{1}{9}\left[(1.02)^2 + (1.06)^2 + (1.10)^2\right] + \frac{(1.10)^2 - (1.02)^2}{3}(.08)(1.06)k$$
$$+ \frac{(1.10)^2 + (1.02)^2}{3}(.0016)k^2 \Bigg]$$
$$= \frac{1}{9}(11.383876 + .04314624k + .01080192k^2)$$

and

$$\text{Var}[a(2)] = \text{E}\left[a(2)^2\right] - \text{E}[a(2)]^2$$
$$= \frac{1}{9}(.02158336 + .02157312k + .01079168k^2).$$

21. At time $t=2$:

$$i=.144 \qquad V=\frac{(.5)(1000)+(.5)(1000)}{1.144}=874.126$$
$$i=.10 \qquad V=\frac{(.5)(1000)+(.5)(1000)}{1.1}=909.091$$
$$i=.06944 \quad V=\frac{(.5)(1000)+(.5)(1000)}{1.06944}=935.069$$

At time $t=1$:

$$i=.12 \qquad V=\frac{(.5)(874.126)+(.5)(909.091)}{1.12}=796.079$$
$$i=.08333 \quad V=\frac{(.5)(909.091)+(.5)(935.069)}{1.08333}=851.153$$

At time $t=0$:

$$i=.10 \quad V=\frac{(.5)(796.079)+(.5)(851.153)}{1.1}=748.74$$

obtaining the same answer as obtained with the other method.

22. (*a*)

Path	Probability	PV	PV^2
10/11/12	.25	.73125	.53473
10/11/10	.25	.74455	.55435
10/9/10	.25	.75821	.57488
10/9/8	.25	.77225	.59637

Value of the bond is

$$1000(.25)[.73125+.74455+.75821+.77225]=751.57.$$

(*b*) $\text{Var}=(1000)^2(.25)[.53473+.55435+.57488+.59637]$
$=232.5664$

and the s.d. $=\sqrt{232.5664}=15.25$.

(*c*) The mean interest rate is $i=.10$ so the value is $1000(1.1)^{-3}=751.31.$

23. (*a*) At time $t=1/2$:

$$i^{(2)}=.10 \qquad V=\frac{(.3)(1045)+(.7)(1045)}{1.05}=995.238$$
$$i^{(2)}=.08 \qquad V=\frac{(.3)(1045)+(.7)(1045)}{1.04}=1004.808.$$

At time $t = 0$:

$$i^{(2)} = .09 \quad V = \frac{(.3)(995.238+45)+(.7)(1004.808+45)}{1.045} = \$1001.85.$$

(*b*) The equation of value is

$$1001.854 = 45v + 1045v^2$$

and solving the quadratic $v = .95789$. Then we have $v = .95789 = e^{-.5\delta}$ and $\delta = .0861,$ or 8.61%.

24. If the interest rate moves down, then call the bond, which gives

$$V = \frac{(.3)(995.238+45)+(.7)(1000+45)}{1.045} = \$998.63.$$

25. At time $t = \frac{1}{2}$:

$$j = .0288 \quad V = \frac{(.4)(1038/1.03458)+(.6)(1038/1.024)}{1.0288} = 981.273$$

$$j = .02 \quad V = \frac{(.4)(1038/1.024)+(.6)(1038/1.01667)}{1.02} = 998.095$$

$$j = .01389 \quad V = \frac{(.4)(1038/1.01667)+(.6)(1038/1.011575)}{1.01389} = 1010.036$$

At time $t = \frac{1}{4}$:

$$j = .024 \quad V = \frac{(.4)(981.273+38)+(.6)(998.095+38)}{1.024} = 1005.24$$

$$j = .01667 \quad V = \frac{(.4)(998.095+38)+(.6)(1010.036+38)}{1.01667} = 1026.1536.$$

At time $t = 0$:

$$j = .02 \quad V = \frac{(.4)(1005.24)+(.6)(1026.1536)}{1.02} = \$997.83.$$

26.

Path	Probability	PV	CV	CV from time 1
10/12/14.4	.16	.7095	1.4094	1.28128
10/12/10	.24	.7379	1.3552	1.23200
10/8.333/10	.24	.7629	1.3108	1.19170
10/8.333/6.944	.36	.7847	1.2744	1.15860

(*a*) $\text{E}[a(3)] = (.16)(1.4094) + (.24)(1.3552) + (.24)(1.3108) + (.36)(1.2744) = 1.326.$

(*b*) $\text{E}[a^{-1}(3)] = .749$.

(*c*) $\text{E}[a(3)] = \text{E}[a^{-1}(3)] + \text{E}[a^{-1}(2)] + \text{E}[a^{-1}(1)]$

$$= .749 + \left[(.4)(1.12)^{-1}(1.1)^{-1} + (.6)(1.08333)^{-1}(1.1)^{-1}\right] + (1.1)^{-1}$$
$$= 2.486.$$

(*d*) $\text{E}\left[\ddot{s}_{\overline{3}|}\right] = 1.326 + 1.2038 + 1.096 = 3.626.$

27. Rendleman – Bartter:

mean

$$\text{E}[\delta_t] = \text{E}[\delta_t - \delta_0 + \delta_0] = \text{E}[\Delta\delta_0] + \text{E}[\delta_0]$$
$$= a\delta_0 t + \delta_0 = \delta_0(1 + at)$$

variance

$$\text{Var}[\delta_t] = a^2\delta^2 t$$

Vasicek:

mean

$$\text{E}[\delta_t] = \text{E}[\Delta\delta_0] + \text{E}[\delta_0]$$
$$= c(b - \delta_0) + \delta_0 = cb + (1 - c)\delta_0$$

variance

$$\text{Var}[\delta_t] = \sigma^2 t$$

Cox – Ingersoll – Ross:

mean

$$\text{E}[\delta_t] = cb + (1 - c)\delta_0$$

variance

$\text{Var}[\delta_t] = \sigma^2\delta_0 t,$ since the process error is proportional to $\sqrt{\delta}$ which squares in computing variances.

28. (*a*) We have

$$d\delta = c(b - \delta)\,dt + \sigma dz$$
$$= 0 + \delta dz \text{ if } c = 0$$
$$= adt + \sigma dz \text{ where } a = 0$$

which is the process for a random walk.

(*b*) We have

$$d\delta = c(b-\delta)\,dt + \sigma dz$$
$$= (b-\delta)\,dt + \sigma dz \quad \text{if} \quad c = 1$$

which is the process for a normal distribution with $\mu = b$.

29. For the random walk model

$$\Delta\delta = a\Delta t + \sigma\Delta z$$

and for the Rendleman-Bartter model

$$\Delta\delta = a\delta\Delta t + \sigma\delta\Delta z$$

Random walk		Rendleman - Bartter	
$\delta_0 = .06$			$\delta_0 = .06$
$\delta_{.5} = .0675$	$\Delta\delta_{.5} = .0075$	$\Delta\delta_{.5} = (.0075)(.06)$	$\delta_{.5} = .06045$
$\delta_1 = .065$	$\Delta\delta_1 = -.0025$	$\Delta\delta_1 = (-.0025)(.06045)$	$\delta_1 = .06030$
$\delta_{1.5} = .063$	$\Delta\delta_{1.5} = -.0020$	$\Delta\delta_{1.5} = (-.002)(.06030)$	$\delta_{1.5} = .06018$
$\delta_2 = .0685$	$\Delta\delta_2 = .0055$	$\Delta\delta_2 = (.0055)(.06018)$	$\delta_2 = .06051$

30. (*a*) We have $\delta_0 = .08$

$$E[\delta_{.5}] = \delta_0 + at = .08 + (.006)(.5) = .083$$

and

$$P = 39e^{-(.08)(.5)} + 1039e^{-(.08)(.5)-(.083)(.5)} = \$995.15.$$

(*b*) We have

$$995.151 = 39v + 1039v^2$$

and solving the quadratic

$$i^{(2)}/2 = .0606 \text{ so that } i^{(2)} = .1212.$$

(*c*) We have

$$\delta_{.5} = .08 + .006(.5) + (.01)(.5)\sqrt{.5} = .08654$$

and

$$P = 39e^{-(.08)(.5)} + 1039e^{-(.08)(.5)-(.08654)(.5)} = \$993.46.$$

31. Rework Examples 12.11-12.14 using ± 2 standard deviations. The following results are obtained:

<u>Random walk</u>

	Max	Min
	$\delta_{.25} = .0790$	$\delta_{.25} = .0590$
	$\delta_{.50} = .0880$	$\delta_{.50} = .0480$
	$\delta_{.75} = .0970$	$\delta_{.75} = .0370$
	$\delta_{1} = .106$	$\delta_{1} = .026$

<u>Rendleman – Bartter</u>

	Max	Min
	$\delta_{.25} = .0790$	$\delta_{.25} = .0590$
	$\delta_{.50} = .0892$	$\delta_{.50} = .0497$
	$\delta_{.75} = .1007$	$\delta_{.75} = .0419$
	$\delta_{1} = .114$	$\delta_{1} = .035$

<u>Vasicek</u>

	Max	Min
	$\delta_{.25} = .0790$	$\delta_{.25} = .0590$
	$\delta_{.50} = .0876$	$\delta_{.50} = .0486$
	$\delta_{.75} = .0957$	$\delta_{.75} = .0386$
	$\delta_{1} = .103$	$\delta_{1} = .029$

<u>Cox-Ingersoll-Ross</u>

	Max	Min
	$\delta_{.25} = .0790$	$\delta_{.25} = .0590$
	$\delta_{.50} = .0923$	$\delta_{.50} = .0494$
	$\delta_{.75} = .1017$	$\delta_{.75} = .0410$
	$\delta_{1} = .111$	$\delta_{1} = .034$

32. (*a*) $(.08)(1.1)^{10} = .2075,$ or 20.75%.

(*b*) $(.08)(.9)^{10} = .0279,$ or 2.79%.

(*c*) $(.08)(1.1)^{5}(.9)^{5} = .0761,$ or 7.61%.

(*d*) A 10% increase followed by a 10% decrease results in a result that is $(1.1)(.9) = 99\%$ of the starting value.

(*e*) $\binom{10}{5}(.5)^{10} = .2461$ using the binomial distribution.

(*f*) 10 increases $(.08)(1.1)^{10} = .2075$

9 increases $(.08)(1.1)^{9}(.9) = .1698$

Probability $= \left[\binom{10}{10} + \binom{10}{9}\right](.5)^{10} = 11(.5)^{10} = .0107.$

33. One year spot rates s_1:

$$i_0 = .070000$$
$$i_1 = .070000e^{1.65(.1)} = .082558$$
$$i_2 = .082558e^{-.26(.1)} = .080439$$
$$i_3 = .080439e^{.73(.1)} = .086530$$
$$i_4 = .086530e^{1.17(.1)} = .097270$$
$$i_5 = .097270e^{.98(.1)} = .1073, \quad \text{or} \quad 10.73\%.$$

Five year spot rates s_5:

$$i_0 = .080000$$
$$i_1 = .080000e^{1.65(.05)} = .086880$$
$$i_2 = .086880e^{-.26(.05)} = .085758$$
$$i_3 = .085758e^{.73(.05)} = .088946$$
$$i_4 = .088946e^{1.17(.05)} = .094304$$
$$i_5 = .094304e^{.98(.05)} = .0990, \quad \text{or} \quad 9.90\%.$$

The yield curve became invested, since 10.73% > 9.90%.

Chapter 11

1. A generalized version of formula (11.2) would be

$$\bar{d} = \frac{t_1 v^{t_1} R_{t_1} + t_2 v^{t_2} R_{t_2} + \cdots + t_n v^{t_n} R_{t_n}}{v^{t_1} R_{t_1} + v^{t_2} R_{t_2} + \cdots + v^{t_n} R_{t_n}}$$

where $0 < t_1 < t_2 < \ldots < t_n$. Now multiply numerator and denominator by $(1+i)^{t_1}$ to obtain

$$\bar{d} = \frac{t_1 R_{t_1} + t_2 v^{t_2 - t_1} R_{t_2} + \cdots + t_n v^{t_n - t_1} R_{t_n}}{R_{t_1} + v^{t_2 - t_1} R_{t_2} + \cdots + v^{t_n - t_1} R_{t_n}}.$$

We now have $\lim_{i \to \infty} \bar{d} = \lim_{v \to 0} \bar{d} = \dfrac{t_1 R_{t_1}}{R_{t_1}} = t_1.$

2. We can apply the dividend discount model and formula (6.28) to obtain

$$P(i) = D(i-k)^{-1}.$$

We next apply formula (11.4) to obtain

$$\bar{v} = -\frac{P'(i)}{P(i)} = \frac{D(i-k)^{-2}}{D(i-k)^{-1}}$$
$$= (i-k)^{-1} = (.08-.04)^{-1}.$$

Finally, we apply formula (11.5)

$$\bar{d} = \bar{v}(1+i) = \frac{1.08}{.08-.04} = 27.$$

3. We can use a continuous version for formula (11.2) to obtain

$$\bar{d} = \frac{\int_0^n t v^t dt}{\int_0^n t v^t dt} = \frac{(\bar{I}\,\bar{a})_{\overline{n}|}}{a_{\overline{n}|}}$$

and then apply formula (11.5)

$$\bar{v} = \frac{\bar{d}}{1+i} = \frac{v(\bar{I}\,\bar{a})_{\overline{n}|}}{\bar{a}_{\overline{n}|}}.$$

4. The present value of the perpetuity is

$$a_{\overline{\infty}|} = \frac{1}{i}.$$

The modified duration of the perpetuity is

$$\bar{v} = \frac{\bar{d}}{1+i} = \frac{v\sum_{t=1}^{\infty} tv^t}{\sum_{t=1}^{\infty} v^t} = \frac{v(Ia)_{\overline{\infty}|}}{a_{\overline{\infty}|}}$$

$$= \frac{v/id}{1/i} = \frac{v}{d} = \frac{v}{iv} = \frac{1}{i}.$$

5. Applying the fundamental definition we have

$$\bar{d} = \frac{10(Ia)_{\overline{8}|} + 800v^8}{10a_{\overline{8}|} + 100v^8} \quad \text{at} \quad i = 8\%$$

$$= \frac{(10)(23.55274) + 800(.54027)}{(10)(5.74664) + 100(.54027)} = 5.99.$$

6. (*a*) We have $\bar{v} = -\frac{P'(i)}{P(i)} = \frac{\bar{d}}{1+i}$

so that $\frac{650}{100} = \frac{\bar{d}}{1.07}$ and $\bar{d} = 6.955.$

(*b*) We have $P(i+h) \approx P(i)[1 - h\bar{v}]$

so that $P(.08) \approx P(.07)[1 - .01\bar{v}]$

$$= 100[1 - (.01)(6.5)] = 93.50.$$

7. Per dollar of annual installment payment the prospective mortgage balance at time $t = 3$ will be $a_{\overline{12}|.06} = 8.38384$. Thus, we have

$$\bar{d} = \frac{\sum tv^t R_t}{\sum v^t R_t} = \frac{(1.06)^{-1} + 2(1.06)^{-2} + 3(9.38384)(1.06)^{-3}}{(1.06)^{-1} + 2(1.06)^{-2} + 9.38384(1.06)^{-3}}$$

$$= \frac{26.359948}{9.712246} = 2.71.$$

8. We have $P(i) = R(1 + i)^{-1}$

$$P'(i) = -R(1 + i)^{-2}$$

$$P''(i) = 2R(1 + i)^{-3}$$

and $\bar{c} = \dfrac{P''(i)}{P(i)} = \dfrac{2R(1+i)^{-3}}{R(1+i)^{-1}} = 2(1+i)^{-2} = \dfrac{2}{(1.08)^2} = 1.715.$

9. (*a*) Rather than using the definition directly, we will find the modified duration first and adjust it, since this information will be needed for part (*b*). We have

$$P(i) = 1000\left[(1+i)^{-1} + (1+i)^{-2}\right]$$

$$P'(i) = 1000\left[-(1+i)^{-2} - 2(1+i)^{-3}\right]$$

$$P''(i) = 1000\left[2(1+i)^{-3} + 6(1+i)^{-4}\right].$$

Now, $\bar{v} = -\dfrac{P'(i)}{P(i)} = \dfrac{(1.1)^{-2} + 2(1.1)^{-3}}{(1.1)^{-1} + (1.1)^{-2}} = \dfrac{1.1+2}{(1.1)^2 + 1.1}$

and $\bar{d} = \bar{v}(1+i) = \dfrac{1.1+2}{1.1+1} = \dfrac{3.1}{2.1} = 1.48.$

(*b*) We have

$$\bar{c} = \frac{P''(i)}{P(i)} = \frac{2(1+i)^{-3} + 6(1+i)^{-4}}{(1+i)^{-1} + (1+i)^{-2}}$$

and multiplying numerator and denominator by $(1+i)^4$

$$\frac{2(1+i)+6}{(1+i)^3 + (1+i)^2} = \frac{2(1.1)+6}{(1.1)^3 + (1.1)^2} = \frac{8.2}{2.541} = 3.23.$$

10. When there is only one payment $\bar{d}$ is the time at which that payment is made for any force of interest. Therefore, $\dfrac{d\bar{d}}{d\delta} = \dfrac{d\bar{v}}{d\delta} = \sigma^2 = 0.$

11. (*a*) $P(i) = 1000\left[(1+i)^{-1} + 2(1+i)^{-2} + 3(1+i)^{-3}\right]$

$$= 1000\left[(1.25)^{-1} + 2(1.25)^{-2} + 3(1.25)^{-3}\right] = \$3616.$$

(*b*) $\bar{d} = \dfrac{1000\left[(1.25)^{-1} + 4(1.25)^{-2} + 9(1.25)^{-3}\right]}{1000\left[(1.25)^{-1} + 2(1.25)^{-2} + 3(1.25)^{-3}\right]} = \dfrac{7968}{3616} = 2.2035.$

(*c*) $\bar{v} = \dfrac{\bar{d}}{1+i} = \dfrac{2.2035}{1.25} = 1.7628.$

(*d*) $\bar{c} = \dfrac{P''(i)}{P(i)} = \dfrac{2(1+i)^{-3} + 12(1+i)^{-4} + 36(1+i)^{-5}}{(1+i)^{-1} + 2(1+i)^{-2} + 3(1+i)^{-3}}$

$$= \frac{2(1.25)^{-3} + 12(1.25)^{-4} + 36(1.25)^{-5}}{3.616} = 4.9048.$$

12. Per dollar of installment payment, we have

$$P(i) = (1+i)^{-1} + (1+i)^{-2} + \cdots + (1+i)^{-n}$$

$$P''(i) = (1)(2)(1+i)^{-3} + (2)(3)(1+i)^{-4} + \cdots + (n)(n+1)(1+i)^{-n-2}.$$

If $i = 0,$ the convexity is

$$\bar{c} = \frac{P''(0)}{P(0)} = \frac{1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1)}{1 + 1 + \cdots + 1}$$

$$= \frac{(1^2 + 2^2 + \cdots + n^2) + (1 + 2 + \cdots + n)}{n}$$

$$= \frac{\frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)}{n} = \frac{n(n+1)(2n+1) + 3n(n+1)}{6n}$$

$$= \frac{n(n+1)(2n+4)}{6n} = \frac{1}{3}(n+1)(n+2).$$

13. We have $P(i) = D(i-k)^{-1}$

$$P''(i) = 2D(i-k)^{-3}$$

so that $\bar{c} = \dfrac{P''(i)}{P(i)} = \dfrac{2D(i-k)^{-3}}{D(i-k)^{-1}} = \dfrac{2}{(i-k)^2} = \dfrac{2}{(.08-.04)^2} = 1250.$

14. From formula (11.10)

$$\frac{d\bar{v}}{di} = \bar{v}^2 - \bar{c} \quad \text{or} \quad -800 = (6.5)^2 - \bar{c} \quad \text{or} \quad \bar{c} = 800 + (6.5)^2 = 842.25.$$

Now applying formula (11.9b), we have

$$P(i+h) \approx P(i)\left[1 - h\bar{v} + \frac{h^2}{2}\bar{c}\right]$$

$$P(.08) \approx 100\left[1 - (.01)(6.5) + \frac{(.01)^2}{2}(842.25)\right]$$

$$= 97.71.$$

15. (*a*) From formula (11.19)

$$\bar{d}_e = \frac{P(i-h) - P(i+h)}{2hP(i)} = \frac{101.6931 - 100.8422}{2(.001)(101.2606)}$$

$$= 4.20.$$

(*b*) From formula (11.20)

$$\bar{c}_e = \frac{P(i-h) - 2P(i) + P(i+h)}{h^2 P(i)}$$

$$= \frac{101.6931 - 2(101.2606) + 100.8422}{(.001)^2(101.2606)} = 139.24.$$

(*c*) From formula (11.22)

$$P(i+h) \approx P(i)\left[1 - h\bar{d}_e + \frac{h^2}{2}\bar{c}_e\right]$$

$$= 101.2606\left[1 - (.0075)(4.20) + \frac{(.0075)^2}{2}(139.24)\right]$$

$$= \$98.47.$$

16. We have:

$$P(.09) = \frac{100{,}000}{a_{\overline{20}|.08}}\left[a_{\overline{10}|.08} + (1.08)^{-10} a_{\overline{10}|.09}\right]$$

$$= 98{,}620.43.$$

$$P(.08) = 100{,}000.00.$$

$$P(.07) = \frac{100{,}000}{a_{\overline{20}|.08}} a_{\overline{10}|.08} + \frac{\left[100{,}000(1.08)^{10} - \frac{100{,}000}{a_{\overline{20}|.08}} s_{\overline{10}|.08}\right] a_{\overline{5}|.07}(1.08)^{-10}}{a_{\overline{5}|.08}}$$

$$= 100{,}852.22.$$

(a) We have

$$\bar{d}_e = \frac{P(i-h)-P(i+h)}{2hP(i)} = \frac{100{,}852.22-98{,}620.43}{2(.01)(100{,}000)} = 1.12$$

(b) We have

$$\bar{c}_e = \frac{P(i-h)-2P(i)+P(i+h)}{h^2P(i)}$$
$$= \frac{100{,}852.22-2(100{,}000)+98{,}620.43}{(.01)^2(100{,}000)} = -52.73$$

17. Using formula (11.22)

$$P(.09) \approx 100{,}000\left[1-(.01)(1.116)+\tfrac{(.01)^2}{2}(-52.734)\right] = \$98{,}620$$

which agrees with the price calculated in Exercise 16.

$$P(.07) \approx 100{,}000\left[1-(.01)(1.116)+\tfrac{(.01)^2}{2}(-52.734)\right] = \$100{,}852$$

which agrees with the price calculated in Exercise 16.

18. We know that $\bar{c} = \dfrac{P''(i)}{P(i)}$. Let $\Delta i = h$.

Then $P'(i) \approx \dfrac{\Delta P(i)}{\Delta i}$, so that

$$P''(i) \approx \frac{\Delta}{\Delta i}\frac{\Delta P(i)}{\Delta i} = \frac{\Delta^2 P(i)}{(\Delta i)^2}$$
$$= \frac{[P(i+h)-P(i)]-[P(i)-P(i-h)]}{(\Delta i)^2}$$

and $\bar{c} = \dfrac{P(i-h)-2P(i)+P(i+h)}{h^2P(i)}$.

19. Directly form formula (11.24), we have

$$\frac{(21.46)(980)+(12.35)(1015)+(16.67)(1000)}{980+1015+1000}$$
$$= 16.77.$$

20. (*a*) Time 1 before payment is made:

$$\bar{d} = \frac{(0)(1)+(1)(1.1)^{-1}+(2)(1.1)^{-2}}{1+(1.1)^{-1}+(1.1)^{-2}} = .9366.$$

Time 1 after payment is made:

$$\bar{d} = \frac{(1)(1.1)^{-1}+(2)(1.1)^{-2}}{(1.1)^{-1}+(1.1)^{-2}} = 1.4762.$$

"Jump" $= 1.4762 - .9366 = .540.$

(*b*) Time 2 before payment is made:

$$\bar{d} = \frac{(0)(1)+(1)(1.1)^{-1}}{1+(1.1)^{-1}} = .4762.$$

Time 2 after payment is made:

$$\bar{d} = \frac{(1)(1.1)^{-1}}{(1.1)^{-1}} = 1.0000$$

"Jump" $= 1.0000 - .4762 = .524.$

(*c*) The numerator is the same before and after the "jump." The denominator is one less after the jump than before. The effect is greater when the numerator is greater.

21. Treasury bills have a stated rate at simple discount, which can be considered to be a discount rate convertible quarterly as they rollover from quarter to quarter. We have

$$\left(1-\frac{d^{(4)}}{4}\right)^{-2} = 1+\frac{i^{(2)}}{2}$$

$$\left(1-\frac{.06}{4}\right)^{-2} = 1+\frac{i^{(2)}}{2} \qquad i^{(2)} = .0613775.$$

Run tests at $i^{(2)} = .0513775$ and $.0713775$.

$$\left(1-\frac{d^{(4)}}{4}\right)^{-2} = 1+\frac{.0513775}{2}, \text{ so } d_L^{(4)} = .0504.$$

$$\left(1-\frac{d^{(4)}}{4}\right)^{-2} = 1+\frac{.0713775}{2}, \text{ so } d_H^{(4)} = .0695.$$

Thus, use 5.04% and 6.95% rates of discount.

22. Macaulay convexity equals to the square of Macaulay duration for single payments. Thus, we have

	Macaulay Duration	Macaulay Convexity	Amount
Bond 1	2	4	10,000
Bond 2	5	25	20,000
Bond 3	10	100	30,000

Then applying formula (11.25)

$$\frac{10,000(4)+20,000(25)+30,000(100)}{10,000+20,000+30,000}=59.$$

23. We set

$$CF_0=-2,948,253$$
$$CF_1=1,105,383$$
$$CF_2=1,149,598$$
$$CF_3=1,195,582$$

and obtain $\text{IRR}=8.18\%$ using a financial calculator.

24. We have

$$\frac{1,105,383}{1.0875}+\frac{1,149,598}{(1.08)^2}+\frac{1,195,582}{(1.07)^3}=\$2,977,990.$$

25. Using absolute matching with zero-coupon bonds, we have

$$\frac{1000}{1.1}+\frac{2000}{(1.12)^2}=\$2503.48.$$

26. Let F_1 and F_2 be the face amount of 1-year and 2-year bonds. At the end of the second year

$$F_2+.06F_2=10,000 \quad \text{and} \quad F_2=9433.96.$$

At the end of the first year

$$(.06)(9433.96)+F_1+.04F_1=10,000 \quad \text{and} \quad F_1=9071.12.$$

The price of the 1-year bond is

$$\frac{9071.12(104)}{105} = 8984.73.$$

The price of the 2-year bond is

$$9433.96\left[\frac{.06}{1.05} + \frac{1.06}{(1.05)^2}\right] = 9609.38.$$

The total price is $8984.73 + 9609.38 = \$18,594$ to the nearest dollar.

27. (*a*) The amount of Bond B is

$$\frac{2000}{1.025} = 1951.220$$

in order to meet the payment due in one year. The amount of Bond A is

$$\frac{1000 - 1951.220(.025)}{1.04} = 914.634$$

in order to meet the payment due in 6 months. The cost of Bond B is

$$P_B = 1951.220\left[\frac{.025}{1.035} + \frac{1.025}{(1.035)^2}\right] = 1914.153.$$

The cost of Bond A is

$$P_A = 914.634\left(\frac{1.04}{1.03}\right) = 923.514.$$

Thus, the total cost to the company is

$$1914.153 + 923.514 = \$2837.67.$$

(*b*) The equation of value is

$$2837.67 = 1000(1+j)^{-1} + 2000(1+j)^{-2}$$

which is a quadratic. Solving the quadratic gives $j = .03402,$ which is a semiannual interest rate. Thus, the nominal IRR convertible semiannually is $2j = 2(.03402) = .0680,$ or 6.80%.

28. (*a*) The liability is a single payment at time $t = 1,$ so $\bar{d} = 1$. We then have

$$\bar{v} = \frac{\bar{d}}{1+i} = \frac{1}{1.10} = .090909$$

which is equal to the modified duration of the assets.

(*b*) We have

$$P(i) = 1100(1+i)^{-1}$$
$$P''(i) = 2200(1+i)^{-3}.$$

Thus, the convexity of the liability is

$$\bar{c} = \frac{P''(.10)}{P(.10)} = \frac{2200(1.1)^{-3}}{1100(1.1)^{-1}}$$
$$= \frac{2}{(1.1)^2} = 1.65289$$

which is less than the convexity of the assets of 2.47934.

29. (*a*) $P(i) = 600 + (400)(1.21)(1+i)^{-2} - 1100(1+i)^{-1}$

(1) $P(.09) = 600 + 484(1.09)^{-2} - 1100(1.09)^{-1} = -1.8012$

(2) $P(.10) = 600 + 484(1.10)^{-2} - 1100(1.10)^{-1} = 0$

(3) $P(.11) = 600 + 484(1.11)^{-2} - 1100(1.11)^{-1} = 1.8346$

(*b*) $P(i) = 400 + (600)(1.21)(1+i)^{-2} - 1100(1+i)^{-1}$

(1) $P(.09) = 400 + 726(1.09)^{-2} - 1100(1.09)^{-1} = 1.8854$

(2) $P(.10) = 400 + 726(1.10)^{-2} - 1100(1.10)^{-1} = 0$

(3) $P(.11) = 400 + 726(1.11)^{-2} - 1100(1.11)^{-1} = -1.7531$

(*c*) In Example 11.14 $P(i) > 0$ for a 1% change in *i* going in either direction, since the portfolio is immunized with 500 in each type of investment. If the investment allocation is changed, the portfolio is no longer immunized.

30. (*a*) From formula (11.27) we have

$$P(i) = A_1(1+i)^{-1} + A_5(1+i)^{-5} - 100\left[(1+i)^{-2} + (1+i)^{-4} + (1+i)^{-6}\right].$$

Now multiplying by $(1+i)^5$ and setting $i = .1$

$$A_1(1.1)^4 + A_5 = 100\left[(1.1)^3 + 1.1 + (1.1)^{-1}\right]$$

or $1.4641A_1 + A_5 = 334.01$.

From formula (11.28) we have

$$P'(i) = -A_1(1+i)^{-2} - 5A_5(1+i)^{-6} + 100\left[2(1+i)^{-3} + 4(1+i)^{-5} + 6(1+i)^{-7}\right].$$

Now multiplying by $(1+i)^6$ and setting $i = .1$

$$A_1(1.1)^4 + 5A_5 = 100\left[2(1.1)^3 + 4(1.1) + 6(1.1)^{-1}\right]$$

or $1.4641A_1 + 5A_5 = 1251.65.$

Solving two equations in two unknowns, we have $A_1 = \$71.44$ and $A_5 = \$229.41$

(*b*) Testing formula (11.29) we have

$$P''(i) = 2A_1(1+i)^{-3} + 30A_5(1+i)^{-7} - 100\left[6(1+i)^{-4} + 20(1+i)^{-6} + 42(1+i)^{-8}\right]$$

and

$$\begin{aligned} P''(.10) &= (2)(71.44)(1.1)^{-3} + (30)(229.41)(1.1)^{-7} \\ &\quad -100\left[(6)(1.1)^{-4} + (20)(1.1)^{-6} + (42)(1.1)^{-8}\right] \\ &= 140.97 > 0. \end{aligned}$$

Yes, the conditions for Redington immunization are satisfied.

31. Adapting formulas (11.30) and (11.31) to rates of interest, we have:

$$A(1.1)^5 + B(1.1)^{-5} - 10,000 = 0$$
$$5A(1.1)^5 - 5B(1.1)^{-5} = 0$$

Solving two equations in two unknowns give the following answers:

(*a*) $A = \$3104.61.$

(*b*) $B = \$8052.56.$

32. Again adapting formulas (11.30) and (11.31) to rates of interest, we have:

$$A(1.1)^a + 6000(1.1)^{-2} - 10,000 = 0$$
$$aA(1.1)^a - 6000(2)(.1)^{-2} = 0$$

or

$$aA(1.1)^a = 9917.36$$
$$A(1.1)^a = 5041.32.$$

Solving two equations in two unknowns gives the following answers:

(*a*) $A = 5041.32(1.1)^{-1.96721} = \$4179.42.$

(*b*) $a = \dfrac{9917.36}{5041.32} = 1.967$.

33. (*a*) If $f = .075$ and $k_1 = .10$, we have

$$A_2 = -.016225 p_1 + .011325 > 0$$

or $p_1 < .6980$.

If $f = .09$ and $k_1 = .90$, we have

$$A_2 = -.000025 p_1 + .015325 > 0$$

or $p_1 < .6130$.

Thus, choose p_1 so that $0 < p_1 < .6980$.

(*b*) If $f = .065$ and $k_1 = .10$, we have

$$A_2 = -.027025 p_1 + .012325 > 0$$

or $p_1 < .4561$.

If $f = .10$ and $k_1 = .90$, we have

$$A_2 = -.010775 p_1 - .007175 > 0$$

or $p_1 > .6613$.

Thus, no solution exists.

34. (*a*) If $f = .07$ and $k_1 = .20$, we have

$$A_2 = -.021625 p_1 + .012825 > 0$$

or $p_1 < .5931$.

If $f = .095$ and $k_1 = .80$, we have

$$A_2 = .005375 p_1 - .001175 > 0$$

or $p_1 > .2186$.

Thus, choose p_1 so that $.2186 < p_1 < .5931$.

(*b*) If $f = .07$ and $k_1 = 0$, we have

$$A_2 = -.021625 p_1 + .010825 > 0$$

or $p_1 < .5006$.

If $f = .095$ and $k_1 = 1$, then

$$A_2 = .005375 p_1 - .004175 > 0$$

or $p_1 > .7767$.

Thus, no solution exists.

35. The present value of the liability at 5% is

$$1,000,000(1.05)^{-4} = 822,703.$$

The present value of the bond at 5% is 1,000,000.

If interest rates decrease by ½%, the coupons will be reinvested at 4.5%. The annual coupon is $822,703(.05) = 41,135$. The accumulated value 12/31/z+4 will be

$$822,703 + 41,135 s_{\overline{4}|.045} = 998,687.$$

The liability value at that point is 1,000,000 creating a loss of 1,000,000-998,687 = $1313.

If the interest rates increase by ½%, the accumulated value 12/31/z+4 will be

$$822,703 + 41,135 s_{\overline{4}|.055} = 1,001,323$$

creating a gain of $1,001,323 - 1,000,000 = \$1323$.

36. (*a*) Under Option A, the 20,000 deposit grows to $20,000(1.1) = 22,000$ at time $t = 1$. Half is withdrawn, so that 11,000 continues on deposit and grows to $11,000(1.1) = 12,100$ at time $t = 2$. The investment in two-year zero coupon bonds to cover this obligation is $12,100(1.11)^{-2} = 9802.63$. Thus, the profit at inception is $10,000 - 9802.63 = \$179.37$.

(*b*) Using the principles discussed in Chapter 10, we have

$$(1.11)^2 = (1.10)(1+f), \text{ so that}$$

$$f = \frac{(1.11)^2}{1.10} = .1201, \text{ or } 12.01\%.$$

37. The present value the asset cash inflow is

$$P_A(i) = 35,000(1.08)^5(1+i)^{-5} + (.08)(50,000)(1/i).$$

The present value the liability cash outflow is

$$P_L(i) = 85,000(1.08)^{10}(1+i)^{-10}.$$

We then have the following derivatives:

$$P_A'(i) = -5(35,000)(1.08)^5(1+i)^{-6} - 4000i^{-2}$$
$$P_L'(i) = -10(85,000)(1.08)^{10}(1+i)^{-11}$$
$$P_A''(i) = (5)(6)(35,000)(1.08)^5(1+i)^{-7} + 8000i^{-3}$$
$$P_L''(i) = (10)(11)(85,000)(1.08)^{10}(1+i)^{-12}.$$

If $i = .08$, we have the following:

$$P_A(.08) = 85,000 \text{ and } P_B(.08) = 85,000$$

$$\bar{v}_A = -\frac{P_A'(.08)}{P_A(.08)} = \frac{787,037.04}{85,000.00} = 9.2593$$

$$\bar{v}_L = -\frac{P_L'(.08)}{P_L(.08)} = \frac{787,037.04}{85,000.00} = 9.2593$$

$$\bar{c}_A = -\frac{P_A''(.08)}{P_A(.08)} = \frac{16,675,026}{85,000} = 196.18$$

$$\bar{c}_L = -\frac{P_L''(.08)}{P_L(.08)} = \frac{8,016,118}{85,000} = 94.31.$$

Thus, the investment strategy is optimal under immunization theory, since

(1) $P_A(.08) = P_B(.08)$

(2) $\bar{v}_A = \bar{v}_L$

(3) $\bar{c}_A > \bar{c}_L$

38. This is a lengthy exercise and a complete solution will not be shown. The approach is similar to Exercise 37. A sketch of the full solution appears below.

When the initial strategy is tested, we obtain the following:

$$P_A(.10) = 37,908 \qquad \bar{v}_A = 2.7273$$
$$P_L(.10) = 37,908 \qquad \bar{v}_L = 2.5547$$

Since $\bar{v}_A \neq \bar{v}_L$, the strategy cannot be optimal under immunization theory.

The superior strategy lets *x*, *y*, *z* be amounts invested in 1-year, 3-year, 5-year bonds, respectively. The three immunization conditions are set up leading to two equations and one inequality in three unknowns.

The solution $x = \$13,223$

$y = \$15,061$

$z = \$9624$ satisfies these three conditions.

39. (*a*) We have

$$\bar{d} = \frac{i(v + 2v^2 + 3v^3 + \cdots + nv^n) + nv^n}{i(v + v^2 + v^3 + \cdots + v^n) + v^n}$$

$$= \frac{i(Ia)_{\overline{n}|} + nv^n}{ia_{\overline{n}|} + v^n} = \frac{\ddot{a}_{\overline{n}|} - nv^n + nv^n}{1 - v^n + v^n} = \ddot{a}_{\overline{n}|}.$$

(*b*) We have $\ddot{a}_{\overline{10}|.08} = 7.25$, as required.

40. (*a*) We have

$$P(i+h) \approx P(i)\left(\frac{1+i}{1+i+h}\right)^{\bar{d}}$$

so that

$$P(.09) \approx P(.08)\left(\frac{1.08}{1.09}\right)^{\bar{d}} = (1)\left(\frac{1.08}{1.09}\right)^{7.2469} = .9354.$$

(*b*) The error in this approach is

$$.9358 - .9354 = .0004\,.$$

Chapter 10

1. (*a*) We have

$$1000\left[(1.095)^{-1}+(1.0925)^{-2}+(1.0875)^{-3}+(1.08)^{-4}+(1.07)^{-5}\right]=\$3976.61.$$

(*b*) The present value is greater than in Example 10.1 (1), since the lower spot rates apply over longer periods while the higher spot rates apply over shorter periods.

2. We have

$$1000\left[1+(1.05)(1+s_1)^{-1}+(1.05)^2(1+s_2)^{-2}+(1.05)^3(1+s_3)^{-3}+(1.05)^4(1+s_4)^{-4}\right]$$

$$=1000\left[1+\frac{1.05}{1.09}+\left(\frac{1.05}{1.081}\right)^2+\left(\frac{1.05}{1.0729}\right)^3+\left(\frac{1.05}{1.06561}\right)^4\right]=\$4786.78.$$

3. Since s_k is differentiable over $0 \le k \le 4$,

$$\frac{d}{dk}s_k=.002-.001k=0 \text{ at } k=2$$

which is a relative maximum or minimum. Computing values for $k=0,1,2,3,4$ we obtain

$$s_0=.09 \quad s_1=.0915 \quad s_2=.092 \quad s_3=.0915 \quad s_4=.09.$$

(*a*) Normal.

(*b*) Inverted.

4.

Payment at	Spot rate	Accumulated value
$t=0$	.095	$(1.095)^5=1.57424$
$t=1$	.0925 − .0025	$(1.09)^4=1.41158$
$t=2$	.0875 − .0050	$(1.0825)^3=1.26848$
$t=3$	.0800 − .0075	$(1.0725)^2=1.15026$
$t=4$	.0700 − .0100	$(1.06)^1=1.06000$
		6.4646

5. Adapting Section 9.4 to fit this situation we have

$$\frac{10,000(1.0925)^4}{(1.05)^2(1.04)^2} = \$11,946.50.$$

6. (*a*) $2P_B - P_A = 2(930.49) - 1019.31 = \$841.67.$

   (*b*) $2C_B - C_A = 2(1000.00) - 1000.00 = \$1000.00.$

   (*c*) We have $s_2 = .09$ and $841.67(1.09)^2 = 1000.00$ confirming the statement.

7. The price of the 6% bond per 100 is

$$P_{.06} = 6a_{\overline{6}|.12} + 100(1.12)^{-6} = 75.33.$$

   The price of the 10% bond per 100 is

$$P_{.10} = 10a_{\overline{6}|.08} + 100(1.08)^{-6} = 109.25.$$

   We can adapt the technique used above in Exercise 6. If we buy 10/6 of the 6% bonds, the coupons will exactly match those of the 10% bond. The cost will be

$$\frac{10}{6}(75.33) = 125.55 \text{ and will mature for } \frac{10}{6}(100).$$

   Thus, we have

$$(125.55 - 109.25)(1 + s_6)^6 = \frac{4}{6}(100)$$

   and solving $s_6 = .2645,$ or 26.45%.

8. Applying formula (10.4)

   (*a*) $\dfrac{1-(1.08)^{-2}}{(1.07)^{-1}+(1.08)^{-2}} = .0796,$ or 7.96%.

   (*b*) $\dfrac{1-(1.09)^{-3}}{(1.07)^{-1}+(1.08)^{-2}+(1.09)^{-3}} = .0888,$ or 8.88%.

   (*c*) The yield curve has a positive slope, so that the at-par yield rate increases with *t*.

9. (*a*) Since $6\% < 8.88\%,$ it is a discount bond.

   (*b*) $P = 60\left[(1.07)^{-1} + (1.08)^{-2} + (1.09)^{-3}\right] + 1000(1.09)^{-3} = 926.03.$

   The amount of discount is $1000.00 - 926.03 = \$73.97.$

10. (*a*) We have

$$1030 = 100\sum_{t=1}^{3}(1+s_t)^{-t} + 1000(1+s_3)^{-3}$$
$$= 100\sum_{t=1}^{3}(1+s_t)^{-t} + 1000(1.08)^{-3}$$

and

$$100\sum_{t=1}^{3}(1+s_t)^{-t} = 1030 - 793.832 = 236.168.$$

Then

$$1035 = 100\sum_{t=1}^{4}(1+s_t)^{-t} + 1000(1+s_4)^{-4}$$
$$= 236.168 + 1100(1+s_4)^{-4}$$

and solving we obtain

$$s_4 = .0833, \text{ or } 8.33\%.$$

(*b*) We have

$$1037 = 100\sum_{t=1}^{5}(1+s_t)^{-t} + 1000(1+s_5)^{-5}$$
$$= 236.168 + 100(1.0833)^{-4} + 1100(1+s_5)^{-5}$$

and solving we obtain

$$s_5 = .0860, \text{ or } 8.60\%.$$

(*c*) $P = 100\sum_{t=1}^{6}(1+s_t)^{-t} + 1000(1+s_6)^{-6}$

$$= 1037 - 100(1.0860)^{-5} + 1100(1.07)^{-6} = \$1107.99.$$

11. (*a*) We have

$$P = 60\left[(1.07)^{-1} + (1.08)^{-2}\right] + 1060(1.09)^{-3} = \$926.03.$$

(*b*) Use a financial calculator setting

$$\text{N} = 3 \quad \text{PV} = -926.03 \quad \text{PMT} = 60 \quad \text{FV} = 1000$$

and CPT $\text{I} = .0892$, or 8.92%.

12. Bond 1: $P_1 = \dfrac{C_1 + Fr_1}{1.08} = \dfrac{C_1 + Fr_1}{1+s_1}$ and $s_1 = .08$, or 8%.

Bond 2: $P_2 = \dfrac{Fr_2}{1.08} + \dfrac{C_2 + Fr_2}{(1.08)^2} = \dfrac{Fr_2}{1+s_1} + \dfrac{C_2 + Fr_2}{(1+s_2)^2}$ and $s_2 = .08,$ or 8%.

Bond 3: $P_2 = \dfrac{Fr_3}{1.08} + \dfrac{Fr_3}{(1.08)^2} + \dfrac{C_3 + Fr_3}{(1.08)^3}$

$$= \frac{Fr_3}{1+s_1} + \frac{Fr_3}{(1+s_2)^2} + \frac{C_3 + Fr_3}{(1+s_3)^3} \text{ and } s_3 = .08, \text{ or } 8\%.$$

13. Consider a $1 bond. We have

$$P = \frac{.08}{1.09} + \frac{.08}{(1.09)^2} + \frac{1.08}{(1.09)^3} = \frac{.08}{1.06} + \frac{.08}{(1.08)^2} + \frac{1.08}{(1+X)^3}$$

$$\text{or } .974687 = .144059 + \frac{1.08}{(1+X)^3}$$

and solving for $X = .0915,$ or 9.15%.

14. We are given $s_t = .09 - .02t,$ so that $s_1 = .07$ and $s_2 = .05.$

Bond A: $P_A = \dfrac{100}{1.07} + \dfrac{1100}{(1.05)^2} = \$1091.19$

and thus $1091.19 = \dfrac{100}{1+i_A} + \dfrac{1100}{(1+i_A)^2}.$

Solving the quadratic gives $i_A = .0509,$ or 5.09%.

Bond B: $P_B = \dfrac{50}{1.07} + \dfrac{1050}{(1.05)^2} = \$999.11$

and thus $999.11 = \dfrac{50}{1+i_B} + \dfrac{1050}{(1+i_B)^2}.$

Solving the quadratic gives $i_B = .0505,$ or 5.05%.

The yield rates go in the opposite direction than in Example 10.2.

15. (*a*) Applying formula (10.10)

$$(1+s_3)^3 = (1+s_1)(1+{}_2f_1)^2$$
$$(1.0875)^3 = (1.07)(1+{}_2f_1)^2$$

and solving ${}_2f_1 = .0964,$ or 9.64%.

(*b*) $\left(1+s_5\right)^5 = \left(1+s_2\right)^2\left(1+{}_3f_2\right)^3$

$(1.095)^5 = (1.08)^2\left(1+{}_3f_2\right)^3$

and solving ${}_3f_2 = .1051,$ or 10.51%.

16. (*a*) We have

$$\left(1+s_4\right)^4 = \left(1+f_0\right)\left(1+f_1\right)\left(1+f_2\right)\left(1+f_3\right)$$
$$=(1.09)(1.09)(1.86)(1.078) = 1.39092$$

and solving $s_4 = .0860,$ or 8.60%.

(*b*) We have

$$\left(1+{}_3f_2\right)^3 = \left(1+f_2\right)\left(1+f_3\right)\left(1+f_4\right)$$
$$=(1.086)(1.078)(1.066) = 1.24797$$

and solving ${}_3f_2 = .0766,$ or 7.66%.

17. We have

$$\left(1+s_2\right)^2 = \left(1+s_1\right)\left(1+f_1\right)$$
$$(1.06)^2 = (1.055)\left(1+f_1\right) \text{ and } 1+f_1 = 1.06502$$

$$\left(1+s_3\right)^3 = \left(1+s_1\right)\left(1+{}_2f_1\right)^2$$
$$(1.065)^3 = (1.055)\left(1+{}_2f_1\right)^2 \text{ and } \left(1+{}_2f_1\right)^2 = 1.14498$$

$$\left(1+s_4\right)^4 = \left(1+s_1\right)\left(1+{}_3f_1\right)^3$$
$$(1.07)^4 = (1.055)\left(1+{}_3f_1\right)^3 \text{ and } \left(1+{}_3f_1\right)^2 = 1.24246.$$

The present value of the 1-year deferred 3-year annuity-immediate is

$$\frac{1000}{1.06502}+\frac{1000}{1.14498}+\frac{1000}{1.24246} = \$2617.18.$$

18. We are given that

$$f_3 = .1076 \quad \text{and} \quad f_4 = .1051$$

and

$$\left(1+{}_2f_3\right)^2 = \left(1+f_3\right)\left(1+f_4\right)$$
$$=(1.1076)(1.1051)$$

and solving ${}_2f_3 = .1063,$ or 10.63%.

19. We have

$$i = f_1 = \frac{(1+s_2)^2}{1+s_1} - 1 = \frac{(1.095)^2}{1.085} - 1 = .1051, \text{ or } 10.51\%.$$

20. We have

$$j = f_4 = \frac{(1+s_5)^5}{(1+s_4)^4} - 1 = \frac{(1.075)^5}{(1.082)^4} - 1 = .0474, \text{ or } 4.74\%.$$

21. The present value of this annuity today is

$$5000\left[\frac{1}{(1.0575)^2} + \frac{1}{(1.0625)^3} + \frac{1}{(1.0650)^4}\right] = 12,526.20.$$

The present value of this annuity one year from today is

$$12,526.20(1+s_1) = 12,526.20(1.05) = \$13,153 \text{ to the nearest dollar.}$$

22. We proceed as follows:

$$(1+{}_4f_1)^4 = \frac{(1+s_5)^5}{1+s_1} = \frac{(1.095)^5}{1.07} = 1.47125$$

$$(1+{}_3f_2)^3 = \frac{(1+s_5)^5}{(1+s_2)^2} = \frac{(1.095)^5}{(1.08)^2} = 1.34966$$

$$(1+{}_2f_3)^2 = \frac{(1+s_5)^5}{(1+s_3)^3} = \frac{(1.095)^5}{(1.0875)^3} = 1.22400$$

$$(1+{}_1f_4)^2 = \frac{(1+s_5)^5}{(1+s_4)^4} = \frac{(1.095)^5}{(1.0925)^4} = 1.10506.$$

We then evaluate $s_{\overline{5|}}$ as

$$s_{\overline{5|}} = 1.47125 + 1.34966 + 1.22400 + 1.10506 + 1 = 6.150.$$

23. For the one-year bond:

$$P = \frac{550}{1.07} = 514.02$$

so, no arbitrage possibility exists.

For the two-year bond:

$$P = \frac{50}{1.07} + \frac{550}{(1.08)^2} = 518.27$$

so, yes, an arbitrage possibly does exist.

Buy the two-year bond, since it is underpriced. Sell one-year \$50 zero coupon bond short for $50/1.07 = \$46.73$. Sell two-year \$550 zero coupon bond short for $550/(1.08)^2 = \$471.54$. The investor realizes an arbitrage profit of $46.73 + 471.54 - 516.00 = \$2.27$ at time $t = 0$.

24. (*a*) Sell one-year zero coupon bond at 6%. Use proceeds to buy a two-year zero coupon bond at 7%. When the one-year coupon bond matures, borrow proceeds at 7% for one year.

(*b*) The profit at time $t = 2$ is

$$1000(1.07)^2 - 1000(1.06)(1.07) = \$10.70.$$

25. The price of the 2-year coupon bond is

$$P = \frac{5.5}{1.093} + \frac{105.5}{(1.093)^2} = 93.3425.$$

Since the yield to maturity rate is greater than either of the two spot rates, the bond is underpriced.

Thus, buy the coupon bond for 93.3425. Borrow the present value of the first coupon at 7% for $5.5/1.07 = 5.1402$. Borrow the present value of the second coupon and maturity value at 9% for $105.5/(1.09)^2 = 88.7972$. There will be an arbitrage profit of $5.1402 + 88.7972 - 93.3425 = \$.59$ at time $t = 0$.

26. (*a*) Applying formula (10.17) we have

$$\lambda_t = \frac{1}{t}\int_0^t \delta_r\,dr = \frac{1}{t}\int_0^t \left(.03 + .008r + .0018r^2\right)dr$$
$$= \frac{1}{t}\left[.03r + .004r^2 + .0006r^3\right]_0^t$$
$$= .03 + .004t + .0006t^2 \quad \text{for } 0 \le t \le 5.$$

(*b*) Applying formula (10.13) we have

$$s_2 = e^{\lambda_2} - 1 = e^{(.03+.008+.0024)} - 1$$
$$= .0412, \quad \text{or } 4.12\%.$$

(c) Similar to (b)

$$s_5 = e^{\lambda_5} - 1 = e^{(.03+.02+.015)} - 1$$
$$= .06716.$$

Now

$$(1+s_5)^5 = (1+s_2)^2 (1+{}_3f_2)^3$$
$$(1.06716)^5 = (1.04123)^2 (1+{}_3f_2)^3$$

and solving ${}_3f_2 = .0848$, or 8.48%.

(d) We have

$$\frac{d\lambda_t}{dt} = .004 + .0012t > 0 \text{ for } t > 0$$

so we have a normal yield curve.

27. Applying formula (10.18) we have

$$\delta_t = \lambda_t + t\frac{d\lambda_t}{dt} = (.05 + .01t) + t(.01)$$
$$= .05 + .02t.$$

The present value is

$$a^{-1}(5) = e^{-\int_0^5 \delta_t dt} = e^{-\int_0^5 [.05+.02t]dt}$$
$$= e^{-[.05t+.01t^2]_0^5} = e^{-.25-.25} = e^{-.50}$$
$$= .6065.$$

Alternatively, λ_5 is a level continuous spot rate for $t = 5$, i.e. $\lambda_5 = .05 + (.01)(5) = .1$. We then have

$$a^{-1}(5) = e^{-5(.1)} = e^{-.5} = .6065.$$

28. Invest for three years with no reinvestment:

$$100,000(1.0875)^3 = 128,614.$$

Reinvest at end of year 1 only:

$$100,000(1.07)(1.10)^2 = 129,470.$$

Reinvest at end of year 2 only:

$$100,000(1.08)^2(1.11) = 129,470.$$

Reinvest at end of both years 1 and 2:

$$100,000(1.07)(1.09)(1.11) = 129,459.$$

29. Year 1: $1+i=(1+i')(1+r_1)$

$1.07=(1.03)(1+r_1)$ and $r_1=.03883$, or 3.9%.

Year 2: $(1+i)^2=(1+i')^2(1+r_1)(1+r_2)$

$(1.08)^2=(1.03)^2(1.03883)(1+r_2)$ and $r_2=.05835$, or 5.8%.

Year 3: $(1+i)^3=(1+i')^3(1+r_1)(1+r_2)(1+r_3)$

$(1.0875)^3=(1.03)^3(1.03883)(1.05835)(1+r_3)$ and $r_3=.07054$, or 7.1%.

Chapter 9

1. A: A direct application of formula (9.7) for an investment of X gives

$$A = X\frac{(1.08)^{10}}{(1.05)^{10}} = X\left(\frac{1.08}{1.05}\right)^{10} = 1.32539X.$$

B: A direct application of formula (9.3a) for the same investment of X gives

$$1+i' = \frac{1.08}{1.05} = 1.028571$$

and the accumulated value is

$$B = X(1.028571)^{10} = 1.32539X.$$

The ratio $A/B = 1.00$.

2. Proceeding similarly to Exercise 1 above:

$$A = \frac{\ddot{s}_{\overline{10}|.08}}{(1.05)^{10}} = 9.60496.$$

$$B = \ddot{s}_{\overline{10}|.028571} = 11.71402.$$

The ratio $A/B = .82$.

3. Again applying formula (9.7) per dollar of investment

$$\frac{(1.07)^5}{(1.10)^5} = .87087$$

so that the loss of purchasing power over the five-year period is

$$1-.87087 = .129, \text{ or } 12.9\%.$$

4. The question is asking for the summation of the "real" payments, which is

$$18,000\left[\frac{1}{1.032}+\frac{1}{(1.032)^2}+\cdots+\frac{1}{(1.032)^{15}}\right]$$

$$= 18,000a_{\overline{15}|.032} = \$211,807 \text{ to the nearest dollar.}$$

5. The last annuity payment is made at time $t = 18$ and the nominal rate of interest is a level 6.3% over the entire period. The "real" rate over the last 12 years is

$$i' = \frac{i-r}{1+r} = \frac{.063-.012}{1+.012} = .0504.$$

Thus, the answer is

$$\begin{aligned} X &= 50(1.063)^{-6}\, a_{\overline{12}|.0504} \\ &= (50)(.693107)(8.84329) = \$306 \text{ to the nearest dollar.} \end{aligned}$$

6. The profitability index (PI) is computed using nominal rates of interest. From formula (9.3a)

$$1.04 = \frac{1+i}{1.035} \quad \text{and} \quad i = (1.04)(1.035) - 1 = .0764.$$

The profitability index is defined in formula (7.20)

$$\text{PI} = \frac{\text{NPV}}{\text{I}} = \frac{2000 a_{\overline{8}|.0764}}{10,000} = 1.17.$$

7. (*a*) Coupon 1 = $10,000(1.04)(.05) = \$520.$

Coupon 2 = $10,000(1.04)(1.05)(.05) = \$546.$

Maturity value = $10,000(1.04)(1.05) = \$10,920.$

(*b*) Nominal yield: The equation of value is

$$-10,500 + 520(1+i)^{-1} + (546 + 10,920)(1+i)^{-2} = 0$$

and solving the quadratic we obtain .0700, or 7.00%.

Real yield: The equation of value is

$$-10,500 + 500(1+i)^{-1} + (500 + 10,000)(1+i)^{-2} = 0$$

and solving the quadratic we obtain .0241, or 2.41%.

8. Bond A: Use a financial calculator and set

$$\text{N} = 5 \quad \text{PV} = -950 \quad \text{PMT} = 40 \quad \text{FV} = 1000 \quad \text{and} \quad \text{CPT I} = 5.16\%.$$

Bond B: The coupons will constitute a geometric progression, so

$$\begin{aligned} P &= 40\left[\left(\frac{1.05}{1.0516}\right) + \left(\frac{1.05}{1.0516}\right)^2 + \cdots + \left(\frac{1.05}{1.0516}\right)^5\right] + 1000(1.05)^5(1.0516)^{-5} \\ &= 40(1.05)\frac{1-\left(\dfrac{1.05}{1.0516}\right)^5}{.0516-.05} + 992.416 = \$1191.50 \end{aligned}$$

9. (*a*) The final salary in the 25th year will have had 24 increases, so that we have

$$10,000(1.04)^{24} = \$25,633 \text{ to the nearest dollar.}$$

(*b*) The final five-year average salary is

$$\frac{10,000}{5}\left[(1.04)^{20} + (1.04)^{21} + \cdots + (1.04)^{24}\right] = \$23,736 \text{ to the nearest dollar.}$$

(*c*) The career average salary is

$$\frac{10,000}{25}\left[1 + (1.04) + (1.04)^2 + \cdots + (1.04)^{24}\right] = \$16,658 \text{ to the nearest dollar.}$$

10. The annual mortgage payment under option A is

$$R_A = \frac{240,000 - 40,000}{a_{\overline{10}|.10}} = 32,549.08.$$

The annual mortgage payment under option B is

$$R_B = \frac{240,000 - 40,000}{a_{\overline{10}|.08}} = 29,805.90.$$

The value of the building in 10 years is

$$240,000(1.03)^{10} = 322,539.93.$$

Thus, the shared appreciation mortgage will result in a profit to Lender B

$$.50(322,539.93 - 240,000) = 41,269.97.$$

(*a*) The present value of payments under Option A is

$$PV_A = 40,000 + 32,549.08a_{\overline{10}|.08} = \$258,407 \text{ to the nearest dollar.}$$

The present value of payments under Option B is

$$PV_B = 40,000 + 29,805.90a_{\overline{10}|.08} + 41,269.97(1.08)^{-10}$$
$$= \$259,116 \text{ to the nearest dollar.}$$

Thus, at 8% choose Option A.

(*b*) Similar to (*a*) using 10%, we have

$$PV_A = 40,000 + 32,549.08a_{\overline{10}|.10} = \$240,000$$

which is just the original value of the property. Then,

$$PV_B = 40,000 + 29,805.90a_{\overline{10}|.10} + 41,269.97(1.10)^{-10}$$
$$= \$223,145 \text{ to the nearest dollar.}$$

Thus, at 10% choose Option B.

11. The price of the 3-year bond is

$$P = 1000(1.03)^{-6} + 40a_{\overline{6}|.03} = 1054.17.$$

The investor will actually pay $P + 12 = 1066.17$ for the bond. Solving for the semiannual yield rate *j* with a financial calculator, we have

$$1066.17 = 1000(1+j)^{-6} + 40a_{\overline{6}|j} \quad \text{and} \quad j = .027871.$$

The yield rate then is $2j = .0557$, or 5.57% compared to 5.37% in Example 9.3. The yield rate is slightly higher, since the effect of the expense is spread over a longer period of time.

12. The actual yield rate to A if the bond is held to maturity is found using a financial calculator to solve

$$910.00 = 1000(1+i)^{-5} + 60a_{\overline{5}|\,i}$$

which gives $i = .0827$, or 8.27%. Thus, if A sells the bond in one year and incurs another $10 commission, the price to yield 8.27% could be found from

$$910.00 = (P + 60 - 10)(1.0827)^{-1}$$

which gives $P = \$935.26.$

13. With no expenses the retirement accumulation is

$$10{,}000(1.075)^{35} = 125{,}688.70.$$

With the 1.5 expense ratio the retirement accumulation becomes

$$10{,}000(1.06)^{35} = 76{,}860.87.$$

The percentage reduction is

$$\frac{125{,}688.70 - 76{,}850.87}{125{,}688.70} = .389, \quad \text{or} \quad 38.9\% \text{ compared to } 34.4\%.$$

14. The expense invested in the other account in year *k* is

$$10{,}000(1.06)^{k-1}(.01) \quad \text{for} \quad k = 1, 2, \ldots, 10.$$

The accumulated value of the account after 10 years will be

$$100\left[(1.09)^9 + (1.06)(1.09)^8 + \cdots + (1.09)^9\right]$$

$$= 100\left[\frac{(1.09)^{10} - (1.06)^{10}}{.09 - .06}\right] = \$1921.73$$

by a direct application of formula (4.34).

15. The daily return rate *j* is calculated from

$$\left(i+j\right)^{365}=1.075 \text{ or } j=.000198.$$

The daily expense ratio *r* is calculated from

$$(1+.000198-r)^{365}=1.075-.015=1.06$$
$$\text{or } r=.00003835.$$

Thus the nominal daily expense ratio is

$$(.00003835)(365)=.0140, \text{ or } 1.40\%.$$

16. Let the underwriting cost each year be \$*X* million. The present value of the cash flows to the corporation equals zero at a 7% effective rate of interest. The equation of value (in millions) at time $t=0$ is given by

$$10-X-X(1.07)^{-1}-(.06)(10)a_{\overline{10}|.07}-10(1.07)^{-10}=0$$

and

$$X=\frac{10-.6a_{\overline{10}|.07}-10(1.07)^{-10}}{1+(.107)^{-1}}$$
$$=.363 \text{ million, or \$363,000 to the nearest \$1000.}$$

17. Basis A:

The interest income $=(1.08)^{20}-1=3.66096$ and the after-tax accumulated value is

$$A=1+(.75)(3.66096)=3.74572.$$

Basis B:

The after-tax accumulated value is

$$B=[1+(.75)(.08)]^{20}=(1.06)^{20}=3.20714.$$

The ratio $A/B=1.168$, or 116.8%.

18. The tax deduction is 35% of the depreciation charge.

Year	Depreciation charge	Tax deduction
1	33,330	11,666
2	44,450	15,558
3	14,810	5,184
4	7,410	2,594

The after-tax yield rate is $(.12)(.65)=.078$.

Thus, the present value of the tax deductions is

$$\frac{11,666}{1.078}+\frac{15,558}{(1.078)^2}+\frac{5184}{(1.078)^3}+\frac{2594}{(1.078)^4}=\$30,267 \text{ to the nearest dollar.}$$

19. (*a*) The semiannual before-tax yield rate j^b can be found from

$$670=700\left(1+j^b\right)^{-10}+35a_{\overline{10|}j^b}$$

which gives $j^b=.05571.$ The before-tax effective yield rate is $(1.05571)^2-1=.1145,$ or 11.45%.

(*b*) The earnings on the bond are $(35)(10)+(700-670)=380$ and the tax on that amount is $(.25)(380)=95.$ Then, $670=(700-95)\left(1+j^a\right)^{-10}+35a_{\overline{10|}j^a}$ giving $j^a=.04432$ and an after-tax effective yield rate of $(1.04432)^2-1=.0906,$ or 9.06%.

20. Before-tax: The equation of value is

$$97.78\left(1+i^b\right)=10+102.50 \quad \text{and} \quad i^b=.151, \quad \text{or} \quad 15.1\%.$$

After-tax: The equation of value is

$$\begin{aligned}97.78\left(1+i^a\right)&=10+102.50-.40(10)-.20(102.50-97.78)\\&=107.556 \text{ and } i^a=.100, \text{ or } 10.0\%.\end{aligned}$$

21. (*a*) The installment payment is

$$R=\frac{10,000}{a_{\overline{30|}.05}}=650.51.$$

The income tax in the 10th year is 25% of the portion of the 10th installment that is interest, i.e.

$$(.25)(650.51)\left(1-v^{30-10+1}\right)=\$104.25.$$

(*b*) The total of the interest paid column in the amortization schedule is

$$650.51\left(30-a_{\overline{30|}.05}\right)=\$9515.37.$$

The total tax on this amount of interest is

$$(.25)(9515.37)=\$2378.84.$$

(c) The payment in the k^{th} year after taxes is

$$650.51\left[v^{30-k+1}+.75\left(1-v^{30-k+1}\right)\right]=650.51\left(.75+.25v^{31-k}\right) \text{ for } k=1,2,\ldots,30.$$

The present value of these payments at the before-tax yield rate is

$$\sum_{k=1}^{30}650.51\left(.75+.25v^{31-k}\right)v^k \quad \text{at} \quad i=5\%$$

$$=(.75)(650.51)a_{\overline{30}|.05}+(.25)(650.51)(30)(1.05)^{-31}$$

$$=\$8575 \text{ to the nearest dollar.}$$

22. The annual installment payment is

$$10,000(.05)+\frac{10,000}{s_{\overline{30}|.04}}=500+178.30=678.30.$$

Total installment payments over 30 years are

$$30(678.30)=20,349.00.$$

Total interest on the sinking fund is

$$10,000-(30)(178.30)=4651.00.$$

Taxes on the sinking fund interest are

$$.25(4651.00)=1162.75.$$

Total cost of the loan to A is

$$20,349.00+1162.75=\$21,512 \text{ to the nearest dollar.}$$

23. The after-tax interest income on the fund is

$$800-(240+200+300)=60.$$

Since the income tax rate is 25%, the before-tax interest income on the fund is

$$60/.75=80.$$

Thus, the fund balance at the beginning of the third year before taxes is actually

$$800+(80-60)=820.$$

The equation of value for the before-tax yield rate is

$$240\left(1+i^b\right)^2+200\left(1+i^b\right)+300=820.$$

Solving the quadratic, we obtain

$$i^b=.113, \text{ or } 11.3\%.$$

24. (a) The NPV for the company is

$$62,000\ddot{a}_{\overline{6}|.08}=\$309,548.$$

(*b*) The NPV for the lessor reflecting taxes and depreciation is

$$(.65)(62,000)\ddot{a}_{\overline{6}|.08} + (.35)(250,000/5)a_{\overline{5}|.08} = \$271,079.$$

25. Applying formula (9.11)

$$1+i^d = (1+i^f)\frac{e^c}{e^e}$$

$$1+.075 = (1+.049)\frac{118}{e^e} \text{ and } e^e = 115.1.$$

26. The change in the real exchange rate is equal to the change in the nominal exchange rate adjusted for the different inflation rates, i.e.

$$\frac{1.5}{1.25} \times \frac{1.03}{1.02} - 1 = +.212, \text{ or } +21.2\%.$$

27. Line 1 - We have

$$1+i^d = (1+i^f)\frac{e^c}{e^e} \text{ or } 1+i^d = (1.0932)\frac{56.46}{60.99} \text{ and } i^d = .012, \text{ or } 1.2\%.$$

Line 2 - We have

$$(1.01)^{.25} = (1+i^f)^{.25}\frac{56.46}{57.61} \text{ and } i^f = .0948, \text{ or } 9.48\%.$$

Line 3 - We have

$$(1.01)^{1/12} = (1.0942)^{1/12}\frac{56.46}{e^e} \text{ and } e^e = 56.84.$$

28. (*a*) For a \$1000 maturity value, the price of the two-year coupon bond is

$$P_0 = 1000(1.0365)^{-2} = \$930.81.$$

The price one year later is

$$P_1 = 1000(1.03)^{-1} = \$970.87.$$

Thus, the one-year return is $\dfrac{970.87 - 930.81}{930.81} = .043$, or 4.3%.

(*b*) Assume the person from Japan buys \$930.81 with yen, which costs

$$(930.81)(120.7) = ¥112,349.$$

After one year the person receives \$970.87 which is worth

$$(970.87)(115) = ¥111,650.$$

Thus, the one year return is

$$\frac{111,650 - 112,349}{112,349} = -.0062, \text{ or } -.62\%.$$

29. (*a*) $\text{NPV} = -80 + 10(1.06)^{-1} + 20(1.06)^{-2} + 23(1.06)^{-3} + 27(1.06)^{-4} + 25(1.06)^{-5}$

$= €6.61$ million.

(*b*) The expected exchange rate expressed in dollars per €1 (not in euros per \$1) at time $t=1$ is $\frac{1.08}{1.06} = \frac{e^e}{1.2}$ and $e^e = 1.223.$ The cash flow at time $t=1$ in dollars then is

$$(10)(1.223) = \$12.23 \text{ million.}$$

Using the same procedure to calculate the expected exchange rate and cash flow in dollars each year gives the following:

Time	0	1	2	3	4	5
e^e	1.2	1.223	1.246	1.269	1.293	1.318
\$ million	-96	12.23	24.91	29.19	34.92	32.94

(*c*) $\text{NPV} = -96 + 12.23(1.08)^{-1} + 24.91(1.08)^{-2} + 29.19(1.08)^{-3} + 34.92(1.08)^{-4}$

$+32.94(1.08)^{-5} = \$7.94$ million.

Interestingly, this answer could also be obtained from the answer in part (*a*), as $(6.613)(1.2) = 7.94$. This demonstrates the internal consistency in using interest rate parity.

30. The equation of value is

$$(1+i)^2 = .4(1+i) + .5 \text{ which simplifies to } i^2 + 1.6i + .1 = 0.$$

Solving the quadratic we obtain

$$i = -.0652, \text{ or } -6.52\%.$$

31. The equation of value is

$$10,000(1+i)^{10} = 1500 s_{\overline{10}|.08} = 21,729.84$$

and

$$i = (2.172984)^{1/10} - 1 = .0807, \text{ or } 8.07\%.$$

32. Use the basic formula for valuing bonds with an adjustment for the probability of default to obtain

$$\begin{aligned} P &= 80a_{\overline{10}|.12} + (.98)(1000)(1.12)^{-10} \\ &= 452.018 + 315.534 = \$767.55. \end{aligned}$$

33. (*a*) $\text{EPV} = \dfrac{(.90)(1000)+(.10)(0)}{1.25} = \$720.$

(*b*) $\text{E}(x^2) = .90\left(\dfrac{1000}{1.25}\right)^2 + .10(0)^2 = 576{,}000$

$\text{Var}(x) = 576{,}000 - (720)^2 = 57{,}600$

$\text{S.D.}(x) = \sqrt{57{,}600} = \$240.$

(*c*) $720 = 1000(1+i)^{-1}$ so that $i = .3889$. Thus, the risk premium is

$$.3889 - .25 = .1389, \text{ or } 13.89\%.$$

34. From formula (9.15) we have

$$\text{EPV} = \sum_{t=1}^{n} R_t \left(\frac{p}{1+i}\right)^t.$$

We can consider p to be the probability of payment that will establish equivalency between the two interest rates. Thus, we have

$$\frac{p}{1.0875} = \frac{1}{1.095} \quad \text{or} \quad p = .99315$$

and the annual probability of default is $1 - p = .00685$.

35. (*a*) $\text{EPV} = \displaystyle\sum_{t=1}^{n} R_t \left(\frac{p}{1+i}\right)^t = \sum_{t=1}^{n} R_t e^{-ct} e^{-\delta t}.$

(*b*) We can interpret the formula in part (*a*) as having force of interest δ, force of default c, and present values could be computed at the higher force of interest $\delta' = \delta + c$. The risk premium is $\delta' - \delta = c$.

(*c*) The probability of default is

$$1 - p = 1 - e^{-c}.$$

(*d*) The probability of non-default over *n* periods is $p^n = e^{-cn}$, so the probability of default is $1 - e^{-cn}$.

36. (*a*) Assume that the borrower will prepay if the interest rate falls to 6%, but not if it rises to 10%. The expected accumulated to the mortgage company is

$$.5[80,000(1.06) + 1,000,000(1.06)] + .5[580,000(1.10) + 500,000(1.08)]$$
$$= .5(1,144,800) + .5(1,178,000) = \$1,161,400.$$

(*b*) $\text{Var} = .5(1,144,800 - 1,161,400)^2 + .5(1,178,000 - 1,161,400)^2 = 275,560,000$ and $\text{S.D.} = \sqrt{275,560,000} = \$16,600.$

(*c*) We have $1,161,400 = 1,000,000(1+i)^2$ which solves for $i = .0777,$ or 7.77%.

(*d*) The option for prepayment by the borrower has a value which reduces the expected yield rate of 8% that the lender could obtain in the absence of this option.

37. (*a*) Form formula (9.15) we have

$$\text{EPV} = \sum_{t=1}^{n} R_t \left(\frac{p}{1+i} \right)^t$$

so that

$$150,000 = \sum_{t=1}^{n} R \left(\frac{.99}{1.12} \right)^t.$$

We can define an adjusted rate of interest i', such that

$$1 + i' = \frac{1.12}{.99} \quad \text{and} \quad i' = .131313.$$

We then obtain $R = \dfrac{150,000}{a_{\overline{15|}.131313}} = 23,368.91.$

If the probability of default doubles, we can define

$$1 + i'' = \frac{1.12}{.98} \quad \text{and} \quad i'' = .142857.$$

We then have

$$\text{EPV} = 23,368.91 a_{\overline{15|}.142857} = \$141,500 \quad \text{to the nearest } \$100.$$

(*b*) We now have

$$1 + i''' = \frac{1.14}{.98} \quad \text{and} \quad i''' = .163265$$

and

$$\text{EPV} = 23{,}368.91 a_{\overline{15}|.163265} = \$128{,}300 \quad \text{to the nearest } \$100.$$

38. If the bond is not called, at the end of the 10 years the investor will have

$$100 s_{\overline{10}|.07} + 1000 = 2381.65.$$

If the bond is called, at the end of 10 years, the investor will have

$$100 s_{\overline{5}|.07} (1.07)^5 + 1050(1.07)^5 = 2279.25.$$

Thus, the expected accumulated value (EAV) is

$$(.75)(2381.65) + (.25)(2279.25) = 2356.05.$$

The expected yield rate to the investor can be obtained from

$$1100(1+i)^{10} = 2356.05$$

and

$$i = \left(\frac{2356.05}{1100}\right)^{1/10} - 1 = .0791, \text{ or } 7.91\%.$$

Chapter 8

1. Let X be the total cost. The equation of value is

$$X = \left(\frac{X}{10}\right)\ddot{a}_{\overline{12}|j} \quad \text{where } j \text{ is the monthly rate of interest or } \ddot{a}_{\overline{12}|j} = 10.$$

The unknown rate j can be found on a financial calculator as 3.503%. The effective rate of interest i is then

$$i = (1+j)^{12} - 1 = (1.03503)^{12} - 1 = .512, \text{ or } 51.2\%.$$

2. Per dollar of loan we have

$$L = 1 \quad K = .12 \quad n = 18 \quad R = 1.12/18$$

and the equation of value

$$\frac{1.12}{18}a_{\overline{18}|j} = 1 \quad \text{or} \quad a_{\overline{18}|j} = 16.07143.$$

The unknown rate j can be found on a financial calculator as .01221. The APR is then

$$\text{APR} = 12j = (12)(.01221) = .1465, \text{ or } 14.65\%.$$

3. The equation of value is

$$7.66a_{\overline{16}|j} = 100 \quad \text{or} \quad a_{\overline{16}|} = 13.05483.$$

The unknown rate j can be found on a financial calculator as .025. The APY is then

$$\text{APY} = (1+j)^{12} - 1 = (1.025)^{12} - 1 = .3449, \text{ or } 34.49\%.$$

4. (*a*) Amount of interest = Total payments − Loan amount

Option A: $13(1000) - 12,000 = 1,000.00.$

Option B: $12 \cdot \left(\dfrac{12,000}{a_{\overline{12}|.01}}\right) - 12,000 = 794.28.$

Difference in the amount of interest = 1,000.00 − 794.28 = \$205.72.

(*b*) The equation of value is

$$12,000 - 1000 = 1000a_{\overline{12}|j} \quad \text{or} \quad a_{\overline{12}|j} = 11.$$

Using a financial calculator $j = .013647$ and $\text{APR} = 12j = .1638$, or 16.38%.

(*c*) $\text{APR} = 12(.01) = 12.00\%,$ since the amortization rate directly gives the APR in the absence of any other fees or charges.

5. Bank 1: $X = \dfrac{L + (2)(.065)L}{24} = .04708L.$

   Bank 2: We have $j = (1.126)^{1/12} - 1 = .00994$ so that $Y = \dfrac{L}{a_{\overline{24|}j}} = .04704L.$

   Bank 3: We have $j = .01$ and $Z = \dfrac{L}{a_{\overline{24|}j}} = .04707L.$

   Therefore $Y < Z < X$.

6. (*a*) The United States Rule involves irregular compounding in this situation. We have

$$B_3 = 8000 - [2000 - (8000)(.03)] = 6240$$
$$B_9 = 6240 - [4000 - (6240)(.06)] = 2614.40$$
$$X = B_{12} = 2614.40(1.03) = \$2692.83.$$

   (*b*) The Merchant's Rule involves simple interest throughout. We have

$$X = 8000(1.12) - 2000(1.09) - 4000(1.03)$$
$$= \$2660.00.$$

7. (*a*) The interest due at time $t = 1$ is $10,000(.1) = 1000$. Since only 500 is paid, the other 500 is capitalized. Thus, the amount needed to repay the loan at time $t = 2$ is $10,500(1.1) = \$11,550.$

   (*b*) Under the United States Rule, the interest is still owed, but is not capitalized. Thus, at time $t = 2$ the borrower owes 500 carryover from year 1, 1000 in interest in year 2, and the loan repayment of 10,000 for a total of $11,500.

8. (*a*) The equation of value is

$$200(1+i)^2 - 1000(1+i) + 1000 = 0$$
$$(1+i)^2 - 5(1+i) + 5 = 0.$$

   Now solving the quadratic we obtain

$$1 + i = \frac{5 \pm \sqrt{(-5)^2 - (4)(1)(5)}}{(2)(1)} = \frac{5 \pm \sqrt{5}}{2}$$
$$= 1.382 \text{ and } 2.618$$

   so that $i = .382$ and 2.618, or 38.2% and 261.8%.

(*b*) The method of equated time on the payments is

$$\bar{t} = \frac{(200)(0)+(1000)(2)}{1200} = \frac{5}{3}.$$

This method then uses a loan of 1000 made at time $t=1$ repaid with 1200 at time $t = 5/3$. The equation of value is $1000(1+j) = 1200$ or $j = .20$ for 2/3 of a year. Thus, the $\text{APR} = 3/2\, j = .30,$ or 30%.

9. Consider a loan $L = a_{\overline{n}|}$ with level payments to be repaid by *n* payments of 1 at regular intervals. Instead the loan is repaid by *A* payments of 1 each at irregular intervals. Thus, $A - a_{\overline{n}|}$ represents the finance charge, i.e. total payments less the amount of loan.

   If *B* is the exact single payment point then $A(1+i)^{-B}$ is the present value of total payments or the amount of the loan. Thus, $A - A(1+i)^{-B}$ is again the finance charge.

   $C/1000$ is the finance charge per 1000 of payment and there are *A* payments. Thus, $C\left(\frac{A}{1000}\right)$ is the total finance charge.

10. The monthly payments are:

$$\text{Option A} = \frac{10,000}{a_{\overline{48}|.10/12}} = 253.626.$$

$$\text{Option B} = \frac{9000}{a_{\overline{48}|.09/12}} = 223.965.$$

   To make the two options equal we have the equation of value

$$(253.626 - 223.965)\, s_{\overline{48}|.09/12} = 1000(1+i)^4$$

   and solving for the effective rate *i*, we obtain $i = .143,$ or 14.3%.

11. (*a*) The monthly payments are

$$\text{Option A} = \frac{16,000}{24} = 666.67.$$

$$\text{Option B} = \frac{15,500}{a_{\overline{24}|.0349/12}} = 669.57.$$

   Option A has the lower payment and thus is more attractive.

(*b*) If the down payment is *D*, then the two payments will be equal if

$$\frac{16,000-D}{24}=\frac{15,500-D}{a_{\overline{24|}.0349/12}}.$$

Therefore,

$$D=\frac{(15,500)(24)-16,000a_{\overline{24|}.0349/12}}{24-a_{\overline{24|}.0349/12}}$$

$$=\$1898 \text{ to the nearest dollar.}$$

12. The monthly rate of interest equivalent to 5% effective is $j=(1.05)^{1/12}-1=.004074$. Thus, the monthly loan payment is

$$R=\frac{15,000}{a_{\overline{48|}.004074}}=344.69.$$

The present value of these payments at 12% compounded monthly is

$$344.69a_{\overline{48|}.01}=13,089.24.$$

Thus, the cost to the dealer of the inducement is

$$15,000-13,089.24=\$1911 \text{ to the nearest dollar.}$$

13. (*a*) Prospective loan balance for A is

$$\frac{20,000}{a_{\overline{48|}.07/12}}a_{\overline{36|}.07/12}=\$15,511>\$15,000.$$

Prospective loan balance for B is

$$\frac{20,000}{a_{\overline{24|}.07/12}}a_{\overline{12|}.07/12}=\$10,349<\$15,000.$$

(*b*) The present value of the cost is the present value of the payments minus the present value of the equity in the automobile.

Cost to A:

$$\frac{20,000}{a_{\overline{48|}.07/12}}a_{\overline{12|}.005}-(15,000-15,511)(1.005)^{-12}=(478.92)(11.62)+(511)(.942)$$

$$=\$6047 \text{ to the nearest dollar.}$$

Cost to B:

$$\frac{20,000}{a_{\overline{24|}.07/12}}a_{\overline{12|}.005}-(15,000-10,349)(1.005)^{-12}=(895.45)(11.62)-(4651)(.942)$$

$$=\$6026 \text{ to the nearest dollar.}$$

14. (*a*) Formula (8.6) is

$$\begin{aligned} R &= B_0 i + \frac{D}{s_{\overline{n}|}} \\ &= (20{,}000)(.005) + \frac{20{,}000 - 13{,}000}{s_{\overline{24}|.005}} \\ &= \$375.24. \end{aligned}$$

(*b*) The equation of value is

$$20{,}000 - 300 = 375.24 a_{\overline{24}|j} + (13{,}000 + 200) v_j^{24}$$
$$19{,}700 = 375.24 a_{\overline{24}|j} + 13{,}200 v_j^{24}.$$

Using a financial calculator, we find that $j = .63\%$ monthly.

(*c*) The equation of value is

$$20{,}000 - 300 = 375.24 a_{\overline{12}|j} + (16{,}000 + 800) v_j^{12}$$
$$19{,}700 = 375.24 a_{\overline{12}|j} + 16{,}800 v_j^{12}.$$

Using a financial calculator, we find that $j = .73\%$ monthly.

15. We modify the formula in Example 8.4 part (2) to

$$19{,}600 - 341.51 = 341.51 a_{\overline{36}|j} + (10{,}750 + 341.51) v_j^{36}$$
$$19{,}258.49 = 341.51 a_{\overline{36}|j} + 11{,}091.51 v_j^{36}.$$

Using a financial calculator, we find that $j = .74\%$ monthly. The nominal rate of interest convertible monthly is $12j = 8.89\%$. This compares with the answer of 7.43% in Example 8.4. Thus, the effect of making a security deposit that does not earn interest is significant.

16. (*a*) The NPV of the "buy" option is

$$50{,}000(1.01)^{-72} - \left(400{,}000 + 4000 a_{\overline{72}|.01}\right) = 24{,}424.80 - 604{,}601.57 = -\$580{,}177.$$

(*b*) The NPV of the "lease" option is

$$-12{,}000 a_{\overline{72}|.01} = -\$613{,}805.$$

(*c*) The "buy" option should be chosen since it is the least negative.

17. (*a*)

Mortgage loan L = $.75(160,000) = 120,000.$

Mortgage payment R = $\dfrac{120,000}{a_{\overline{360}|.0075}} = 965.55.$

Interest as points Q = $(.015)(120,000) = 1800.00.$

September 16 interest = $.09\left(\dfrac{15}{365}\right)(120,000) = 443.84.$

November 1 interest = $(.0075)(120,000) = 900.00.$

December 1 interest = $.0075[120,000 - (965.55 - 900.00)]$
= $899.51.$

Total interest = $1800.00 + 443.84 + 900.00 + 899.51$
= $\$4043.35.$

(*b*) Interest as points $Q = (.015)(120,000) = 1800.00$

Adjusted loan $L^* = 120,000 - 1800 = 118,200$

APR calculation $a_{\overline{360}|j} = \dfrac{L^*}{R} = \dfrac{118,200}{965.55} = 122.41728.$

Use a financial calculator to find $j = .007642$, $\text{APR} = 12j = 12(.007642) = .0917$, or 9.17%.

18. The interest saved by this payment scheme is the interest in each even-numbered payment in the original $12 \times 15 = 180$ payment amortization schedule. Thus, we have

$$
\begin{aligned}
&1000\left[(1-v^{179}) + (1-v^{177}) + \cdots + (1-v)\right] \\
&= 1000\left[90 - (v + v^3 + \cdots + v^{177} + v^{179})\right] \\
&= 90,000 - 1000v(1 + v^2 + \cdots + v^{176} + v^{178}) \\
&= 90,000 - 1000v\frac{1-v^{180}}{1-v^2} \\
&= 90,000 - 1000(1+i)\left[\frac{1-v^{180}}{(1+i)^2 - 1}\right] \\
&= 90,000 - 1000\frac{\ddot{a}_{\overline{180}|}}{s_{\overline{2}|}}.
\end{aligned}
$$

19. At time $t = 2$ the accumulated value of the construction loan is

$$1,000,000(1.075)^4 + 500,000(1.075)^3 + 500,000(1.075)^2 = 2,534,430.08$$

which becomes the present value of the mortgage payments. Thus, we have the equation of value

$$2,534,430.08 = P\ddot{a}_{\overline{60}|.01} + 2P\ddot{a}_{\overline{300}|.01}(1.01)^{-60}$$

$$\text{and} \quad P = \frac{2,534,430.08}{\ddot{a}_{\overline{60}|.01} + 2(1.01)^{-60}\,\ddot{a}_{\overline{300}|.01}} = \$16,787$$

to the nearest dollar. The 12th mortgage payment is equal to *P*, since it is before the payment doubles. Also, note the annuity-due, since the first mortgage payment is due exactly two years after the initial construction loan disbursement.

20. The loan origination fee is $.02(100,000) = 2000.$

The mortgage payment is $R = \dfrac{100,000}{a_{\overline{30}|.08}} = 8882.74.$

Loan balance at $t = 1$: $B_1 = 100,000(1.08) - 8882.74 = 99,117.26.$

Loan balance at $t = 2$: before any payments $B_2' = 99,117.26(1.08) = 107,046.64.$

Adjusted loan $L^* = 100,000 - 2000 = 98,000.$

Thus, the equation of value becomes

$$98,000 = 8882.74v + 107,046.64v^2$$

and solving the quadratic

$$v = \frac{-8882.74 \pm \sqrt{(8882.74)^2 - (4)(107,046.64)(-98,000)}}{2(107,046.64)}$$

$$= .91622 \text{ rejecting the negative root.}$$

Finally, $i = \dfrac{1}{v} - 1 = .0914,$ or 9.14%.

21. There are $10 \times 4 = 40$ payments on this loan. The quarterly interest rates are $j_1 = \dfrac{.12}{4} = .03$ and $\dfrac{j_2}{4} = .035.$ The loan balance $B_{12} = 1000a_{\overline{28}|.03} = 18,764.12.$ The loan balance after 12 more payments is

$$B_{24} = (18,764.12)(1.035)^{12} - 1000s_{\overline{12}|.035}$$

$$= \$13,752 \text{ to the nearest dollar.}$$

22. (*a*) The equation of value is

$$100{,}000 = R\left[v + 1.05v^2 + (1.05)^2 v^3 + (1.05)^3 v^4 + (1.05)^4 v^5 + (1.05)^4 v^6 + (1.05)^4 v^7 + \cdots + (1.05)^4 v^{30}\right]$$

$$= \frac{R}{1.09}\left[1 + \frac{1.05}{1.09} + \left(\frac{1.05}{1.09}\right)^2 + \left(\frac{1.05}{1.09}\right)^3 + \left(\frac{1.05}{1.09}\right)^4 \left(1 + v + \cdots + v^{25}\right)\right]$$

and $R = \dfrac{100{,}000(1.09)}{\dfrac{1-\left(\frac{1.05}{1.09}\right)^5}{1-\left(\frac{1.05}{1.09}\right)} + \left(\frac{1.05}{1.09}\right)^4 a_{\overline{25|}.09}} = \$8318$ to the nearest dollar.

(*b*) $I_1 = (.09)(100{,}000) = 9000$ and

$R_1 = \$8318;$ so, yes, negative amortization does occur.

23. The payment on the assumed mortgage is

$$R_1 = \frac{60{,}000}{a_{\overline{30|}.08}} = 5329.64.$$

The loan balance $B_{10} = 5329.64 a_{\overline{20|}.08} = 52{,}327.23.$ The amount of the "wraparound" mortgage is $(.85)(120{,}000) - 52{,}327.23 = 49{,}672.77.$ The payment on the "wraparound" mortgage is $R_2 = \dfrac{49{,}672.77}{a_{\overline{20|}.10}} = 5834.54.$ The total payment required is $R_1 + R_2 = \$11{,}164$ to the nearest dollar.

24. The equity in the house will be

$$100{,}000(1.06)^5 - 500 s_{\overline{60|}.01} = 133{,}882.56 - 40{,}834.83 = \$92{,}988$$

to the nearest dollar.

25. The monthly payment is $\dfrac{1200+108}{12} = 109.$

(*a*) All the early payments are principal, so

$$B_4 = 1200 - 4(109) = \$764.$$

(*b*) All interest is paid from the first payment, so

$$B_4 = 1200 - (109 - 108) - 3(109) = \$872.$$

(*c*) The ratio 1200/1308 of each payment is principal, so

$$B_4 = 1200 - 4\left(\frac{1200}{1308}\right)(109) = \$800.$$

(*d*) The interest in the first four payments is

$$\left(\frac{12+11+10+9}{78}\right)(108) = 58.15, \text{ so}$$
$$B_4 = 1200 - 4(109) + 58.15 = \$822.15.$$

26. Under the direct ratio method

$$I_2 = K\frac{8}{S_9} = 20 \quad \text{and} \quad I_8 = K \cdot \frac{2}{S_9}.$$

Therefore $I_8 = \frac{2}{8}(20) = \$5.$

27. The total payments are $6(50) + 6(75) = 750$. Now, $K = 750 - 690 = 60$, so that $60/750 = .08$ of each payment is interest and .92 is principal. Therefore, principal payments are 46 for the first six months and 69 for the last 6 months. The 12 successive loan balances are:

690, 644, 598, 552, 506, 460, 414, 345, 276, 207, 138, 69

which sum to 4899. We then have

$$i^{cr} = \frac{(12)(60)}{4899} = .147, \text{ or } 14.7\%.$$

28. We are given:

$$i^{max} = \frac{2mK}{L(n+1) - K(n-1)} = .20 \quad \text{and} \quad i^{min} = \frac{2mK}{L(n+1) + K(n-1)} = .125.$$

Taking reciprocals

$$\frac{L(n+1)}{2mK} - \frac{K(n-1)}{2mK} = 5 \quad \text{and} \quad \frac{L(n+1)}{2mK} + \frac{K(n-1)}{2mK} = 8.$$

We have two equations in two unknowns which can be solved to give

$$\frac{L(n+1)}{2mK} = 6.5 \quad \text{and} \quad \frac{K(n-1)}{2mK} = 1.5.$$

Now taking the reciprocal of formula (8.19)

$$\frac{1}{i^{dr}} = \frac{L(n+1) + \frac{1}{3}K(n-1)}{2mK} = 6.5 + \tfrac{1}{3}(1.5) = 7$$

so that $i^{dr} = \frac{1}{7} = .143,$ or 14.3%.

29. For annual installments R, we have $m = 1$ and $n = 5$. The finance charge is $K = 5R - L$. We then have

$$B_2^{dr} = 3P - \frac{6}{15}(5R - L) = R + \frac{6}{15}L.$$

For the amortized loan, we have

$$B_2^P = Ra_{\overline{3}|.05} = 2.72317R.$$

Equating the two we have

$$\frac{6}{15}L = 1.72317R \quad \text{or} \quad L = 4.30793R.$$

However, since $L = Ra_{\overline{5}|\,i},$ we have $a_{\overline{5}|\,i} = 4.31.$

30. We have

$$I_1 = \frac{i}{m}\cdot L$$

$$I_2 = \frac{i}{m}\left[L - \frac{L+K}{n}\right]$$

$$\vdots$$

$$I_n = \frac{i}{m}\left[1 - (n-1)\left(\frac{L+K}{n}\right)\right].$$

However, these interest payments do not earn additional interest under simple interest. The finance charge is the sum of these interest payments

$$K = \sum_{t-0}^{n-1}\frac{i}{m}\left[1 - t\cdot\frac{L+K}{n}\right] = \frac{i}{m}\left[Ln - \frac{L+K}{n}\cdot\frac{n(n-1)}{2}\right]$$

which can be solved to give formula (8.14)

$$i^{max} = \frac{2mK}{L(n+1) - K(n-1)}.$$

31. The reciprocal of the harmonic mean is the arithmetic mean of the two values given. In symbols,

$$\frac{1}{2}\left[\frac{1}{i^{max}} + \frac{1}{i^{min}}\right] = \frac{1}{2}\left[\frac{L(n+1) - K(n-1)}{2mK} + \frac{L(n+1) + K(n-1)}{2mK}\right]$$
$$= \frac{1}{2}\left[\frac{2L(n+1)}{2mK}\right] = \frac{L(n+1)}{2mK} = \frac{1}{i^{cr}}.$$

32. (*a*) The outstanding loan balances are

$$L, L - \left(\frac{L+K}{n}\right), L - 2\left(\frac{L+K}{n}\right), \ldots, L - (n-r)\left(\frac{L+K}{n}\right)$$

after $n-r$ payments have been made. Since *r* payments are enough to pay *K*, then $B_{n-r+1} = 0$. The denominator of formula (8.13) then becomes

$$(n-r+1)L - \left(\frac{L+K}{n}\right)\left[\frac{(n-r)(n-r+1)}{2}\right].$$

Finally, applying formula (8.13) and multiplying numerator and denominator by $2n$, we obtain

$$i^{max} = \frac{2mnK}{2n(n-r+1)L - (n-r)(n-r+1)(L+K)}.$$

(*b*) The first $r-1$ payments are all interest, so that the outstanding balances are all equal to *L* followed by

$$(n-r)\left(\frac{L+K}{n}\right), (n-r+1)\left(\frac{L+K}{n}\right), \ldots, \frac{L+K}{n}.$$

Again applying formula (8.13)

$$i = \frac{mK}{rL + r\left(\frac{L+K}{n}\right)\left[\frac{(n-r)(n-r+1)}{2}\right]} = \frac{2mnK}{2nrL + (n-r)(n-r+1)(L+K)}.$$

33. (*a*) (1) $D_3 = \dfrac{A}{s_{\overline{10|}}}(1+j)^2$ and $D_9 = \dfrac{A}{s_{\overline{10|}}}(1+j)^8$

Therefore, $D_9 = D_3(1+j)^6 = (1000)(1.05)^6 = \$1340.10$.

(2) $D_9 = D_3 = \$1000.00$.

(3) $D_3 = \dfrac{8A}{S_{10}}$ and $D_9 = \dfrac{2A}{S_{10}}$.

Therefore, $D_9 = \dfrac{1}{3}D_3 = \dfrac{1}{4}(1000) = \$250.00$.

(*b*) (1) $D_3 = \dfrac{A}{s_{\overline{10|}}}(1.05)^2 = 1000$, so that $A = 1000 s_{\overline{10|}} v^2 = \$11,408.50$.

(2) $D_3 = \dfrac{A}{10} = 1000$, so that $A = \$10,000.00$

(3) $D_3 = \dfrac{8A}{S_{10}} = \dfrac{8A}{\frac{1}{2}(10)(11)} = 1000$, so that $A = \dfrac{(1000)(10)(11)}{(2)(8)} = \$6875.00$.

34. The present value of the depreciation charges is

$$\sum_{t=1}^{10} \frac{2000-400}{s_{\overline{10|}\,i}}(1+i)^{t-1} v_i^t = \sum_{t=1}^{10} \frac{1600}{s_{\overline{10|}\,i}(1+i)} = \frac{16,000}{\ddot{s}_{\overline{10|}\,i}} = 1000, \text{ or } \ddot{s}_{\overline{10|}\,i} = 16.$$

Using a financial calculator, we obtain $i = .0839$, or 8.39%.

35. We have the following:

(*i*) $D = \dfrac{X-Y}{n} = 1000$ or $X - Y = 1000n$.

(*ii*) $D_3 = \dfrac{n-3+1}{S_n}(X-Y) = \dfrac{n-2}{\frac{1}{2}n(n+1)}(X-Y) = 800$

or $(n-2)(X-Y) = 400n(n+1)$.

Now substituting (*i*) into (*ii*), we have

$$1000n(n-2) = 400n(n+1)$$
$$1000n - 2000 = 400n + 400$$
$$600n = 2400 \text{ or } n = 4.$$

Therefore, $X - Y = 4000$.

$$(iii)\ d = 1 - \left(\frac{Y}{X}\right)^{.25} = .33125 \quad \text{or} \quad \left(\frac{Y}{X}\right)^{.25} = .66875$$

$$\frac{Y}{X} = (.66875)^4 = .2 \qquad\qquad Y = .2X.$$

Therefore, $X - .2X = 4000,$ and $X = \$5000.$

36. Under the constant percentage method

$$\begin{aligned} D_1 &= .2B_0 = .2(20{,}000) &= 4000 \\ D_2 &= .2B_1 = .2(16{,}000) &= 4000(.8) \\ D_3 &= .2B_2 = .2(12{,}800) &= 4000(.8)^2 \\ &\vdots & \vdots \\ D_{15} & &= 4000(.8)^{14} \end{aligned}$$

The depreciation charges constitute an annuity whose payments vary in geometric progression. The accumulated value is

$$4000\left[(1.06)^{14} + (.8)(1.06)^{13} + \cdots + (.8)^{13}(1.06) + (.8)^{14}\right]$$

$$= 4000\frac{\left[\frac{(1.06)^{15}}{.8} - (.8)^{14}\right]}{\frac{1.06}{.8} - 1} = \$36{,}329 \text{ to the nearest dollar.}$$

37. Under the sum of the years digits method

$$(5000 - S)\frac{10 + 9 + 8 + 7}{55} = 5000 - 2218 = 2782$$

and solving $S = 5000.$ The level depreciation charge over the next six years will be

$$\frac{2218 - 500}{6} = \$286.33.$$

38. Machine I: $B_{18} = S + \dfrac{S_2}{S_{20}}(A - S) = 5000 + \dfrac{3}{210}(35{,}000) = 5500.$

Machine II: $B_{18} = A - \dfrac{A - S}{s_{\overline{20}|}} s_{\overline{18}|} = 5346.59 + .86633S.$

Equating the two and solving for S gives

$$S = \frac{5500 - 5346.59}{.86633} = \$177 \text{ to the nearest dollar.}$$

39. Under the compound interest method

$$B_{10} = A - \left(\frac{A-S}{s_{\overline{15}|}}\right) s_{\overline{10}|} = 15{,}000 - \left(\frac{13{,}000}{21.5786}\right)(12.5778) = 7422.52.$$

Continuing thereafter on the straight-line method gives

$$B_{12} = 7422.52 - \frac{2}{5}(7442.52 - 2000) = \$5253 \text{ to the nearest dollar.}$$

40. Machine A: $D = \dfrac{2450-1050}{14} = 100$

and the present value of these depreciation charges is

$$100 a_{\overline{14}|.10} = 736.67.$$

Machine B: $S_{14} = \dfrac{1}{2}(14)(15) = 105.$

The pattern of depreciation charges is

$$\frac{14}{105}(Y-1050), \frac{13}{105}(Y-1050), \ldots, \frac{1}{105}(Y-1050).$$

The present value of these depreciation charges is

$$\frac{Y-1050}{105}\left(14v^{14} + 13v^{13} + \cdots + v^{14}\right) = \frac{Y-1050}{105}(Da)_{\overline{14}|}.$$

Now evaluating $(Da)_{\overline{14}|} = \dfrac{14 - a_{\overline{14}|.1}}{.1} = 66.3331$

we obtain

$$\frac{(Y-1050)(66.3331)}{105} = 736.67$$

and solving $Y = \$2216$ to the nearest dollar.

41. We have

$$\frac{d}{dt}\left(B_t^{SL} - B_t^{CP}\right) = \frac{d}{dt}\left[\left\{A - \frac{t}{n}(A-S)\right\} - A(1-d)^t\right]$$

$$= \frac{-A-S}{n} - A(1-d)^t \ln(1-d) = 0.$$

Now $A(1-d)^n = S$, so that $1-d = (S/A)^{1/n}$. Substituting for $1-d$, we obtain

$$\frac{A-S}{n}=-A\left[(S/A)^{1/n}\right]^t \ln\left[(S/A)^{1/n}\right].$$

After several steps of algebraic manipulation we find that

$$t=n\frac{\ln(1-S/A)-\ln\left[-\ln(S/A)\right]}{n\ln(S/A)}.$$

42. (*a*) $H=10{,}000(.05)+\dfrac{9000}{s_{\overline{10}|.05}}+500=\$1715.55.$

(*b*) $K=\dfrac{1715.55}{.05}=\$34{,}311$ to the nearest dollar.

43. Equating periodic charges, we have

$$1000i+\frac{950}{s_{\overline{9}|}}=1100i+\frac{900}{s_{\overline{9}|}}.$$

This simplifies to

$$\frac{50}{s_{\overline{9}|}}=100i \qquad \text{or} \qquad 50=100\left[(1+i)^9-1\right]$$

$$(1+i)^9=1.5 \quad \text{and} \quad i=(1.5)^{1/9}-1=.0461, \text{ or } 4.61\%.$$

44. Plastic trays:

To cover 48 years, six purchases will be necessary at the prices:

$$20, 20(1.05)^8, 20(1.05)^{16}, 20(1.05)^{24}, 20(1.05)^{32}, 20(1.05)^{40}.$$

The present value of these purchases is

$$20\left[1+\left(\frac{1.05}{1.1025}\right)^8+\left(\frac{1.05}{1.1025}\right)^{16}+\cdots+\left(\frac{1.05}{1.1025}\right)^{40}\right]$$

$$=20\left[1+(1.05)^{-8}+(1.05)^{-16}+\cdots+(1.05)^{-40}\right]$$

$$=20\frac{1-v^{48}}{1-v^8}=55.939.$$

Metal trays:

Two purchases will be necessary at the prices: $X, X(1.05)^{24}$.

The present value of these purchases is

$$X\frac{1-v^{48}}{1-v^{24}}=1.3101X.$$

Therefore, $1.3101X=55.939$ or $X=\$42.70.$

45. Without preservatives the periodic charge for the first 14 years is

$$H = 100i + \frac{100}{s_{\overline{14|}}} = \frac{100}{a_{\overline{14|}}}.$$

For the next 14 years it is $H(1.02)^{14}$, continuing indefinitely. Thus, the capitalized cost is

$$K = Ha_{\overline{14|}} + H(1.02)^{14} v^{14} a_{\overline{14|}} + \cdots$$

$$= Ha_{\overline{14|}}\left[1 + \left(\frac{1.02}{1.04}\right)^{14} + \cdots\right] = 100\left[\frac{1}{1-\left(\frac{1.02}{1.04}\right)^{14}}\right] = 420.108.$$

With preservatives we replace 100 with $100 + X$ and 14 with 22 to obtain

$$K = (100 + X)\left[\frac{1}{1-\left(\frac{1.02}{1.04}\right)^{22}}\right] = 2.87633(100 + X).$$

Equating and solving for X we obtain

$$X = \frac{420.108}{2.87633} - 100 = \$46.06.$$

46. We can equate periodic charges to obtain

$$1000(.035) + \frac{950}{s_{\overline{10|}.035}} = (1000 + X)(.035) + \frac{950 + X}{s_{\overline{15|}.035}}$$

$$\frac{950}{s_{\overline{10|}.035}} = X(.035) + \frac{950 + X}{s_{\overline{15|}.035}}$$

$$80.9793 = .035X + 49.2338 + .05183X$$

and $X = \frac{31.7455}{.08683} = \$365.63.$

47. Machine 1:

For the first 20 years periodic charges are

$$H_1 = 100{,}000i + \frac{100{,}000}{s_{\overline{20|}}} + 3000(1.04)^{t-1} = \frac{100{,}000}{a_{\overline{20|}}} + 3000(1.04)^{t-1}$$

for $t = 1, 2, \ldots, 20.$

The present value is

$$100{,}000 + 3000\left[1 + \left(\frac{1.04}{1.08}\right) + \left(\frac{1.04}{1.08}\right)^2 + \cdots + \left(\frac{1.04}{1.08}\right)^{19}\right] = 142{,}921.73.$$

For the next 20 years it is $H(1.04)^{20}$ continuing indefinitely. Thus, the capitalized cost is

$$142,921.73\left[1+\left(\frac{1.04}{1.08}\right)^{20}+\left(\frac{1.04}{1.08}\right)^{40}+\cdots\right]=269,715.55.$$

Machine 2:

$$H_2=\frac{A}{a_{\overline{15}|}}+10,000(1.04)^{t-1} \quad \text{for} \quad t=1,2,\ldots,15.$$

The present value is

$$X+10,000\left[1+\left(\frac{1.04}{1.08}\right)+\cdots+\left(\frac{1.04}{1.08}\right)^{14}\right]=116,712.08+X.$$

The capitalized cost is

$$(116,712.08+A)\left[1+\left(\frac{1.04}{1.08}\right)^{15}+\left(\frac{1.04}{1.08}\right)^{30}+\cdots\right]=(2.31339)(116,712.08+A).$$

Since Machine 2 produces output twice as fast as Machine 1, we must divide by 2 before equating to Machine 1. Finally, putting it all together we obtain

$$A=\frac{2(269,715.55)}{2.31339}-116,712.08=\$116,500 \text{ to the nearest } \$100.$$

48. The sinking fund deposit is

$$D=\frac{A-S}{s_{\overline{n}|j}}.$$

From (*i*), (*ii*), and (*iii*) we obtain

$$B_6=A-Ds_{\overline{6}|.09} \quad \text{or} \quad 55,216.36=A-7.52334D.$$

From (*ii*), (*v*), and (*vi*) we obtain

$$H=Ai+\frac{A-S}{s_{\overline{n}|j}}+M \quad \text{or}$$

$$11,749.22=.09A+D+3000.$$

Thus, we have two equations in two unknowns which can be solved to give

$$D=2253.74 \quad \text{and} \quad A=\$72,172.$$

Chapter 7

1. The maintenance expense at time $t=6$ is $3000(1.06)^{6-0}=4255.56$. The projected annual return at time $t=6$ is $30,000(.96)^{6-1}=24,461.18$. Thus,

$$R_6 = 24,461.18 - 4255.56 = \$20,206 \text{ to the nearest dollar.}$$

2. (*a*) $P(i) = -7000 + 4000v_i - 1000v_i^2 + 5500v_i^3$.

   Thus, $P(.09) = 1000\left[-7 + 4(.91743) - (.91743)^2 + 5.5(.91743)^3\right] = 75.05.$

   (*b*) $P(.10) = 1000\left[-7 + 4(.90909) - (.90909)^2 + 5.5(.90909)^3\right] = -57.85.$

3. Net cash flows are:

Time	NCF
0	− 3000
1	2000 − 1000 = 1000
2	4000

The IRR is found by setting $P(i)=0$, i.e.

$$-3000 + 1000v + 4000v^2 = 0$$

$$4v^2 + v - 3 = (4v-3)(v+1) = 0$$

so that $v = \frac{3}{4}$, rejecting the root $v=-1$. Finally, $1+i = \frac{4}{3}$, and $i = \frac{1}{3}$, so $n=3$.

4. The equation of value equating the present values of cash inflows and cash outflows is

$$2,000,000 + Xv^5 = 600,000a_{\overline{10}|} - 300,000a_{\overline{5}|} \text{ at } i = 12\%.$$

Therefore,

$$X = \left[600,000a_{\overline{10}|} - 300,000a_{\overline{5}|} - 2,000,000\right](1.12)^5$$
$$= \$544,037.$$

5. Project P: $P(i) = -4000 + 2000v + 4000v^2$.

   Project Q: $P(i) = 2000 + 4000v - Xv^2$.

   Now equating the two expressions, we have

$$(X+4000)v^2 - 2000v - 6000 = 0$$

$$(X+4000) - 2000(1.1) - 6000(1.1)^2 = 0$$

and

$$X = 2200 + 7260 - 4000 = \$5460.$$

6. (*a*) This Exercise is best solved by using the NPV functionality on a financial calculator. After entering all the NCF's and setting $\mathrm{I} = 15\%$, we compute $\mathrm{NPV} = P(.15) = -\$498,666.$

   (*b*) We use the same NCF's as in part (*a*) and compute $\mathrm{IRR} = 13.72\%$.

7. (*a*) The formula for $P(i)$ in Exercise 2 has 3 sign changes, so the maximum number of positive roots is 3.

   (*b*) Yes.

   (*c*) There are no sign changes in the outstanding balances, i.e.

$$7000 \quad \text{to} \quad 3000 \quad \text{to} \quad 4000 \quad \text{at} \quad i = 0.$$

   Taking into account interest in the range of 9% to 10 % would not be significant enough to cause any sign changes.

8. The equation of value at time $t = 2$ is

$$100(1+r)^2 - 208(1+r) + 108.15 = 0$$
$$(1+r)^2 - 2.08(1+r) + 1.0815$$

   which can be factored as

$$[(1+r) - 1.05][(1+r) - 1.03].$$

   Thus, $r = .05$ and .03, so that $|i - j| = .02.$

9. Using one equation of value at time $t = 2,$ we have

$$\begin{matrix} 1000(1.2)^2 + A(1.2) + B = 0 \\ 1000(1.4)^2 + A(1.4) + B = 0 \end{matrix} \quad \text{or} \quad \begin{matrix} 1.2A + B = -1440 \\ 1.4A + B = -1960. \end{matrix}$$

   Solving two equations in two unknowns gives $A = -2600$ and $B = 1680.$

10. (*a*) Adapting formula (7.6) we have:

    Fund A: 10,000

    Fund B: $600 s_{\overline{5|}.04}(1.04)^5 = (600)(5.416323)(1.216653) = 3953.87$

    Fund C: $600 s_{\overline{5|}.05} = (600)(5.525631) = 3315.38$.

    A+B+C = $10,000 + 3953.87 + 3315.38 = \$17,269$ to the nearest dollar.

(*b*) We then have the equation of value

$$10,000(1+i')^{10} = 17,269$$

so that

$$i' = (1.7269)^{1/10} - 1 = .0562, \text{ or } 5.62\%.$$

11. If the deposit is D, then the reinvested interest is $.08D,\ .16D,\ .24D, \ldots,\ .80D$. We must adapt formula (7.7) for an annuity-due rather than an annuity-immediate. Thus, we have the equation of value

$$10D + .08D(Is)_{\overline{10}|.04} = 1000$$

so that

$$D = \frac{1000}{10 + \frac{.08}{.04}\left(\ddot{s}_{\overline{10}|.04} - 10\right)} = \frac{1000}{2\ddot{s}_{\overline{10}|.04} - 10} = \frac{1000}{\ddot{s}_{\overline{11}|.04} - 12}.$$

12. The lender will receive a total accumulated value of $1000s_{\overline{20}|.05} = 33,065.95$ at the end of 20 years in exchange for the original loan of 10,000. Thus, we have the equation of value applying formula (7.9)

$$10,000(1+i')^{20} = 33,065.95$$

and

$$i' = (3.306595)^{1/20} - 1 = .0616, \quad \text{or} \quad 6.16\%.$$

13. From formula (7.7) the total accumulated value in five years will be

$$5(1000) + 40\frac{s_{\overline{5}|.03} - 5}{.03} = 5412.18.$$

The purchase price P to yield 4% over these five years is

$$P = 5412.18(1.04)^{-5} = \$4448 \text{ to the nearest dollar.}$$

14. Applying formula (7.10) we have

$$110(1+i')^{24} = 5s_{\overline{24}|.035} + 100 = 283.3326$$

so that

$$(1+i')^{24} = 2.57575 \quad \text{and} \quad i' = (2.57575)^{1/24} - 1 = .04021.$$

The answer is

$$2i' = 2(.04021) = .0804, \text{ or } 8.04\%.$$

15. The yield rate is an annual effective rate, while the bond coupons are semiannual. Adapting formula (7.10) for this situation we have

$$1000(1.07)^{10} = 30s_{\overline{20}|j} + 1000$$

and

$$s_{\overline{20}|j} = 32.23838.$$

We now use a financial calculator to solve for the unknown rate j to obtain $j = .047597$. The answer is the annual effective rate i equivalent to j, i.e. $i = (1+j)^2 - 1 = .0975$, or 9.75%.

16. The equation of value is

$$300\ddot{s}_{\overline{20}|.08} = (20)(300) + 300i(Is)_{\overline{20}|i/2}$$

or

$$14,826.88 = 6000 + 300i\left(\frac{s_{\overline{21}|i/2} - 21}{i/2}\right)$$

$$= 6000 + 600s_{\overline{21}|i/2} - 12,600$$

and

$$s_{\overline{21}|i/2} = 35.711467.$$

We now use a financial calculator to solve for the unknown rate $i/2$ to obtain $i/2 = .050$, so that $i = .100$, or 10.0%.

17. The loan is 25,000 and if it is entirely repaid at the end of one year the amount paid will be

$$25,000(1.08) = 27,000.$$

This money can be reinvested by the lender at only 6% for the next three years. Thus, over the entire four-year period we have a lender yield rate of

$$25,000(1+i')^4 = 27,000(1.06)^3 = 32,157.43$$

or

$$i' = (1.286)^{1/4} - 1 = .0649, \text{ or } 6.49\%.$$

18. The accumulated value of the 50,000 payments at time $t = 4$ is

$$50,000s_{\overline{3}|.08} = 162,300.$$

Thus we have

$$\text{NPV} = P(.1) = -100,000 + (1.1)^{-4}(162,300) = \$10,867 \text{ to the nearest dollar.}$$

19. We have

$$B = 1000(1.04) + 200\left[1 + \frac{3}{4}(.04)\right] - 300\left[1 + \frac{1}{4}(.04)\right]$$
$$= \$943.$$

20. First, we apply formula (7.11)

$$B = A + C + I$$
$$10,636 = 10,000 + 1800 - K + 900 + I$$

so that $I = K - 2064$. Next, we apply formula (7.15)

$$i = .06 = \frac{I}{A + \sum_t C_t(1-t)} = \frac{K - 2064}{10,000 + 1800\left(\frac{5}{6}\right) - K\left(\frac{1}{2}\right) + 900\left(\frac{1}{3}\right)} = \frac{K - 2064}{11,800 - \frac{1}{2}K}$$

and solving for K

$$.06\left(11,800 - \frac{1}{2}K\right) = K - 2064$$
$$1.03K = 2772 \quad \text{giving} \quad K = \$2691 \quad \text{to the nearest dollar.}$$

21. We have

$$2,000,000 = .08(25,000,000) + .04(X - 2,200,000 - 750,000)$$
$$= 1,882,000 + .04X$$
$$\text{and} \quad X = 2,950,000.$$

Now

$$B = 25,000,000 + 2,950,000 + 2,000,000 - 2,200,000 - 750,000 = 27,000,000.$$

Finally, we apply formula (7.16) to obtain

$$i = \frac{2I}{A + B - I} = \frac{(2)(2,000,000)}{25,000,000 + 27,000,000 - 2,000,000} = .08, \text{ or } 8\%.$$

22. Under compound interest theory

$$(1 + {}_t i_0)(1 + {}_{1-t} i_t) = 1 + i$$

without approximation.

(*a*) $${}_t i_0 = \frac{1+i}{1+(1-t)i} - 1 = \frac{ti}{1+(1-t)i}.$$

(*b*) $${}_{1-t} i_t = \frac{1+i}{1+ti} - 1 = \frac{(1-t)i}{1+ti}.$$

23. We combine formula (7.11)

$$B = A + C + I$$

and formula (7.15) with one term in the denominator to obtain

$$i \approx \frac{I}{A + \sum_t C_t(1-t)} = \frac{I}{A + C(1-k)}$$

$$= \frac{I}{A + (B - A - I)(1-k)} = \frac{I}{kA + (1-k)B - (1-k)I}.$$

24. (*a*) Yes. The rate changes because the new dates change the denominator in the calculation of i^{DW}.

(*b*) No. The rate does not change because the calculation of i^{TW} depends on the various fund balances, but not the dates of those balances.

25. (*a*) The equation of value is

$$1000(1+i)^2 + 1000(1+i) = 2200$$

$$\text{or } (1+i)^2 + (1+i) - 2.2.$$

Solving the quadratic $1+i = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-2.2)}}{(2)(1)} = 1.06524$ rejecting the negative root. Thus, $i^{DW} = i = .0652$, or 6.52%.

(*b*) Over the two-year time period formulas (7.18) and (7.19) give

$$1 + i^{TW} = \left(\frac{1200}{1000}\right)\left(\frac{2200}{2200}\right) = 1.2.$$

The equivalent annual effective rate is

$$i = (1 + i^{TW})^{1/2} - 1 = (1.2)^{.5} - 1 = .0954, \text{ or } 9.54\%.$$

26. Dollar-weighted calculation:

$$2000(1+i) + 1000\left(1 + \frac{1}{2}i\right) = 3200$$

$$i^{DW} = i = \frac{200}{2500} = .08.$$

Time-weighted calculation:

$$i^{TW} = i^{DW} + .02 = .08 + .02 = .10$$

$$\text{and } 1 + i = 1.1 = \frac{X}{2000} \cdot \frac{3200}{X + 1000} = 1.6\frac{X}{X + 1000}.$$

Solving for *X* we obtain $X = \$2200$.

27. (*a*) The equation of value is

$$2000(1+i)+1000(1+i)^{1/2}=3213.60$$
$$2+(1+i)+(1+i)^{1/2}-3.2136=0$$

which is a quadratic in $(1+i)^{1/2}$. Solving the quadratic

$$(1+i)^{1/2}=\frac{-1\pm\sqrt{1^2-(4)(2)(-3.2136)}}{(2)(2)}=1.042014$$

rejecting the negative root. Finally, $i^{DW}=i=(1.042014)^2-1=.0857$, or 8.57%.

(*b*) $1+i=\left(\frac{2120}{2000}\right)\left(\frac{3213.60}{3120}\right)=1.0918$

so $i^{TW}=.0918$, or 9.81%.

28. The 6-month time-weighted return is

$$i^{TW}=\left(\frac{40}{50}\right)\left(\frac{80}{60}\right)\left(\frac{157.50}{160}\right)-1=.05.$$

The equivalent annual rate is

$$(1.05)^2-1=.1025.$$

The 1-year time-weighted return is

$$i^{TW}=\left(\frac{40}{50}\right)\left(\frac{80}{60}\right)\left(\frac{175}{160}\right)\left(\frac{X}{250}\right)-1=.1025.$$

and solving, we obtain $X=236.25$.

29. Time-weighted return:

$$i^{TW}=0 \quad \text{means} \quad \frac{12}{10}\cdot\frac{X}{12+X}=1 \quad \text{so} \quad X=60.$$

Dollar-weighted return:

$$I=X-X-10=-10$$

so that

$$i^{DW}=Y=\frac{-10}{10+(.5)(60)}=-25\%.$$

30. (*a*) Dollar-weighted:

$$A\left(1+i^{DW}\right)=C \quad \text{and} \quad i^{DW}=\frac{C}{A}-1=\frac{C-A}{A}.$$

Time-weighted:

$$1+i^{TW}=\left(\frac{B}{A}\right)\left(\frac{C}{B}\right) \quad \text{and} \quad i^{DW}=\frac{C}{A}-1=\frac{C-A}{A}.$$

(*b*) Dollar-weighted:

The interest earned is $I=C-A-D$

and the "exposure" is $A+\frac{1}{2}D,$ so $i^{DW}=\dfrac{C-A-D}{A+\frac{1}{2}D}.$

Time-weighted:

$$i^{TW}=\left(\frac{B}{A}\right)\left(\frac{C}{B+D}\right)-1.$$

(*c*) Dollar-weighted:

same as part (*b*), so $i^{DW}=\dfrac{C-A-D}{A+\frac{1}{2}D}.$

Time-weighted:

$$i^{TW}=\left(\frac{B-D}{A}\right)\left(\frac{C}{B}\right)-1$$

(*d*) Dollar-weighted calculations do not involve interim fund balances during the period of investment. All that matters are cash flows in or out of the fund and the dates they occur.

(*e*) Assume $i_b^{TW}\le i_c^{TW}$, then

$$\left(\frac{B}{A}\right)\left(\frac{C}{B+D}\right)\le\left(\frac{B-D}{A}\right)\left(\frac{C}{B}\right) \quad \text{or} \quad \frac{B}{B+D}\le\frac{B-D}{B}$$

which implies that $B^2\le B^2-D^2,$ a contradiction. Therefore we must have $i_b^{TW}>i_c^{TW}$.

31. We have

$$B_2=10,000(1.0825)(1.0825)=11,718.06$$
$$B_6=10,000(1.0825)(1.0825)(1.0840)(1.0850)(1.0850)(1.0835)=16,202.18$$

so the amount of interest earned is

$$B_6-B_2=16,202.18-11,718.06=\$4484.12.$$

32. Deposit in $z+3$ $(1.090)(1.090)(1.091)(1.091)(1.092) = 1.54428$
Deposit in $z+4$ $(1.090)(1.091)(1.092)(1.093) = 1.41936$
Deposit in $z+5$ $(1.0925)(1.0935)(1.095) = 1.30814$
Deposit in $z+6$ $(1.095)(1.095) = 1.19903$
Deposit in $z+7$ $1.100 = \underline{1.10000}$
$6.5708.$

33. $P = 1000(1.095)(1.095)(1.096) = 1314.13$
$Q = 1000(1.0835)(1.086)(1.0885) = 1280.82$
$R = 1000(1.095)(1.10)(1.10) = 1324.95$
Thus, $R > P > Q$.

34. Let $i = .01j$. Interest earned on:
Deposit in z $100(1.1)(1.1)(1+i)(.08) = 9.68(1+i)$
Deposit in $z+1$ $100(1.12)(1.05)(.10) = 11.76$
Deposit in $z+2$ $100(1.08)(i-.02) = 108(i-.02)$.
Thus, total interest is
$$9.68 + 9.68i + 11.76 + 108i - 2.16 = 28.40$$
$$117.68i = 9.12 \quad \text{and} \quad i = .0775.$$
The answer is $j = 100i = 7.75\%$.

35. The accumulated value is
$$1000\left(1+i_1^5\right)\left(1+i_2^5\right)\left(1+i_3^5\right) = 1000(1.085)^{1.05}(1.090)^{1.05}(1.095)^{1.05}$$
$$= 1000(1.31185).$$
The equivalent level effective rate is
$$i = (1.31185)^{1/3} - 1 = .0947, \quad \text{or} \quad 9.47\%.$$

36. (a) $\delta_{s,t} = \dfrac{\frac{\partial}{\partial t} a(s,t)}{a(s,t)} = \dfrac{\partial}{\partial t} \ln a(s,t).$

(b) $a(s,s) = 1$ and $a(s,t) = e^{\int_0^t \delta_{s,r} dr}$.

(c) Using an average portfolio rate
$$a(s)a(s,t) = a(t) \quad \text{and} \quad a(s,t) = \frac{a(t)}{a(s)}.$$

(*d*) $a(0,t) = (1+i)^t$.

(*e*) $a(t,t) = 1,$ since no interest has yet been earned.

37. The margin is $1000m$ and the interest on it is $(.08)(1000m) = 80m$. The net profit is $200 + 80m - 60 = 140 + 80m$ on a deposit of $1000m$. Thus, the yield rate is

$$\frac{140 + 80m}{1000m} = \frac{7 + 4m}{50m}.$$

38. The margin is $(.08)(50) = 40$

Interest on margin $= 4$

Dividend on stock $= 2$

Profit on short sale $= 50 - X$

Thus, $.2 = \dfrac{(50 - X) + 4 - 2}{40}$ and $X = 44$.

39. The margin is $(.40)(25{,}000) = 10{,}000$

Interest on margin $= (.08)(10{,}000) = 800$

Profit on short sale $= 25{,}000 - X$

Thus, $.25 = \dfrac{(25{,}000 - X) + 800}{10{,}000}$ and $X = \$23{,}300$.

40. A's transaction:

The margin is $(.50)(1000) = 500$
Interest on margin $= (.60)(500) = 30$
Dividend on the stock $= X$
Profit on short sale $= 1000 - P$
Thus, $.21 = \dfrac{(1000 - P) + 30 - X}{500}$.

B's transaction:

$$.21 = \frac{(1000 - P + 25) + 30 - 2X}{500}.$$

Solving the two equations in two unknowns gives

$$X = \$25 \quad \text{and} \quad P = \$900.$$

41. Earlier receipt of dividends. Partial release of margin.

42. The yield rate in Exercise 2 is between 9% and 10% and thus less than the interest preference rate of 12%. Thus, the investment should be rejected.

43. The yield rate of the financing arrangement can be determined from the equation of value

$$5000 = 2400 + 1500v + 1500v^2$$
$$\text{or } 1.5v + 1.5v - 2.6 = 0.$$

Solving the quadratic, we have

$$v = \frac{-1.5 \pm \sqrt{(1.5)^2 (-4)(1.5)(-2.6)}}{(2)(1.5)} = .90831$$

rejecting the negative root. Thus, $i = .10095$. Since the buyer would be financing at a rate higher than the interest preference rate of 10%, the buyer should pay cash.

44. (*a*) In Example 7.4 we have

$$P(i) = -100 + 200v - 101v^2 = 0$$
$$\text{or } P(i) = 100(1+i)^2 - 200(1+i) + 101$$
$$= 1 + 100i^2 = 0$$

The graph has a minimum at (0,1) and is an upward quadratic in either direction.

(*b*) There are no real roots, since the graph does not cross the *x*-axis.

45. Option (*i*):

$$800(1+i) = 900 \qquad i = \frac{900}{800} - 1 = .125.$$

Option (*ii*):

$$1000(1+i) = 1125 \qquad i = \frac{1125}{1000} - 1 = .125.$$

Thus they are equivalent, but both should be rejected. They both exceed the borrower's interest preference rate of 10%.

46. We have

$$P(i) = -100 + 230(1+i)^{-1} - 132(1+i)^{-2}$$

and

$$P'(i) = -230(1+i)^{-2} + 264(1+i)^{-3} = 0$$

so that

$$(1+i)^{-1} = \frac{230}{264} \qquad i = \frac{264}{230} - 1 = .1478, \text{ or } 14.78\%.$$

47. The following is an Excel spreadsheet for this Exercise.

Year	Contributions	Returns	PV Factors	PV Contributions	PV Returns
0	10,000	0	1.0000000	10,000.00	0.00
1	5,000	0	0.9090909	4,545.45	0.00
2	1,000	0	0.8264463	826.45	0.00
3	1,000	0	0.7513148	751.31	0.00
4	1,000	0	0.6830135	683.01	0.00
5	1,000	0	0.6209213	620.92	0.00
6	1,000	8,000	0.5644739	564.47	4,515.79
7	1,000	9,000	0.5131581	513.16	4,618.42
8	1,000	10,000	0.4665074	466.51	4,665.07
9	1,000	11,000	0.4240976	424.10	4,665.07
10		12,000	0.3855433	0.00	4,626.52
	23,000	50,000		19,395.39	23,090.88
				PI =	1.191

48. We have

$$100 + 132(1.08)^{-2} = 230(1+i)^{-1}$$
$$213.16872 = 230(1+i)^{-1}$$

so that

$$1 + i = \frac{230}{213.16872} = 1.0790.$$

Thus, the $\text{MIRR} = 7.90\%$, which is less than the required return rate of 8%. The project should be rejected.

49. The investor is in lender status during the first year, so use $r=.15$. Then $B_1=100(1.15)-230=-115$. The investor is now in borrower status during the second year, so use *f*. Then $B_2=0=-115(1+f)+132$ and $f=\frac{132}{115}-1=.1478,$ or 14.78%.

50. We compute successive balances as follows:

$$B_0=1000$$
$$B_1=1000(1.15)+2000=3150$$
$$B_2=3150(1.15)-4000=-377.50$$
$$B_3=-377.50(1.1)+3000=2584.75$$
$$B_4=2584.75(1.15)-4000=-1027.54$$
$$B_5=-1027.54(1.1)+5000=\$3870 \text{ to the nearest dollar.}$$

51. The price of the bond is

$$1000(1.03)^{-20}+40a_{\overline{20}|.03}=1148.77.$$

Thus, the loan and interest paid is

$$1148.77(1.05)^{10}=1871.23.$$

The accumulated bond payments are

$$1000+40s_{\overline{20}|.02}1971.89.$$

Thus, the net gain is $1971.89-1871.23=\$100.66.$

52. A withdrawal of $W=1000$ would exactly exhaust the fund at $i=.03$. We now proceed recursively:

$$F_0=1000a_{\overline{10}|.03}$$
$$F_1=F_0(1.04)=1000\ddot{a}_{\overline{10}|.03}(1.04)$$
$$W_1=\frac{1000a_{\overline{10}|.03}(1.04)}{\ddot{a}_{\overline{10}|.03}}=\frac{1000(1.04)}{1.03}$$
$$F_1-W_1=1000(1.04)\left[a_{\overline{10}|.03}-v_{.03}\right]=\frac{1000(1.04)a_{\overline{9}|.03}}{1.03}$$
$$F_2=\frac{1000(1.04)^2a_{\overline{9}|.03}}{1.03}$$
$$W_2=\frac{1000(1.04)^2a_{\overline{9}|.03}}{1.03\ddot{a}_{\overline{9}|.03}}=\frac{1000(1.04)^2}{(1.03)^2}$$

Continuing this recursive process 8 more times and reflecting the interest rate change at time $t = 4,$ we arrive at

$$W_{10} = \frac{1000(1.04)^4(1.05)^6}{(1.03)^{10}} = \$1167 \text{ to the nearest dollar.}$$

53. We are given:

$$A = 273,000 \quad B = 372,000 \quad I = 18,000$$

so that

$$C = B - A - I = 81,000.$$

Using the simple interest approximation $273,000(1.06) + 81,000(1 + .06t) = 372,000$ which can be solved to give $t = 1/3$. Thus, the average date for contributions and withdrawals is September 1, i.e. the date with four months left in the year.

54. The accumulation factor for a deposit made at time *t* evaluated at time *n*, where $0 \le t \le n,$ is

$$e^{\int_t^n \delta_r dr} = e^{\int_t^n \frac{dr}{1+r}} = e^{\ln(1+n) - \ln(1+t)}$$
$$= \frac{1+n}{1+t}.$$

Then, the accumulated value of all deposits becomes

$$1 \cdot \left(\frac{1+n}{1+0}\right) \int_0^n (1+t)\left(\frac{1+n}{1+t}\right) dt = (1+n) + n(1+n) = (n+1)^2.$$

Chapter 6

1. (*a*) $P = 1000(1.10)^{-10} = \$385.54$.

   (*b*) $P = 1000(1.09)^{-10} = \$422.41$.

   (*c*) The price increase percentage is $\dfrac{422.41 - 385.54}{385.54} = .0956, \quad \text{or} \quad 9.56\%.$

2. The price is the present value of the accumulated value, so we have

$$P = 1000\left(1 + \frac{.08}{2}\right)^{20}(1.1)^{-10} = \$844.77.$$

3. (*a*) The day counting method is actual/360. In 26 weeks there are $26 \times 7 = 182$ days. Using the simple discount method, we have

$$9600 = 10,000\left(1 - \frac{182}{360}d\right) \text{ and } d = .0791, \text{ or } 7.91\%.$$

   (*b*) An equation of value with compound interest is

$$9600 = 10,000(1+i)^{-1/2} \text{ and } i = .0851, \text{ or } 8.51\%.$$

4. We have $F = 100$, $C = 105$, $r = .05$, $g = 5/105$, $i = .04$, $G = 5/.04 = 125$,

   $K = 105(1.04)^{-20} = 47.921$, and $n = 20$.

   Basic: $P = 5a_{\overline{20}|} + 105v^{20} = 5(13.59031) + 105(.45639) = \$115.87$.

   Premium/discount: $P = 105 + (5 - 4.2)a_{\overline{20}|} = \$115.87$.

   Base amount: $P = 125 + (105 - 125)(1.04)^{-20} = \$115.87$.

   Makeham: $P = 47.921 + \dfrac{5}{.04(105)}(105 - 47.921) = \$115.87$.

5. We apply the premium/discount formula to the first bond to obtain

$$1136.78 = 1000 + 1000(.025 - .02)a_{\overline{n}|}$$

which can be solved to obtain $a_{\overline{n}|} = 27.356$. Now apply the premium/discount formula to the second bond to obtain

$$P = 1000 + 1000(.0125 - .02)(27.356) = \$794.83.$$

6. Since the present value of the redemption value is given, we will use Makeham's formula. First, we find

$$g = \frac{Fr}{C} = \frac{45}{1125} = .04.$$

Now

$$P = K + \frac{g}{i}(C - K) = 225 + \frac{.04}{.05}(1125 - 225) = \$945.$$

7. Since $K = Cv^n$, we have $450 = 1000v^n$ and $v^n = .45$. Now we will apply the base amount formula

$$P = G + (C - G)v^n = G(1 - v^n) + Cv^n$$

and substituting values

$$1110 = G(1 - .45) + 450 \quad \text{and} \quad G = \$1200.$$

8. The price of the 10-year bond is

$$P = 1000(1.035)^{-20} + 50a_{\overline{20}|.035} = 1213.19\,.$$

The price of the 8-year bond is

$$P = F\,(1.035)^{-16} + .03Fa_{\overline{16}|.035} = 1213.19$$

and solving

$$F = \frac{1213.19}{.576706 + (.03)(12.09412)} = \$1291 \text{ to the nearest dollar.}$$

9. Since n is unknown, we should use an approach in which n only appears once. We will use the base amount formula. First, we have

$$G = \frac{Fr}{i} = \frac{1000(.06)}{.05} = 1200$$

and

$$P = 1200 + (1000 - 1200)v^n = 1200 - 200v^n.$$

If we double the term of the bond we have

$$P + 50 = 1200 + (1000 - 1200)v^{2n} = 1250 - 200v^n.$$

Thus we have a quadratic which reduces to

$$200v^{2n} - 200v^n + 50 = 0$$

or

$$4v^{2n} - 4v^n + 1 = 0$$

and factoring

$$(2v^n - 1)^2 = 0.$$

Thus, $v^n = .5$ and $P = 1200 - 200(.5) = \$1100$.

10. (*a*) The nominal yield is the annualized coupon rate of 8.40%.

(*b*) Here we want the annualized modified coupon rate, so

$$2g = 2\left(\frac{Fr}{C}\right) = 2\left(\frac{42}{1050}\right) = 8.00\%.$$

(*c*) Current yield is the ratio of annualized coupon to price or $\frac{84}{919.15} = 9.14\%$.

(*d*) Yield to maturity is given as 10.00%.

11. Using the premium/discount formula, we have

$$P_1 = 1 + p = 1 + (1.5i - i)a_{\overline{n}|} = 1 + .5ia_{\overline{n}|}$$

and

$$P_2 = 1 + (.75i - i)a_{\overline{n}|} = 1 - .25ia_{\overline{n}|} = 1 - .5p.$$

12. Let *X* be the coupon amount and we have $X = 5 + .75X$ so $X = 20$.

13. We have $n = 20$ and are given that $P_{19} = C(i - g)v^2 = 8$. We know that the principal adjustment column is a geometric progression. Therefore, we have

$$\sum_{t=1}^{8} P_t = 8(v^{11} + v^{12} + \cdots + v^{18}) \text{ at } 4.5\%$$

$$= 8v^{10}a_{\overline{8}|} = 8(1.045)^{-10}(6.59589) = \$33.98.$$

14. Since $i > g$, the bond is bought at a discount. Therefore, the total interest exceeds total coupons by the amount of the discount. We have

$$\Sigma I_t = n \cdot Cg + d = (10)(50) + 1000(.06 - .05)\, a_{\overline{10}|.06}$$
$$= 500 + 10(7.36009) = \$573.60.$$

15. We have semiannual yield rate j

(*i*) $X = (40 - 1000j)\, a_{\overline{20}|}$

(*ii*) $Y = -(45 - 1000j)\, a_{\overline{20}|}$

(*iii*) $2X = -(50 - 1000j)\, a_{\overline{20}|}$.

By inspection, we have $2(X + Y) = X + 2X$, so that $2Y = X$ and $Y = \dfrac{X}{2}$.

16. (*a*) The total premium is $1037.17 - 1000 = 37.17$ amortized over four periods, with each amortization equal to $37.17/4 = 9.2925$. Thus, we have

$$B_0 = 1037.17$$
$$B_1 = 1037.17 - 9.2925 = 1027.88$$
$$B_2 = 1027.8775 - 9.2925 = 1018.59$$
$$B_3 = 1018.585 - 9.2925 = 1009.29$$
$$B_4 = 1009.2925 - 9.2925 = 1000.00$$

(*b*) The total discount is $1000 - 964.54 = 35.46$ amortized over four periods, with each amortization equal to $35.46/4 = 8.865$. Thus, we have

$$B_0 = 964.54$$
$$B_1 = 965.54 + 8.865 = 973.41$$
$$B_2 = 973.405 + 8.865 = 982.27$$
$$B_3 = 982.27 + 8.865 = 991.14$$
$$B_4 = 991.135 + 8.865 = 1000.00$$

(*c*) For premium bonds the straight line values are less than true book values. For discount bonds the opposite is the case.

17. (*a*) Since $k < 1$, then $1 + ki > (1+i)^k$, so

Theoretical = Semi-Theoretical < Practical.

(b) Since $\dfrac{(1+i)^k - 1}{i} < k,$ then for the accrued coupon, we have

Theoretical < Semi-Theoretical = Practical.

Finally, $B^m = B^f - AC$ and combining results

Semi-Theoretical < Theoretical
Semi-Theoretical < Practical

but Practical $\lesseqgtr$ Theoretical is indeterminate.

18. Theoretical method:

$$B^f_{1/3} = 964.54(1.05)^{1/3} = 980.35$$

$$AC = 40\left[\frac{(1.05)^{1/3} - 1}{.05}\right] = 13.12$$

$$B^m_{1/3} = 980.35 - 13.12 = 967.23$$

Practical method:

$$B^f_{1/3} = 964.54\left[1 + \left(\frac{1}{3}\right)(.05)\right] = 980.62$$

$$AC = \frac{1}{3}(40) = 13.33$$

$$B^m_{1/3} = 980.62 - 13.33 = 967.29$$

Semi-Theoretical:

$$B^f_{1/3} = 964.54(1.05)^{1/3} = 980.35$$

$$AC = \frac{1}{3}(40) = 13.33$$

$$B^m_{1/3} = 980.35 - 13.33 = 967.02$$

19. From Appendix A

April 15	is	Day 105
June 28	is	Day 179
October 15	is	Day 288

The price on April 15, Z is

$$P = 1000 + (30 - 35)a_{\overline{31}|.035} = 906.32.$$

The price on June 25, Z is

$$906.32\left[1 + \frac{179 - 105}{288 - 105}(.035)\right] = \$919.15.$$

20. (*a*) Using a financial calculator

$$\text{N} = 12 \times 2 = 24$$

$$\text{PMT} = 100\left(\frac{.10}{2}\right) = 5$$

$$\text{FV} = 100$$

$$\text{PV} = -110$$

$$\text{and CPT I} = 4.322.$$

Answer $= 2(4.322) = 8.64\%.$

(*b*) Applying formula (6.24), we have

$$i \approx \frac{g - \frac{k}{n}}{1 + \frac{n+1}{2n}k} \quad \text{where} \quad k = \frac{P-C}{C} = \frac{110-100}{100} = .1$$

$$= \frac{.05 - .1/24}{1 + \frac{25}{48}(.1)} = .04356.$$

Answer = $2(.04356) = .0871, \text{ or } 8.71\%$.

21. Bond 1: $P = 500 + (45 - 500i)a_{\overline{40|}}$.

Bond 2: $P = 1000 + (30 - 1000i)a_{\overline{40|}}$.

We are given that

$$(45 - 500i)a_{\overline{40|}} = 2(1000i - 30)a_{\overline{40|}}$$

so

$$45 - 500i = 2000i - 60$$

and

$$i = \frac{105}{2500} = .042.$$

The answer is $2i = .084$, or 8.4%.

22. Using the premium/discount formula

$$92 = 100\left[1 - .01a_{\overline{15|}\,i}\right]$$

so that

$$a_{\overline{15|}\,i} = 8.$$

Using a financial calculator and the technique in Section 3.7 we have

$$i = 9.13\%.$$

23. Using the basic formula, we have

$$P = 1000v^n + 42a_{\overline{n}|}$$

$$(i)\ \ P + 100 = 1000v^n + 52.50a_{\overline{n}|}$$

$$(ii)\ \ 42a_{\overline{n}|} = 1000v^n.$$

Subtracting the first two above

$$10.50a_{\overline{n}|} = 100 \quad \text{or} \quad a_{\overline{n}|} = 9.52381.$$

From (*ii*)

$$42a_{\overline{n}|} = 42(9.52381) = 400 = 1000v^n$$

$$= 1000\left(1 - ia_{\overline{n}|}\right) = 1000 - 9523.81i$$

so that $i = \dfrac{1000-400}{9523.81} = .063,$ or 6.3%.

24. (*a*) Premium bond, assume early:

$$P = 1000 + (40-30)a_{\overline{20}|.03} = \$1148.77\,.$$

(*b*) Discount bond, assume late:

$$P = 1000 + (40-50)a_{\overline{30}|.05} = \$846.28\,.$$

(*c*) Use a financial calculator:

$\text{N} = 20 \quad \text{PMT} = 40 \quad \text{FV} = 1000 \quad \text{PV} = -846.28 \quad \text{and} \quad \text{CPT I} = 5.261.$

Answer $= 2(5.261) = 10.52\%.$

(*d*) Premium bond, assume late:

$$P = 1000 + (40-30)a_{\overline{30}|.03} = \$1196.00\,.$$

(*e*) Discount bond, assume early:

$$P = 1000 + (40-50)a_{\overline{20}|.05} = \$875.38\,.$$

25. Note that this bond has a quarterly coupon rate and yield rate. The price assuming no early call is

$$P = 1000(1.015)^{-40} + 20a_{\overline{40}|.015} = 1149.58.$$

The redemption value at the end of five years to produce the same yield rate would have to be

$$1149.58 = C(1.015)^{-20} + 20a_{\overline{20}|.015}$$

$$\text{and } C = 1149.58(1.015)^{20} - 20s_{\overline{20}|.015}$$

$$= \$1086 \text{ to the nearest dollar.}$$

26. In Example 6.8 we had a premium bond and used the earliest possible redemption date in each interval. In this Exercise we have a discount bond and must use the latest possible redemption date in each interval:

At year 6: $P = 1050 + (20 - 26.25)a_{\overline{12}|.025} = 985.89$

At year 9: $P = 1025 + (20 - 25.625)a_{\overline{18}|.025} = 944.26$

At year 10: $P = 1000 + (20 - 25)a_{\overline{20}|.025} = 922.05$

Assume no early call, so the price is \$922.05. If the bond is called early, the yield rate will be higher than 5%.

27. Using Makeham's formula $g = \frac{1000(.045)}{1100} = \frac{.045}{1.1}$.

Now, $P = K + \frac{g}{i}(C - K)$ and we have

$$918 = 1100v^n + \frac{.045}{(1.1)(.05)}(1100 - 1100v^n)$$

$$= 200v^n + 900$$

$$v^n = \frac{18}{200} = .09 \quad \text{and} \quad n = \frac{-\ln(.09)}{\ln(1.05)} = 49.35.$$

The number of years to the nearest integer $= \frac{49.35}{2} = 25.$

28. The two calculated prices define the endpoints of the range of possible prices. Thus, to guarantee the desired yield rate the investor should pay no more than \$897.
The bond is then called at the end of 20 years at 1050. Using a financial calculator, we have

$\text{N} = 20 \quad \text{PMT} = 80 \quad \text{FV} = 1050 \quad \text{PV} = -897 \quad \text{and} \quad \text{CPT I} = 9.24\%.$

29. Use Makeham's formula

$$P = \sum_{t=1}^{10} 1000 v_{.04}^t + \frac{.06}{.04}\left[10,000 - \sum_{t=1}^{10} 1000 v_{.04}^t\right]$$

$$= 1000 a_{\overline{10}|.04} + \frac{3}{2}\left[10,000 - 1000 a_{\overline{10}|.04}\right]$$

$$= 15,000 - 500 a_{\overline{10}|.04} = \$10,945 \text{ to the nearest dollar.}$$

30. Use Makeham's formula

$P = K + \dfrac{.06}{.10}\left[10,000 - K\right]$ where $K = 500\left(a_{\overline{25}|} - a_{\overline{5}|}\right) = 2643.13$ and

$P = 6000 + .4(2643.13) = \$7057$ to the nearest dollar.

31. Use Makeham's formula

$$P = K + \frac{g}{i}(C - K) = \frac{g}{i}C + \left(1 - \frac{g}{i}\right)K$$

where

$$\frac{g}{i} = \frac{g}{1.25g} = .8 \qquad C = 100,000$$

and

$$K = 10,000\left(v^{10} + v^{16} + v^{22} + 2v^{28} + 2v^{34} + 3v^{40}\right).$$

Applying formula (4.3) in combination with the technique presented in Section 3.4 we obtain

$$K = 10,000\left[\frac{3a_{\overline{46}|} - a_{\overline{40}|} - a_{\overline{28}|} - a_{\overline{10}|}}{a_{\overline{6}|}}\right].$$

Thus, the answer is

$$80,000 + 2000\left[\frac{3a_{\overline{46}|} - a_{\overline{40}|} - a_{\overline{28}|} - a_{\overline{10}|}}{a_{\overline{6}|}}\right].$$

32. From the first principles we have

$$P = 105v^n + 8a_{\overline{n}|}^{(2)} = 105v^n + \frac{8\left(1 - v^n\right)}{i^{(2)}}$$

$$= \left(105 - \frac{8}{i^{(2)}}\right)v^n + \frac{8}{i^{(2)}}.$$

Thus, $A = 105i^{(2)} - 8$ and $B = 8$.

33. From first principles we have

$$P = 1000(1.06)^{-20} + 40a_{\overline{20}|.06} + 10a_{\overline{10}|.06}$$
$$= 311.8047 + 458.7968 + 73.6009 = \$844.20.$$

34. From first principles we have

$$P = 1050(1.0825)^{-20} + 75\left[\frac{1}{1.0825} + \frac{1.03}{(1.0825)^2} + \cdots + \frac{(1.03)^{19}}{(1.0825)^{20}}\right]$$
$$= 1050(1.0825)^{-20} + \frac{75}{1.0825}\left[\frac{1-\left(\frac{1.03}{1.0825}\right)^{20}}{1-\frac{1.03}{1.0825}}\right]$$
$$= \$1115 \text{ to the nearest dollar.}$$

35. Applying formula (6.28)

$$P = \frac{D}{i-g} = \frac{10}{.12-.05} = 142.857.$$

The level dividend that would be equivalent is denoted by D and we have

$$142.857 = Da_{\overline{\infty}|} = \frac{D}{.12} \text{ or } D = \$17.14.$$

36. Modifying formula (6.28) we have

$$P = v^5 \frac{D}{i-g} = (1.15)^{-5}\frac{(.5)(6)(1.08)^6}{.15-.08} = \$33.81.$$

37. If current earnings are E, then the earnings in 6 years will be $1.6E$. The stock price currently is $10E$ and in 6 years will be $15(1.6E) = 24E$. Thus, the yield rate can be determined from

$$10E(1+i)^6 = 24E$$

which reduces to

$$i = (2.4)^{1/6} - 1 = .157, \text{ or } 15.7\%.$$

38. The price at time $t = 0$ would be

$$2.50a_{\overline{\infty}|.02} = \frac{2.50}{.02} = 125.$$

The bond is called at the end of 10 years. Using a financial calculator we have

$$\text{N} = 40 \quad \text{PMT} = 2.50 \quad \text{FV} = 100 \quad \text{PV} = -125 \quad \text{and} \quad \text{CPT I} = .016424.$$

The answer is $4(.016424) = .0657$, or 6.57%.

39. (*a*) MV for the bonds $= 1000(900) = 900,000.$
MV for the stocks $= 10,000(115) = 1,150,000.$
Total MV = \$2,050,000.

(*b*) BV for the bonds = 1,000,000, since the yield rate equals the coupon rate.
BV for the stocks = 1,000,000, their cost.
Total BV = \$2,000,000.

(*c*) $\text{BV}_B + \text{MV}_S = 1,000,000 + 1,150,000 = \$2,150,000.$

(*d*) $\text{PV}_B = 40,000 a_{\overline{15}|.05} + 1,000,000 v_{.05}^{15} = 896,208.$

$$\text{PV}_S = 60,000 a_{\overline{\infty}|.05} = \frac{60,000}{.05} = 1,2000,000.$$

Total $\text{PV} = \$2,096,200$ to the nearest \$100.

40. From first principles we have

$$P = 9\bar{a}_{\overline{12}|} + 100v^{12} = 9\left(\frac{1-v^{12}}{\delta}\right) + 100v^{12}$$
$$= 9\left(\frac{1-e^{-12\delta}}{\delta}\right) + 100e^{-12\delta}$$
$$= \frac{1}{\delta}\left[(100\delta - 9)e^{-12\delta} + 9\right].$$

41. From the premium/discount formula we have

$$p = (g-i)a_{\overline{n}|} \quad \text{and} \quad q = \left(\frac{1}{2}g - i\right)a_{\overline{n}|}.$$

We then have

$$(2g-i)a_{\overline{n}|} = Ap + Bq = A(g-i)a_{\overline{n}|} + B\left(\frac{1}{2}g - i\right)a_{\overline{n}|}.$$

Equating coefficients gives

$$A+\frac{1}{2}B=2$$
$$A+B=1.$$

Solving these simultaneous equations gives $A=3$ and $B=-2$.

42. Using Makeham's formula for the first bond

$$P=K+\frac{g}{i}(C-K)=Cv_{.04}^{5}+\frac{.06}{.04}\left(C-Cv_{.04}^{5}\right)$$
$$=C\left[1.5-.5(1.04)^{-5}\right]=1.089036C.$$

Using Makeham's formula again for the second bond

$$1.089036C=C\left[1.25-.25(1.04)^{-n}\right].$$

Thus $(1.04)^{-n}=.643854$ and $n=\dfrac{-\ln(.643854)}{\ln 1.04}=11.23$ or 11 years to the nearest year.

43. Since $r=g>i,$ the bond is a premium bond. Therefore $B_{19}>C=1000.$ We then have $P_{20}=B_{19}-1000$ and $I_{20}=iB_{19}$ so that

$$Fr=1000r=P_{20}+I_{20}$$
$$=B_{19}-1000+iB_{19}=B_{19}(1+i)-1000.$$

Thus, we have

$$B_{19}=1000\frac{1+r}{1+i}=1000\frac{1.03+i}{1+i}.$$

We are also given

$$i\cdot B_{19}=.7\left(B_{19}-1000\right) \text{ so that } B_{19}=\frac{700}{.7-i}.$$

Therefore

$$1000\frac{1.03+i}{1+i}=\frac{700}{.7-i}$$

which can be solved to obtain $i=.02$. Finally, we can obtain the price of the bond as

$$P=1000+1000(.05-.02)a_{\overline{20}|.02}$$
$$=1000+30(16.35149)=\$1490.54.$$

44. If suspended coupon interest accrues at the yield rate, then there is no difference between the restructured bond and the original bond. We have

$$\begin{aligned} P &= 33.75 a_{\overline{20}|.037} + 1000(1.037)^{-20} \\ &= 33.75(13.95861) + 1000(.483532) \\ &= \$955 \text{ to the nearest dollar.} \end{aligned}$$

45. The redemption value C is the same for both bonds.
Bond X: Use the base amount formula. We have $Fr = Gi,$ so that

$$G = F\frac{r}{i} = 1000(1.03125) = \$1031.25$$

and $K = Cv^n = 381.50.$

Bond Y: We have

$$Cv^{n/2} = 647.80.$$

Taking the ratio $\dfrac{Cv^n}{Cv^{n/2}} = v^{n/2} = \dfrac{381.50}{647.80} = .5889163$

so $v^n = (.5889163)^2 = .3468224$

and $C = \dfrac{381.50}{.3468224} = 1100.$

Finally,

$$\begin{aligned} P_X &= G + (C - G)v^n \\ &= 1031.25 + (1100 - 1031.25)(.3468224) \\ &= \$1055 \text{ to the nearest dollar.} \end{aligned}$$

46. (*a*) Prospectively, $B_t = C + (Fr - Ci)a_{\overline{n-t}|}$ so that

$$\begin{aligned} i\sum_{t=0}^{n-1} B_t &= \sum_{t=0}^{n-1}\left[Ci + (Fr - Ci)(1 - v^{n-t})\right] \\ &= \sum_{t=0}^{n-1}\left[Civ^{n-t} + Fr(1 - v^{n-t})\right] \\ &= Cia_{\overline{n}|} + nFr - Fra_{\overline{n}|}. \end{aligned}$$

However $P = C + (Fr - Ci)a_{\overline{n}|}$ so that

$$P + i\sum_{t=0}^{n-1} B_t = C + n \cdot Fr.$$

(*b*) In a bond amortization schedule

- $i\sum_{t=0}^{n-1} B_t$ is the sum of the interest earned column.
- $P - C = C(g-i)a_{\overline{n}|}$ is the sum of the principal adjustment column.
- $n \cdot Fr$ is the sum of coupon column.

The sum of the first two is equal to the third.

47. (*a*) From Exercise 50 in Chapter 4

$$\frac{d}{di}a_{\overline{n}|} = -v(Ia)_{\overline{n}|}.$$

Then $\frac{dP}{di} = \frac{d}{di}\left[Cga_{\overline{n}|} + Cv^n\right] = Cg\left[-v\left(Ia_{\overline{n}|}\right)\right] - Cv^{n+1} = -Cv\left[g(Ia)_{\overline{n}|} + nv^n\right].$

(*b*) $\frac{dP}{dg} = \frac{d}{dg}\left[Cga_{\overline{n}|} + Cv^n\right] = Ca_{\overline{n}|}.$

Chapter 5

1. The quarterly interest rate is $j = .06/4 = .015$. The end of the second year is the end of the eighth quarter. There are a total of 20 installment payments, so

$$R = \frac{1000}{a_{\overline{20}|.015}}$$

and using the prospective method

$$B_8^p = Ra_{\overline{12}|.015} = \frac{1000a_{\overline{12}|.015}}{a_{\overline{20}|.015}} = \frac{1000(10.90751)}{17.16864} = \$635.32.$$

2. Use the retrospective method to bypass having to determine the final irregular payment. We then have

$$\begin{aligned} B_5^r &= 10,000(1.12)^5 - 2000s_{\overline{5}|.12} \\ &= (10,000)(1.76234) - (2000)(6.35283) \\ &= \$4918 \quad \text{to the nearest dollar.} \end{aligned}$$

3. The quarterly interest rate is $j = .10/2 = .025$. Applying the retrospective method we have $B_4^r = L(1+j)^4 - Rs_{\overline{4}|j}$ and solving for L

$$\begin{aligned} L = \frac{B_4^r + Rs_{\overline{4}|j}}{(1+j)^4} &= \frac{12,000 + 1500(4.15252)}{1.10381} \\ &= \$16,514 \quad \text{to the nearest dollar.} \end{aligned}$$

4. The installment payment is $R = \dfrac{20,000}{a_{\overline{12}|}}$ and the fourth loan balance prospectively is

$$B_4^p = \frac{20,000}{a_{\overline{12}|}} a_{\overline{8}|} = \frac{20,000(1-v^8)}{1-v^{12}} = \frac{20,000(1-2^{-2})}{1-2^{-3}} = \$17,143 \text{ to the nearest dollar.}$$

5. We have

$$R = \frac{20,000}{a_{\overline{20}|}} \quad \text{and} \quad B_5^P = Ra_{\overline{15}|}.$$

The revised loan balance at time $t = 7$ is $B_7' = B_5^p(1+i)^2$, since no payments are made for two years. The revised installment payment thus becomes

$$R' = \frac{B_7}{a_{\overline{13}|}} = 20,000 \frac{a_{\overline{15}|}(1+i)^2}{a_{\overline{20}|}a_{\overline{13}|}}.$$

6. The installment payment is $R=\frac{L}{a_{\overline{n}|}}=\frac{1}{a_{\overline{25}|}}$. Using the original payment schedule

$$B_5^p=Ra_{\overline{20}|}=\frac{a_{\overline{20}|}}{a_{\overline{25}|}}$$

and using the revised payment schedule $B_5^p=Ra_{\overline{15}|}+Ka_{\overline{5}|}$. Equating the two and solving for K we have

$$K=\frac{1}{a_{\overline{5}|}}\left(\frac{a_{\overline{20}|}}{a_{\overline{25}|}}-\frac{a_{\overline{15}|}}{a_{\overline{25}|}}\right)=\frac{a_{\overline{20}|}-a_{\overline{15}|}}{a_{\overline{25}|}a_{\overline{5}|}}.$$

7. We have

$$R=\frac{150,000}{a_{\overline{15}|.065}}=\frac{150,000}{9.4026689}=15,952.92$$

and

$$B_5^p=Ra_{\overline{10}|.065}(15,952.92)(7.1888302)=114,682.83.$$

The revised fifth loan balance becomes

$$B_5'=114,682.83+80,000=194,682.83$$

and the revised term of the loan is $n'=15-5+7=17$. Thus, the revised installment payment is

$$R'=\frac{194,682.83}{a_{\overline{17}|.075}}=\frac{194,682.83}{9.4339598}=\$20,636 \quad \text{to the nearest dollar.}$$

8. The quarterly interest rate is $j=.12/4=.03.$ Directly from formula (5.5), we have

$$P_6=1000v_{.03}^{20-6+1}=1000(1.03)^{-15}=\$641.86.$$

9. The installment payment is

$$R=\frac{10,000}{a_{\overline{20}|}}$$

and applying formula (5.4) we have

$$I_{11}=\frac{10,000}{a_{\overline{20}|}}\left(1-v^{20-11+1}\right)=\frac{10,000(.1)\left(1-v^{10}\right)}{1-v^{20}}$$

$$=\frac{1000\left(1-v^{10}\right)}{\left(1-v^{10}\right)\left(1+v^{10}\right)}=\frac{1000}{1+v^{10}}.$$

10. The quarterly interest rate is $j = .10/4 = .025$. The total number of payments is $n = 5 \times 4 = 20$. Using the fact that the principal repaid column in Table 5.1 is a geometric progression, we have the answer

$$100\left[(1+i)^{13} + (1+i)^{14} + (1+i)^{15} + (1+i)^{16} + (1+i)^{17}\right]$$
$$= 100\left(s_{\overline{18}|} - s_{\overline{13}|}\right) = 100(22.38635 - 15.14044) = \$724.59.$$

11. (*a*) We have $B_4^p = a_{\overline{6}|\,i} + v_i^6 a_{\overline{10}|\,j}$ so that $I_5 = i \cdot B_4 = i\left(a_{\overline{6}|\,i} + v_i^6 a_{\overline{10}|\,j}\right)$.

(*b*) After 10 years, the loan becomes a standard loan at one interest rate. Thus applying formula (5.5)

$$P_{15} = v_j^{20-15+1} = v_j^6.$$

12. After the seventh payment we have $B_7^p = a_{\overline{13}|}$. If the principal $P_8 = v^{20-8+1} = v^{13}$ in the next line of the amortization schedule is also paid at time $t = 7$; then, in essence, the next line in the amortization schedule drops out and we save $1 - v^{13}$ in interest over the life of the loan. The loan is exactly prepaid one year early at time $t = 19$.

13. (*a*) The amount of principal repaid in the first 5 payments is

$$B_0 - B_5 = L - B_5^p = L - \left(\frac{L}{a_{\overline{10}|}}\right) a_{\overline{5}|} = L\left(1 - \frac{a_{\overline{5}|}}{a_{\overline{10}|}}\right) = L\left(1 - \frac{1 - v^5}{1 - v^{10}}\right) = L\left(1 - \frac{1 - 2/3}{1 - 4/9}\right) = .4L.$$

(*b*) The answer is

$$B_5(1+i)^5 = (L - .4L)\frac{3}{2} = .9L.$$

14. We are given

$$I_8 = R\left(1 - v^{28}\right) = 135 \quad \text{and} \quad I_{22} = R\left(1 - v^{14}\right) = 108.$$

Taking the ratio

$$\frac{I_8}{I_{22}} = \frac{1 - v^8}{1 - v^{14}} = 1 + v^{14} = \frac{135}{108} = 1.25$$

so that $v^{14} = .25$.

Now, we can solve for R

$$R = \frac{108}{1-v^{14}} = \frac{108}{.75} = 144.$$

Finally,

$$I_{29} = R\left(1-v^7\right) = 144\left[1-(.25)^{.5}\right] = \$72.$$

15. We have

$$L = 1000 a_{\overline{10}|}.$$

and using the column total from Table 5.1

$$L = 1000\left(10 - a_{\overline{10}|}\right)$$

Equating the two we have $a_{\overline{10}|} = 5$ and solving for the unknown rate of interest using a financial calculator, we have $i = 15.0984\%$. Thus, the answer is

$$I_1 = iL = (.150984)(5000) = \$754.95.$$

16. We know that $X = Ra_{\overline{n}|.125}$.

From (i) we have

$$R(1-v) = 153.86 \quad \text{so that} \quad R = 1384.74.$$

From (ii) we have

$$\begin{aligned} X &= 6009.12 + (1384.74 - 153.86) = 7240.00 \\ &= 1384.74 a_{\overline{n}|.125}. \end{aligned}$$

Therefore, $a_{\overline{n}|.125} = 7240/1384.74 = 5.228$ and solving for the unknown n using a financial calculator we obtain $n = 9$.

From (iii) we have

$$Y = Rv^{9-1+1} = 1384.74(1.125)^{-9} = \$479.73.$$

17. (*a*) $.10(10,000) = \$1000.$

(*b*) $1500 - 1000 = \$500.$

(*c*) $1000 - .08(5000) = \$600.$

(*d*) $1500 - 600 = \$900$.

(*e*) $5000(1.08) + 500 = \$5900$.

As a check, note that $5900 - 5000 = 900,$ the answer to part (*d*).

18. (*a*) $B_5 = 1000(1.08)^5 - 120s_{\overline{5}|}$ by the retrospective definition of the outstanding loan balance.

(*b*) $B_5 = 1000\left[1 + is_{\overline{5}|}\right] - 120s_{\overline{5}|} = 1000 + 80s_{\overline{5}|} - 120s_{\overline{5}|} = 1000 - 40s_{\overline{5}|}$. The total annual payment 120 is subdivided into 80 for interest on the loan and 40 for the sinking fund deposit. After five years the sinking fund balance is $40s_{\overline{5}|}$. Thus, B_5 is the original loan less the amount accumulated in the sinking fund.

19. We have $X\ddot{s}_{\overline{10}|.07} = 10,000$ so that $X = \dfrac{10,000}{\ddot{s}_{\overline{10}|.07}} = \dfrac{10,000}{14.7836} = \$676.43.$

20. Amortization payment: $\dfrac{.5L}{a_{\overline{10}|.05}}$.

Sinking fund payment: $(.05)(.5L) + \dfrac{.5L}{s_{\overline{10}|.04}}$.

The sum of the two is equal to 1000. Solving for L we obtain

$$L = \frac{1000}{\frac{.5}{a_{\overline{10}|.05}} + .025 + \frac{.5}{s_{\overline{10}|.04}}} = \frac{1000}{.06475 + .025 + .04165} = \$7610 \quad \text{to the nearest dollar.}$$

21. The interest on the loan and sinking fund deposits are as follows:

Years	Interest	SFD
1 - 10	$.06(12,000) = 720$	$1000 - 720 = 280$
11 - 20	$.05(12,000) = 600$	$1000 - 600 = 400$

The sinking fund balance at time $t = 20$ is

$$\begin{aligned} 280s_{\overline{10}|.04}(1.04)^{10} + 400s_{\overline{10}|.04} &= (280)(12.00611)(1.48024) + (400)(12.00611) \\ &= 9778.57. \end{aligned}$$

Thus the shortage in the sinking fund at time $t = 20$ is $12,000 - 9778.57 = \$2221$ to the nearest dollar.

22. (*a*) The total payment is

$$3000(.04) + \frac{\frac{1}{3}(3000)}{s_{\overline{20}|.025}} + \frac{\frac{2}{3}(3000)}{s_{\overline{20}|.035}} = 120 + 39.15 + 70.72 = \$229.87\,.$$

(*b*) We have

$$\frac{1}{3}Ds_{\overline{20}|.025}+\frac{2}{3}Ds_{\overline{20}|.035}=3000$$

so that

$$D=\frac{3000}{8.5149+18.8531}=109.62\,.$$

and the total payment is

$$120+109.62=\$229.62$$

(*c*) In part (*a*) more than $1/3$ of the sinking fund deposit goes into the lower-earning sinking fund, whereas in part (*b*) exactly $1/3$ does. Therefore, the payment in part (*a*) must be slightly higher than in part (*b*) to make up for the lesser interest earned.

23. We have

$$36,000=400,000i+\frac{400,000}{s_{\overline{31}|.03}}=400,000i+8000$$

and

$$i=\frac{36,000-8000}{400,000}=.07, \quad \text{or} \quad 7\%.$$

24. We have $P=\frac{1000}{a_{\overline{10}|.10}}=\frac{1000}{6.14457}=162.745.$

The interest on the loan is $.10(1000)=100,$ so that $D=162.745-100=62.745$. The accumulated value in the sinking fund at time $t=10$ is

$$62.745s_{\overline{10}|.14}=(62.745)(19.3373)=1213.32\,.$$

Thus, the excess in the sinking fund at time $t=10$ is $1213.32-1000=\$213.32$.

25. Total interest = total payments minus the loan amount, so

$$(500)(4)(10)-(500)(4)a^{(4)}_{\overline{10}|.08}=20,000-2000(6.90815)$$
$$=\$6184 \quad \text{to the nearest dollar.}$$

26. Semiannual interest payment = $(10,000)(.12/2) = 600$.

    Annual sinking fund deposit = $\dfrac{10,000}{s_{\overline{5|}.08}} = \dfrac{10,000}{5.8666} = 1704.56$.

    Total payments = $(600)(2)(5) + (1704.56)(5) = \$14,523$ to the nearest dollar.

27. The quarterly interest rate is $j = .10/4 = .025$. We are given $R = 3000$ and $I_3 = 2000$, so therefore $P_3 = 1000$. There are $3 \times 4 = 12$ interest conversion periods between P_3 and P_6. Therefore $P_6 = P_3(1+j)^{12} = 1000(1.025)^{12} = \$1344.89$.

28. The quarterly interest rate on the loan is $j_1 = .10/4 = .025$. The semiannual interest rate on the sinking fund is $j_2 = .07/2 = .035$. The equivalent annual effective rate is $i_2 = (1.035)^2 - 1 = .07123$. Thus, the required annual sinking fund deposit is

$$D = \frac{5000(1.025)^{40}}{s_{\overline{10|}.07123}} = \frac{5000(2.865064)}{13.896978} = \$966.08.$$

29. There are 17 payments in total. We have $B_3 = 300a_{\overline{14|}} + 50(Ia)_{\overline{14|}}$

    and

$$\begin{aligned} P_4 &= 350 - iB_3 \\ &= 350 - 300(1 - v^{14}) - 50\left(\ddot{a}_{\overline{14|}} - 14v^{14}\right) \\ &= 50 + 1000v^{14} - 50\ddot{a}_{\overline{14|}} \\ &= 50 + 1000(.577475) - 50(10.9856) \\ &= \$78.20. \end{aligned}$$

30. The semiannual loan interest rate is $j_1 = .06/2 = .03$. Thus, the semiannual interest rate payments are $30, 27, 24, \ldots, 3$. The semiannual yield rate is $j_2 = .10/2 = .05$. The price is the present value of all the payments at this yield rate, i.e.

$$\begin{aligned} &100a_{\overline{10|}.05} + 3(Da)_{\overline{10|}.05} \\ &= 100a_{\overline{10|}.05} + (3)(20)\left(10 - a_{\overline{10|}.05}\right) \\ &= 600 + 40a_{\overline{10|}.05} = 600 + 40(7.7217) = \$908.87. \end{aligned}$$

31. (*a*) Retrospectively, we have

$$B_3 = 2000(1.1)^3 - 400\left[(1.1)^2 + (1.04)(1.1) + (1.04)^2\right] = \$1287.76\ .$$

(*b*) Similarly to part (*a*)

$$B_2 = 2000(1.1)^2 - 400(1.1 + 1.04) = 1564.00$$

so that

$$P_3 = B_2 - B_3 = 1564.00 - 1287.76 = \$276.24.$$

32. A general formula connecting successive book values is given by

$$B_t = B_{t-1}(1+i) - (1.625t)\left(i \cdot B_{t-1}\right).$$

Letting $t = 16$, we have

$$B_{16} = B_{15}(1+i) - 26iB_{15} = 0$$

since the fund is exactly exhausted. Therefore $1 + i - 26i = 0$ and $i = \dfrac{1}{25}$ or 4%.

33. Under option (*i*)

$$P = \frac{2000}{a_{\overline{10}|.0807}} = \frac{2000}{6.68895} = 299$$

and total payments $= 299(10) = 2990.$

Under option (*ii*) the total interest paid needs to be $2990 - 2000 = 990$. Thus, we have

$$990 = i(2000 + 1800 + 1600 + \cdots + 200) = 11{,}000i$$

so that

$$i = \frac{990}{11{,}000} = .09, \text{ or } 9\%.$$

34. There are a total of 60 monthly payments. Prospectively B_{40} must be the present value of the payments at times 41 through 60. The monthly interest rate is $j = .09/12 = .0075$. Payments decrease 2% each payment, so we have

$$B_{40} = 1000\left[(.98)^{40}(1.0075)^{-1} + (.98)^{41}(1.0075)^{-2} + \cdots + (.98)^{59}(1.0075)^{-20}\right]$$

$$= 1000(.98)^{40}(1.0075)^{-1}\frac{1-(.98/1.0075)^{20}}{1-(.98/1.0075)}$$

$= \$6889$ to the nearest dollar upon summing the geometric progression.

35. We have $B_0 = 1000$. For the first 10 years only interest is paid, so we have $B_{10} = 1000$. For the next 10 years each payment is equal to 150% of the interest due. Since the lender charges 10% interest, 5% of the principal outstanding will be used to reduce the principal each year. Thus, we have $B_{20} = 1000(1-.05)^{10} = 598.74$. The final 10 years follows a normal loan amortization, so

$$X = \frac{598.74}{a_{\overline{10|}.10}} = \frac{598.74}{6.14457} = \$97.44.$$

36. We have $B_t = \bar{a}_{\overline{25-t|}}$ and the interest paid at time t is $\delta B_t\, dt$ by applying formulas (5.12) and (5.14). Thus, the interest paid for the interval $5 \le t \le 10$ is

$$\int_5^{10} \delta \bar{a}_{\overline{25-t|}} dt = \int_5^{10} \left(1 - v^{25-t}\right) dt = \left[t - \frac{v^{25-t}}{\delta}\right]_5^{10} = (10-5) - \frac{1}{\delta}\left(v^{15} - v^{20}\right).$$

Evaluating this expression for $i = .05,$ we obtain

$$5 - \frac{1}{\ln(1.05)}\left[(1.05)^{-15} - (1.05)^{-20}\right] = 2.8659.$$

37. (*a*) $$(1+i)^t - \frac{\bar{s}_{\overline{t|}}}{\bar{a}_{\overline{n|}}} = (1+i)^t - \frac{(1+i)^t - 1}{1-v^n} = \frac{(1+i)^t - v^{n-t} - (1+i)^t + 1}{1-v^n} = \frac{1-v^{n-t}}{1-v^n} = \frac{\bar{a}_{\overline{n-t|}}}{\bar{a}_{\overline{n|}}}.$$

(*b*) The LHS is the retrospective loan balance and the RHS is the prospective loan balance for a loan of 1 with continuous payment $1/\bar{a}_{\overline{n|}}$.

38. The loan is given by

$$L = \int_0^n t v^t dt = (\bar{I}\,\bar{a})_{\overline{n|}}.$$

(*a*) $$B_k^p = \int_k^n t v^{t-k} dt = \int_0^{n-k} (k+s) v^s ds = k\bar{a}_{\overline{n-k|}} + (\bar{I}\,\bar{a})_{\overline{n-k|}}.$$

(*b*) $$B_k^r = L(1+i)^k - \int_0^k t(1+i)^{k-t}\, dt = (\bar{I}\,\bar{a})_{\overline{n|}}(1+i)^k - (\bar{I}\,\bar{s})_{\overline{k|}}.$$

39. (*a*) Since $B_0 = 1$ and $B_{10} = 0$ and loan balances are linear, we have

$$B_t = 1 - t/10 \quad \text{for} \quad 0 \le t \le 10.$$

The principal repaid over the first 5 years is $B_0 - B_5 = 1 - .5 = .5.$

(*b*) The interest paid over the first 5 years is

$$\int_0^5 \delta B_t\, dt = \int_0^5 \delta\left(1 - \frac{t}{10}\right) = \delta\left[t - \frac{t^2}{20}\right]_0^5 = .10\left(5 - \frac{25}{20}\right) = .375.$$

40. (*a*) The undiscounted balance is given by

$$B_t = \int_t^\infty P(s)\,ds = \alpha e^{-Bt}.$$

The rate of payment is the rate of change in B_t, i.e.

$$P(t) = -\frac{d}{dt}B_t = -\frac{d}{dt}\alpha e^{-Bt} = \alpha\beta e^{-Bt}.$$

(*b*) This is $B_0 = \alpha e^{-Bt}\big|_{t=0} = \alpha.$

(*c*) The present value of the payment at time $t=0$ is

$$\int_0^\infty v^t P(t)\,dt = \int_0^\infty e^{-\delta t}\alpha\beta e^{-Bt}dt = \alpha\beta\int_0^\infty e^{-(\beta+\delta)t}dt = \frac{\alpha\beta}{\beta+\delta}.$$

(*d*) Similarly to part (*c*)

$$\int_t^\infty v^{s-t}P(s)\,ds = \alpha\beta\int_t^\infty e^{-\delta(s-t)}e^{-Bs}ds = \alpha\beta\int_t^\infty e^{\delta t}e^{-(\beta+\delta)s}ds = \frac{\alpha\beta}{\beta+\delta}e^{-\beta t}.$$

41. The quarterly interest rate is $.16/4 = .04$ on the first 500 of loan balance and $.14/4 = .035$ on the excess. Thus, the interest paid at time t is $I_t = (.04)(500) + .035(B_{t-1} - 500) = 2.50 + .035B_{t-1}$ as long as $B_t \geq 500$. We can generate values recursively as follows:

$$I_1 = 2.50 + .035(2000) = 72.50$$
$$P_1 = P - I_1 = P - 72.50$$
$$B_1 = B_0 - P_1 = 2072.50 - P$$

$$I_2 = 2.50 + .035(2072.50 - P) = 75.04 - .035P$$
$$P_2 = P - I_2 = 1.035P - 75.04$$
$$B_2 = B_1 - P_2 = 2147.54 - 2.035P$$

$$I_3 = 2.50 + .035(2147.54 - 2.035P) = 77.664 - .071225P$$
$$P_3 = P - I_3 = 1.071225P - 77.664$$
$$B_3 = B_2 - P_3 = 2225.204 - 3.106225P$$

$$I_4 = 2.50 + .035(2225.204 - 3.106225P) = 80.382 - .108718P$$
$$P_4 = P - I_4 = 1.108718P - 80.382$$
$$B_4 = B_3 - P_4 = 2305.586 - 4.214943P = 1000.00$$

Solving for $P = \dfrac{2305.586 - 1000.00}{4.214943} = \$310$ to the nearest dollar.

42. The quarterly interest rate is $.12/4=.03$ on the first 500 of loan balance and $.08/4=.02$ on the excess.

(*a*) For each payment of 100, interest on the first 500 of the loan balance is $.03(500)=15$. Thus, the remaining loan balance of $1000-500=500$ is amortized with payments of $100-15=85$ at 2% interest. Retrospectively,

$$B_3 = 500(1.02)^3 - 85s_{\overline{3}|.02} = 270.46$$
$$I_4 = .02(270.46) = 5.41$$
$$P_4 = 85 - I_4' = 85 - 5.41 = \$79.59.$$

(*b*) Prior to the crossover point, the successive principal repayments form a geometric progression with common ratio 1.02 (see Table 5.5 for an illustration).

43. We have $B_0 = 3000$.

Proceeding as in Exercise 41, we find that

$$B_5 = 3191.289 - 5.101005P.$$

Proceeding further, we find that

$$B_9 = 3364.06 - 9.436502P.$$

However, prospectively we also know that

$$B_9 = Pa_{\overline{3}|.015}.$$

Equating the two expressions for B_9, we have

$$P = \frac{3364.06}{9.436502 + a_{\overline{3}|.015}} = \$272.42.$$

44. (*a*) We have

$$a_{\overline{n}|} + i\sum_{t=0}^{n-1} a_{\overline{n-t}|} = a_{\overline{n}|} + \sum_{t=0}^{n-1}\left(1-v^{n-t}\right) = a_{\overline{n}|} + n - a_{\overline{n}|} = n.$$

(*b*) For a loan $L = a_{\overline{n}|}$, then $a_{\overline{n}|}$ is the sum of the principal repaid column. The summation of $ia_{\overline{n-t}|}$ is the sum of the interest paid column. The two together sum to the total installment payments which is *n*.

45. (*a*) Prospectively, we have

$$B_t = Ra_{\overline{n-t}|} = \frac{R}{i}\left(1-v^{n-t}\right) = \frac{R}{i}\left(1-v^n - v^{n-t} + v^n\right)$$
$$= \frac{R}{i}\left[\left(1-v^n\right) - v^n\left\{(1+i)^t - 1\right\}\right] = R\left(a_{\overline{n}|} - v^n s_{\overline{t}|}\right).$$

(*b*) The outstanding loan balance B_t is equal to the loan amount $Ra_{\overline{n}|}$ minus the sum of the principal repaid up to time *t*.

46. The initial fund is $B_0 = 10,000a_{\overline{10}|.035}$. After 5 years, fund balance is retrospectively

$$B_5' = 10,000a_{\overline{10}|.035}(1.05)^5 - 10,000s_{\overline{5}|.05}.$$

The outstanding balance on the original schedule is

$$B_5 = 10,000a_{\overline{5}|.035}.$$

Thus, the excess interest at time $t = 5$ is

$B_5' - B_5 = 10,000\left[(8.3166)(1.27628) - 5.5256 - 4.5151\right] = \$5736$ to the nearest dollar.

47. (*a*) The original deposit is

$$D_1 = \frac{10,000}{\ddot{s}_{\overline{10}|.05}} = \frac{10,000}{13.20679} = \$757.19.$$

(*b*) After 5 years the balance is

$$D_1\ddot{s}_{\overline{5}|.05} = (757.19)(5.80191) = 4393.14\,.$$

Then, the revised deposit is

$$D_2 = \frac{10,000 - 4393.14(1.04)^5}{\ddot{s}_{\overline{5}|.04}} = \$826.40.$$

48. We have $R_L = \dfrac{L}{a_{\overline{30}|.04}}$.

The payment for loan *M* is $L/30$ in principal plus a declining interest payment. The loan balances progress linearly as

$$L, \frac{29L}{30}, \frac{28L}{30}, \cdots, \frac{31-k}{30}L$$

in year *k*. We have $P_L = P_M$, so that

$$\frac{L}{a_{\overline{30}|.04}} = \frac{L}{30} + \frac{.04(31-k)L}{30}$$

$$\text{or} \quad \frac{L}{a_{\overline{30}|.04}} = \frac{2.24 - .04k}{30}$$

and solving, we obtain $k = 12.63$. Thus, P_L first exceeds P_M at time $t = 13$.

49. (*a*) We have on the original mortgage

$$R = \frac{80,000}{a_{\overline{20}|.08}} \quad \text{and} \quad B_9 = \left(\frac{80,000}{a_{\overline{20}|.08}}\right) a_{\overline{11}|.08}$$

and on the revised mortgage

$$B_9' = B_9 - 5000 = R' a_{\overline{9}|.09}.$$

Thus,

$$R' = \frac{\left(\dfrac{80,000}{a_{\overline{20}|.08}}\right) a_{\overline{11}|.08} - 5000}{a_{\overline{9}|.09}}.$$

(*b*) We have at issue

$$80,000 = \left(\frac{80,000}{a_{\overline{20}|.08}}\right) a_{\overline{9}|.09} + 5000 v_{.09}^9 + R' v_{.09}^9 a_{\overline{9}|.09}$$

so that

$$R' = \frac{80,000(1.09)^9 - \left(\dfrac{80,000}{a_{\overline{20}|.08}}\right) s_{\overline{9}|.09} - 5000}{a_{\overline{9}|.09}}.$$

50. The regular payment is

$$R = \frac{1000}{a_{\overline{10}|.05}} = \frac{1000}{7.72173} = 129.50.$$

The penalties are:

$$.02(300 - 129.50) = 3.41 \text{ at } t = 1$$

$$\text{and } .02(250 - 129.50) = 2.41 \text{ at } t = 2.$$

Thus, only 296.59 and 247.59 go toward principal and interest. Finally,

$B_3 = 1000(1.05)^3 - 296.59(1.05)^2 - 247.59(1.05) = \$571$ to the nearest dollar.

Chapter 4

1. The nominal rate of interest convertible once every two years is *j*, so that

$$1+j=\left(1+\frac{.07}{2}\right)^4 \quad \text{and} \quad j=(1.035)^4-1=.14752.$$

The accumulated value is taken 4 years after the last payment is made, so that

$$\begin{aligned} &2000 s_{\overline{8}|j}(1+j)^2 = 2000(13.60268)(1.31680) \\ &= \$35{,}824 \quad \text{to the nearest dollar.} \end{aligned}$$

2. The quarterly rate of interest j is obtained from

$$(1+j)^4 = 1.12 \quad \text{so that} \quad j = .02874.$$

The present value is given by

$$\begin{aligned} &600\ddot{a}_{\overline{40}|j} - 200\ddot{a}_{\overline{20}|j} \\ &= 600(24.27195) - 200(15.48522) \\ &= \$11{,}466 \quad \text{to the nearest dollar.} \end{aligned}$$

3. The equation of value at time $t=8$ is

$$100\left[(1+8i)+(1+6i)+(1+4i)+(1+2i)\right]=520$$

so that

$$4+20i=5.2, \text{ or } 20i=1.2, \quad \text{and} \quad i=.06, \text{ or } 6\%.$$

4. Let the quarterly rate of interest be *j*. We have

$$400a_{\overline{40}|j}=10{,}000 \quad \text{or} \quad a_{\overline{40}|j}=25.$$

Using the financial calculator to find an unknown *j*, set $\text{N}=40 \quad \text{PV}=25 \quad \text{PMT}=-1$ and CPT I to obtain $j=.02524,$ or 2.524%. Then

$$\left(1+\frac{i^{(12)}}{12}\right)^{12}=(1.02524)^4 \quad \text{and} \quad i^{(12)}=.100, \text{ or } 10.0\%.$$

5. Adapting formula (4.2) we have

$$\begin{aligned} &2000\frac{s_{\overline{32}|.035}}{s_{\overline{4}|.035}}(1.035)^8 \\ &=(2000)\left(\frac{57.33450}{4.21494}\right)(1.31681)=\$35{,}824 \text{ to the nearest dollar.} \end{aligned}$$

6. (*a*) We use the technique developed in Section 3.4 that puts in imaginary payments and then subtracts them out, together with adapting formula (4.1), to obtain

$$\frac{200}{s_{\overline{4}|}}\left(a_{\overline{176}|} - a_{\overline{32}|}\right).$$

Note that the number of payments is $\frac{176-32}{4} = 36,$ which checks.

(*b*) Similar to part (*a*), but adapting formula (4.3) rather than (4.1), we obtain

$$\frac{200}{a_{\overline{4}|}}\left(a_{\overline{180}|} - a_{\overline{36}|}\right).$$

Again we have the check that

$$\frac{180-36}{4} = 36.$$

7. The monthly rate of discount is $d_j = \frac{d^{(12)}}{12} = \frac{.09}{12} = .0075$ and the monthly discount factor is $v_j = 1 - d_j = .9925.$ From first principles, the present value is

$$300\left[1 + (.9925)^6 + (.9925)^{12} + \cdots + (.9925)^{114}\right] = 300\frac{1-(.9925)^{120}}{1-(.9925)^6}$$

upon summing the geometric progression.

8. Using first principles and summing an infinite geometric progression, we have

$$v^3 + v^6 + v^9 + \ldots = \frac{v^3}{1-v^3} = \frac{1}{(1+i)^3 - 1} = \frac{125}{91}$$

and

$$(1+i)^3 - 1 = \frac{91}{125} \quad \text{or} \quad (1+i)^3 = \frac{216}{125}$$

$$\text{and} \quad 1+i = \left(\frac{216}{125}\right)^{1/3} = \frac{6}{5} = 1.2 \quad \text{which gives} \quad i = .20, \quad \text{or} \quad 20\%.$$

9. Using first principles with formula (1.31), we have the present value

$$100\left[1 + e^{-.02} + e^{-.04} + \cdots + e^{-.38}\right]$$

and summing the geometric progression

$$100\frac{1-e^{-.4}}{1-e^{-.02}}.$$

10. This is an unusual situation in which each payment does not contain an integral number of interest conversion periods. However, we again use first principles measuring time in 3-month periods to obtain $1+v^{4/3}+v^{8/3}+\cdots+v^{140/3}$ and summing the geometric progression, we have

$$\frac{1-v^{48}}{1-v^{4/3}}.$$

11. Adapting formula (4.9) we have

$$2400\ddot{a}^{(4)}_{\overline{10|}.12}-800\ddot{a}^{(4)}_{\overline{5|}.12}.$$

Note that the proper coefficient is the "annual rent" of the annuity, not the amount of each installment. The nominal rate of discount $d^{(4)}$ is obtained from

$$\left(1-\frac{d^{(4)}}{4}\right)^{-4}=1+i=1.12 \quad \text{and} \quad d^{(4)}=4\left[1-(1.12)^{-1/4}\right]=.11174.$$

The answer is

$$2400\cdot\frac{1-(1.12)^{-10}}{.11174}-800\frac{1-(1.12)^{-5}}{.11174}=\$11,466 \text{ to the nearest dollar.}$$

12. (*a*) $$\frac{1}{m}\sum_{t=1}^{m}v^{t/m}\ddot{a}_{\overline{n|}}=\ddot{a}_{\overline{n|}}\sum_{t=1}^{m}\frac{1}{m}v^{1/m}=\ddot{a}_{\overline{n|}}a^{(m)}_{\overline{1|}}=\frac{1-v^n}{d}\cdot\frac{1-v}{i^{(m)}}=\frac{1-v^n}{i^{(m)}}=a^{(m)}_{\overline{n|}}.$$

(*b*) The first term in the summation is the present value of the payments at times $1/m, 1+1/m, \ldots, n-1+1/m$. The second term is the present value of the payments at times $2/m, 1+2/m, \ldots, n-1+2/m$. This continues until the last term is the present value of the payments at times $1,2,\ldots,n$. The sum of all these payments is $a^{(m)}_{\overline{n|}}$.

13. The equation of value is

$$1000\,{}_{n|}\ddot{a}^{(2)}_{\overline{\infty|}}=10,000 \quad \text{or} \quad {}_{n|}\ddot{a}^{(2)}_{\overline{\infty|}}=10,$$

where n is the deferred period. We then have

$${}_{n|}\ddot{a}^{(2)}_{\overline{\infty|}}=v^n\ddot{a}^{(2)}_{\overline{\infty|}}=\frac{v^n}{d^{(2)}}=10 \quad \text{or} \quad v^n=10d^{(2)}.$$

Now expressing the interest functions in terms of d, we see that

$$v=1-d \quad \text{and} \quad d^{(2)}=2\left[1-(1-d)^{1/2}\right].$$

We now have

$$(1-d)^n = 20\left[1-(1-d)^{.5}\right]$$
$$n\ln(1-d) = \ln 20\left[1-(1-d)^{.5}\right]$$
$$\text{and} \quad n = \frac{\ln 20\left[1-(1-d)^{.5}\right]}{\ln(1-d)}.$$

14. We have

$$3a_{\overline{n}|}^{(2)} = 2a_{\overline{2n}|}^{(2)} = 45s_{\overline{1}|}^{(2)}$$
$$\text{or} \quad 3\left(\frac{1-v^n}{i^{(2)}}\right) = 2\left(\frac{1-v^{2n}}{i^{(2)}}\right) = 45\frac{i}{i^{(2)}}.$$

Using the first two, we have the quadratic

$$3(1-v^n) = 2(1-v^{2n}) \quad \text{or} \quad 2v^{2n} - 3v^n + 1 = 0$$

which can be factored $(2v^n - 1)(v^n - 1) = 0$ or $v^n = 1/2$, rejecting the root $v = 1$. Now using the first and third, we have

$$3(1-v^n) = 45i \quad \text{or} \quad i = \frac{3\left(1-1/2\right)}{45} = \frac{1}{30}.$$

15. Using a similar approach to Exercise 10, we have

$$1 + v^{3/4} + v^{6/4} + \cdots + v^{141/4} = \frac{1-v^{36}}{1-v^{3/4}}.$$

16. Each of the five annuities can be expressed as $1-v^n$ divided by $i, i^{(m)}, \delta, d^{(m)}$, and d, respectively. Using the result obtained in Exercise 32 in Chapter 1 immediately establishes the result to be shown. All five annuities pay the same total amount. The closer the payments are to time $t = 0$, the larger the present value.

17. The equation of value is

$$2400\bar{a}_{\overline{n}|} = 40,000 \quad \text{or} \quad \bar{a}_{\overline{n}|} = 50/3.$$

Thus

$$\bar{a}_{\overline{n}|} = \frac{1-e^{-.04n}}{.04} = \frac{50}{3}$$

or

$$1-e^{-.04n}=\frac{2}{3} \qquad e^{-.04n}=\frac{1}{3}$$

and

$$.04n=\ln 3=1.0986, \quad \text{so that} \quad n=27.47.$$

18. We have

$$\bar{a}_{\overline{n}|}=\frac{1-v^n}{\delta}=4 \quad \text{or} \quad v^n=1-4\delta$$

and

$$\bar{s}_{\overline{n}|}=\frac{(1+i)^n-1}{\delta}=12 \quad \text{or} \quad (1+i)^n=1+12\delta.$$

Thus, $1+12\delta=\dfrac{1}{1-4\delta}$ leading to the quadratic $1+8\delta-48\delta^2=1,$ so that $\delta=\dfrac{8}{48}=\dfrac{1}{6}.$

19. Using formula (4.13) in combination with formula (1.27), we have

$$\bar{a}_{\overline{n}|}=\int_0^n v^t dt=\int_0^n e^{-\int_0^t \delta_r dr}=\int_0^n e^{-\int_0^t (1+r)^{-1}dr}dt$$

Now

$$e^{-\int_0^t (1+r)^{-1}dr}=e^{-\ln(1+t)}=(1+t)^{-1}.$$

Thus,

$$\bar{a}_{\overline{n}|}=\int_0^n (1+t)^{-1}dt=\ln(1+t)\Big]_0^n=\ln(n+1).$$

20. Find t such that $v^t=\bar{a}_{\overline{1}|}=\dfrac{1-v}{\delta}=\dfrac{iv}{\delta}.$ Thus, $t\ln v=\ln v+\ln\dfrac{i}{\delta}$ and $t=1-\dfrac{1}{\delta}\ln\dfrac{i}{\delta}.$

21. Algebraically, apply formulas (4.23) and (4.25) so that $(Ia)_{\overline{n}|}=\dfrac{\ddot{a}_{\overline{n}|}-nv^n}{i}$ and $(Da)_{\overline{n}|}=\dfrac{n-a_{\overline{n}|}}{i}.$ Thus,

$$(Ia)_{\overline{n}|}+(Da)_{\overline{n}|}=\frac{1}{i}\left(\ddot{a}_{\overline{n}|}-nv^n+n-a_{\overline{n}|}\right)$$

$$=\frac{1}{i}\left(a_{\overline{n}|}+1-v^n-nv^n+n-a_{\overline{n}|}\right)=(n+1)\left(\frac{1-v^n}{i}\right)=(n+1)a_{\overline{n}|}.$$

Diagrammatically,

Time:	0	1	2	3	…	$n-1$	n
$(Ia)_{\overline{n}\rvert}$:		1	2	3	…	$n-1$	n
$(Da)_{\overline{n}\rvert}$:		n	$n-1$	$n-2$	…	2	1
Total:		$n+1$	$n+1$	$n+1$	…	$n+1$	$n+1$

22. Applying formula (4.21) directly with $P=6$, $Q=1$, and $n=20$

$$Pa_{\overline{n}\rvert}+Q\frac{a_{\overline{n}\rvert}-nv^n}{i}=6a_{\overline{20}\rvert}+\frac{a_{\overline{20}\rvert}-20v^{20}}{i}.$$

23. The present value is

$$\begin{aligned}v^4(Da)_{\overline{10}\rvert}&=\frac{v^4}{i}\left(10-a_{\overline{10}\rvert}\right)=\frac{1}{i}\left(10v^4-a_{\overline{14}\rvert}+a_{\overline{4}\rvert}\right)\\&=\frac{1}{i}\left[10\left(1-ia_{\overline{4}\rvert}\right)-a_{\overline{14}\rvert}+a_{\overline{4}\rvert}\right]\\&=\frac{1}{i}\left[10-a_{\overline{14}\rvert}+a_{\overline{4}\rvert}(1-10i)\right].\end{aligned}$$

24. Method 1:

$$\begin{aligned}\text{PV}&=(Ia)_{\overline{n}\rvert}+v^n na_{\overline{\infty}\rvert}=\frac{1}{i}\left(\ddot{a}_{\overline{n}\rvert}-nv^n+nv^n\right)\\&=\frac{\ddot{a}_{\overline{n}\rvert}}{i}=\frac{(1+i)a_{\overline{n}\rvert}}{i}=\frac{a_{\overline{n}\rvert}}{d}.\end{aligned}$$

Method 2:

$$\begin{aligned}\text{PV}&=(Ia)_{\overline{\infty}\rvert}-v^n(Ia)_{\overline{\infty}\rvert}=(1-v^n)\left(\frac{1}{i}+\frac{1}{i^2}\right)\\&=\left(\frac{1-v^n}{i}\right)\left(1+\frac{1}{i}\right)=\frac{1-v^n}{id}=\frac{a_{\overline{n}\rvert}}{d}.\end{aligned}$$

25. We are given that $11v^6=13v^7$ from which we can determine the rate of interest. We have $11(1+i)=13,$ so that $i=2/11.$ Next, apply formula (4.27) to obtain

$$\frac{P}{i}+\frac{Q}{i^2}=\frac{1}{i}+\frac{2}{i^2}=\frac{11}{2}+2\left(\frac{11}{2}\right)^2=66.$$

26. We are given:

$$X=va_{\overline{\infty}\rvert}=\frac{v}{i}\quad\text{and}\quad 20X=v^2(Ia)_{\overline{\infty}\rvert}=\frac{v^2}{id}.$$

Therefore,

$$X = \frac{v}{i} = \frac{v^2}{20id} \quad \text{or} \quad 20d = v = 1 - d \quad \text{and} \quad d = 1/21.$$

27. The semiannual rate of interest $j = .16/2 = .08$ and the present value can be expressed as

$$300a_{\overline{10}|.08} + 50(Da)_{\overline{10}|.08} = 300a_{\overline{10}|.08} + 50\left(\frac{10 - a_{\overline{10}|}}{.08}\right)$$
$$= 300A + 50\left(\frac{10 - A}{.08}\right) = 6250 - 325A.$$

28. We can apply formula (4.30) to obtain

$$\text{PV} = 600\left[1 + \frac{1.05}{1.1025} + \left(\frac{1.05}{1.1025}\right)^2 + \cdots + \left(\frac{1.05}{1.1025}\right)^{19}\right]$$
$$= 600\left[\frac{1 - (1.05/1.1025)^{20}}{1 - (1.05/1.1025)}\right] = \$7851 \text{ to the nearest dollar.}$$

29. We can apply formula (4.31)

$$i' = \frac{i - k}{1 + k} = \frac{.1025 - .05}{1 + .05} = .05, \quad \text{or} \quad 5\%,$$

which is the answer.

Note that we could have applied formula (4.32) to obtain $\text{PV} = 600\ddot{a}_{\overline{20}|.05} = \$7851$ as an alternative approach to solve Exercise 28.

30. The accumulated value of the first 5 deposits at time $t = 10$ is

$$1000\ddot{s}_{\overline{5}|.08}(1.08)^5 = (1000)(6.33593)(1.46933) = 9309.57.$$

The accumulated value of the second 5 deposits at time $t = 10$ is

$$1000\left[(1.05)(1.08)^5 + (1.05)^2(1.08)^4 + \cdots + (1.05)^5(1.08)\right]$$
$$= 1000(1.05)(1.08)^5\left[\frac{1 - (1.05/1.08)^5}{1 - 1.05/1.08}\right] = 7297.16.$$

The total accumulated value is $9309.57 + 7297.16 = \$16,607$ to the nearest dollar.

31. We have the equation of value

$$4096 = 1000\left[\frac{1}{(1.25)^5} + \frac{1+.01k}{(1.25)^6} + \frac{(1+.01k)^2}{(1.25)^7} + \cdots\right]$$

or

$$4.096 = \frac{1/(1.25)^5}{1-(1+.01k)/1.25} = \frac{1}{(1.25)^4(.25-.01k)}$$

upon summing the infinite geometric progression. Finally, solving for k

$$10 = \frac{1}{.25-.01k} \quad \text{and} \quad k = 15\%.$$

32. The first contribution is $(40,000)(.04) = 1600$. These contributions increase by 3% each year thereafter. The accumulated value of all contributions 25 years later can be obtained similarly to the approach used above in Exercise 30. Alternatively, formula (4.34) can be adapted to an annuity-due which gives

$$1600\frac{(1.05)^{25}-(1.03)^{25}}{.05-.03}(1.05) = \$108,576 \text{ to the nearest dollar.}$$

33. Applying formula (4.30), the present value of the first 10 payments is

$$100\left[\frac{1-(1.05/1.07)^{10}}{.07-.05}\right](1.07) = 919.95.$$

The 11th payment is $100(1.05)^9(.95) = 147.38$. Then the present value of the second 10 payments is $147.38\left[\frac{1-(.95/1.07)^{10}}{.07-.05}\right](1.07)(1.07)^{-10} = 464.71$. The present value of all the payments is $919.95 + 464.71 = \$1385$ to the nearest dollar.

34. We have

$$\text{PV} = \frac{1}{m^2}\left[v^{\frac{1}{m}} + 2v^{\frac{2}{m}} + \cdots + nmv^{\frac{nm}{m}}\right]$$

$$\text{PV}(1+i)^{\frac{1}{m}} = \frac{1}{m^2}\left[1 + 2v^{\frac{1}{m}} + \cdots + nmv^{n-\frac{1}{m}}\right]$$

$$\text{PV}\left[(1+i)^{\frac{1}{m}} - 1\right] = \frac{1}{m^2}\left[1 + v^{\frac{1}{m}} + \cdots + v^{n-\frac{1}{m}} - nmv^{\frac{nm}{m}}\right]$$

$$= \frac{1}{m}\left[\ddot{a}_{\overline{n}|}^{(m)} - nv^n\right].$$

Therefore

$$\text{PV} = \frac{\ddot{a}^{(m)}_{\overline{n}|} - nv^n}{m\left[(1+i)^{\frac{1}{m}} - 1\right]} = \frac{\ddot{a}^{(m)}_{\overline{n}|} - nv^n}{i^{(m)}}.$$

35. (*a*) $\frac{1}{12}\left[(\overset{(12 \text{ terms})}{1+1+\cdots+1}) + (\overset{(12 \text{ terms})}{2+2+\cdots+2})\right] = \frac{1}{12}(12+24) = 3.$

(*b*) $\frac{1}{144}\left[(1+2+\cdots+12)+(13+14+\cdots+24)\right] = \frac{(24)(25)}{(2)(144)} = \frac{25}{12}.$

36. We have

$$\begin{aligned}\text{PV} &= \left[v^5 + v^6 + 2v^7 + 2v^8 + 3v^9 + 3v^{10} + \cdots\right] \\ &= \left(v^5 + v^7 + v^9 + \cdots\right)\ddot{a}_{\overline{\infty}|} = \frac{v^5}{1-v^2}\cdot\frac{1}{d} \\ &= \frac{v^5}{1-v^2}\cdot\frac{1}{iv} = \frac{v^4}{i-vd}.\end{aligned}$$

37. The payments are 1,6,11,16,…. This can be decomposed into a level perpetuity of 1 starting at time $t=4$ and on increasing perpetuity of $1,2,3,\ldots$ starting at time $t=8$. Let i_4 and d_4 be effective rates of interest and discount over a 4-year period. The present value of the annuity is

$$\frac{1}{i_4} + 5(1+i_4)^{-1}\left(\frac{1}{i_4\cdot d_4}\right) \quad \text{where} \quad i_4 = (1+i)^4 - 1.$$

We know that

$$(1+i)^4 = (.75)^{-1} = 4/3, \text{ or } i_4 = \frac{4}{3} - 1 = \frac{1}{3} \quad \text{and} \quad d_4 = \frac{1/3}{1+1/3} = \frac{1}{4}.$$

Thus, the present value becomes

$$3 + (5)\left(\frac{3}{4}\right)\left(\frac{1}{\frac{1}{3}\cdot\frac{1}{4}}\right) = 3 + 45 = 48.$$

38. Let j be the semiannual rate of interest. We know that $(1+j)^2 = 1.08,$ so that $j = .03923$. The present value of the annuity is

$$1 + \frac{1.03}{1.03923} + \left(\frac{1.03}{1.03923}\right)^2 + \cdots = \frac{1}{1 - 1.03/1.03923} = 112.59$$

upon summing the infinite geometric progression.

39. The ratio is

$$\frac{\int_5^{10} t\,dt}{\int_0^5 t\,dt} = \frac{\frac{1}{2}t^2\big]_5^{10}}{\frac{1}{2}t^2\big]_0^5} = \frac{75/2}{25/2} = 3.$$

40. Taking the limit of formula (4.42) as $n \to \infty$, we have

$$(\bar{I}\bar{a})_{\overline{\infty}|} = \frac{\bar{a}_{\overline{\infty}|}}{\delta} = \frac{1}{\delta^2} = \frac{1}{(.08)^2} = 156.25.$$

41. Applying formula (4.43) we have the present value equal to

$$\int_0^\infty f(t)v^t dt = \int_0^\infty \left(\frac{1+k}{1+i}\right)^t dt = \frac{\left(\frac{1+k}{1+i}\right)^t}{\ln\left(\frac{1+k}{1+i}\right)}\Bigg]_0^\infty$$

$$= -\frac{1}{\ln\left(\frac{1+k}{1+i}\right)} = \frac{1}{\ln(1+i) - \ln(1+k)} = \frac{1}{\delta_i - \delta_k}.$$

Note that the upper limit is zero since $i > k$.

42. (*a*) $(\bar{D}\bar{a})_{\overline{n}|} = \int_0^n (n-t)v^t dt.$

(*b*) $\int_0^n (n-t)v^t dt = n\bar{a}_{\overline{n}|} - (\bar{I}\bar{a})_{\overline{n}|}$

$$= \frac{n(1-v^n)}{\delta} - \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta} = \frac{n - \bar{a}_{\overline{n}|}}{\delta}.$$

The similarity to the discrete annuity formula (4.25) for $(Da)_{\overline{n}|}$ is apparent.

43. In this exercise we must adapt and apply formula (4.44). The present value is

$$\int_1^{14} (t^2-1) e^{-\int_0^t (1+r)^{-1} dr} dt.$$

The discounting function was seen to be equal to $(1+t)^{-1}$ in Exercise 19. Thus, the answer is

$$\int_1^{14} \frac{t^2-1}{t+1} dt = \int_1^{14} \frac{(t-1)(t+1)}{t+1} dt = \int_1^{14} (t-1)\,dt$$

$$= \left[\frac{1}{2}t^2 - t\right]_1^{14} = (98-14) - \left(\frac{1}{2} - 1\right) = 84.5.$$

44. For perpetuity #1 we have

$$1+v^{.5}+v+v^{1.5}+\cdots=\frac{1}{1-v^{.5}}=20$$

$$\text{so that} \quad 1-v^{.5}=.05 \quad \text{and} \quad v^{.5}=.95.$$

For perpetuity #2, we have

$$X\left[1+v^2+v^4+\cdots\right]=X\,\frac{1}{1-v^2}=20$$

$$\text{so that} \quad X=20\left(1-v^2\right)=20\left[1-(.95)^4\right]=3.71.$$

45. We have

$$\int_0^n \bar{a}_{\overline{t}|}dt=\frac{1}{\delta}\int_0^n\left(1-v^t\right)\Big]dt=\frac{n-\bar{a}_{\overline{n}|}}{\delta}=\frac{n-(n-4)}{.1}=40.$$

46. For each year of college the present value of the payments for the year evaluated at the beginning of the year is

$$1200a^{(12)}_{\overline{9/12}|}.$$

The total present value for the payments for all four years of college is

$$1200a^{(12)}_{\overline{9/12}|}\left(1+v+v^2+v^3\right)=1200\ddot{a}_{\overline{4}|}a^{(12)}_{\overline{9/12}|}.$$

47. For annuity #1, we have $\text{PV}_1=\dfrac{P}{i}$.

For annuity #2, we have $\text{PV}_2=q\left(\dfrac{1}{i}+\dfrac{1}{i^2}\right)$.

Denote the difference in present values by *D*.

$$D=\text{PV}_1-\text{PV}_2=\frac{p-q}{i}-\frac{q}{i^2}.$$

(*a*) If $D=0$, then

$$\frac{p-q}{i}-\frac{q}{i^2}=0 \quad \text{or} \quad p-q=\frac{q}{i} \quad \text{or} \quad i=\frac{q}{p-q}.$$

(*b*) We seek to maximize *D*.

$$\frac{dD}{di}=\frac{d}{di}\left[(p-q)i^{-1}-qi^{-2}\right]$$
$$=-(p-q)i^{-2}+2qi^{-3}=0.$$

Multiply through by i^3 to obtain

$$-(p-q)i + 2q = 0 \quad \text{or} \quad i = \frac{2q}{p-q}.$$

48. We must set soil (S) posts at times 0,9,18,27. We must set concrete posts (C) at times 0,15,30. Applying formula (4.3) twice we have

$$\text{PV}_S = 2\frac{a_{\overline{36}|}}{a_{\overline{9}|}} \quad \text{and} \quad \text{PV}_C = (2+X)\frac{a_{\overline{45}|}}{a_{\overline{15}|}}.$$

Equating the two present values, we have

$$2\frac{a_{\overline{36}|}}{a_{\overline{9}|}} = (2+X)\frac{a_{\overline{45}|}}{a_{\overline{15}|}} \quad \text{so that}$$

$$X = 2\left[\frac{a_{\overline{36}|}}{a_{\overline{9}|}} - \frac{a_{\overline{45}|}}{a_{\overline{15}|}}\right] \cdot \frac{a_{\overline{45}|}}{a_{\overline{15}|}} = 2\left(\frac{a_{\overline{36}|}a_{\overline{15}|}}{a_{\overline{9}|}a_{\overline{45}|}} - 1\right).$$

49. We know $\bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta} = a$, so that $v^n = 1 - a\delta$. Similarly, $\bar{a}_{\overline{2n}|} = \frac{1-v^{2n}}{\delta} = b$, so that $v^{2n} = 1 - b\delta$. Therefore, $1 - b\delta = (1-a\delta)^2 = 1 - 2a\delta + a^2\delta^2$, or $a^2\delta^2 = (2a-b)\delta$ so that $\delta = \frac{2a-b}{a^2}$. Also we see that $n\ln v = \ln(1-a\delta) - n\delta = \ln(1-a\delta)$ so that $n = \frac{\ln(1-a\delta)}{-\delta}$. From formula (4.42) we know that $(\bar{I}\bar{a})_{\overline{n}|} = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta}$. We now substitute the identities derived above for $\bar{a}_{\overline{n}|}, n, v^n$, and δ. After several steps of tedious, but routine, algebra we obtain the answer

$$\frac{a^3}{(2a-b)^2}\left[2a - b - (b-a)\ln\left(\frac{a}{b-a}\right)\right].$$

50. (*a*) (1) $\frac{d}{di}a_{\overline{n}|} = \frac{d}{di}\sum_{t=1}^{n} v^t = \frac{d}{di}\sum_{t=1}^{n}(1+i)^{-t} = -\sum_{t=1}^{n} t(1+i)^{-t-1} = -v\sum_{t=1}^{n} tv^t = -v(Ia)_{\overline{n}|}.$

(2) $\frac{d}{di}a_{\overline{n}|}\Big|_{i=0} = -v(Ia)_{\overline{n}|}\Big|_{i=0} = -\sum_{t=1}^{n} t = -\frac{n(n+1)}{2}.$

(*b*) (1) $\frac{d}{di}\bar{a}_{\overline{n}|} = \frac{d}{di}\int_0^n v^t dt = \frac{d}{di}\int_0^n (1+i)^{-t} dt = -\int_0^n t(1+i)^{-t-1} dt = -v\int_0^n tv^t dt = -v(\bar{I}\bar{a})_{\overline{n}|}.$

(2) $\frac{d}{di}\bar{a}_{\overline{n}|}\Big|_{i=0} = -v(\bar{I}\bar{a})_{\overline{n}|}\Big|_{t=0} = -\int_0^n t\,dt = -\frac{n^2}{2}.$

Chapter 3

1. The equation of value using a comparison date at time $t = 20$ is
$$50,000 = 1000 s_{\overline{20|}} + X s_{\overline{10|}} \quad \text{at} \quad 7\%.$$
Thus,
$$X = \frac{50,000 - 1000 s_{\overline{20|}}}{s_{\overline{10|}}} = \frac{50,000 - 40,995.49}{13.81645} = \$651.72.$$

2. The down payment (*D*) plus the amount of the loan (*L*) must equal the total price paid for the automobile. The monthly rate of interest is $j = .18/12 = .015$ and the amount of the loan (*L*) is the present value of the payments, i.e.
$$L = 250 a_{\overline{48|}.015} = 250(34.04255) = 8510.64.$$
Thus, the down payment needed will be
$$D = 10,000 - 8510.64 = \$1489.36.$$

3. The monthly interest rate on the first loan (L_1) is $j_1 = .06/12 = .005$ and
$$L_1 = 500 a_{\overline{48|}.005} = (500)(42.58032) = 21,290.16.$$
The monthly interest rate on the second loan (L_2) is $j_2 = .075/12 = .00625$ and
$$L_2 = 25,000 - L_1 = 25,000 - 21,290.16 = 3709.84.$$
The payment on the second loan (*R*) can be determined from
$$3709.84 = R a_{\overline{12|}.00625}$$
giving
$$R = \frac{3709.84}{11.52639} = \$321.86.$$

4. A's loan: $20,000 = R a_{\overline{8|}.085}$
$$R = \frac{20,000}{5.639183} = 3546.61$$
so that the total interest would be
$$(8)(3546.61) - 20,000 = 8372.88.$$

   B's loan: The annual interest is
$$(.085)(20,000) = 1700.00$$
so that the total interest would be
$$(8)(1700.00) = 13,600.00.$$
Thus, the difference is
$$13,600.00 - 8372.88 = \$5227.12.$$

5. Using formula (3.2), the present value is

$$na_{\overline{n|}} = \frac{n\left[1-(1+i)^{-n}\right]}{i} \quad \text{where} \quad i = \frac{1}{n}.$$

This expression then becomes

$$\frac{n\left[1-\left(\frac{n+1}{n}\right)^{-n}\right]}{\frac{1}{n}} = n^2\left[1-\left(\frac{n}{n+1}\right)^n\right].$$

6. We are given $a_{\overline{n|}} = \frac{1-v^n}{i} = x,$ so that $v^n = 1-ix.$ Also, we are given $a_{\overline{2n|}} = \frac{1-v^{2n}}{i} = y,$ so that $v^{2n} = 1-iy.$ But $v^{2n} = (v^n)^2$ so that $1-iy = (1-ix)^2$. This equation is the quadratic $x^2i^2 - (2x-y)i = 0$ so that $i = \frac{2x-y}{x^2}.$ Then applying formula (1.15a), we have $d = \frac{i}{1+i} = \frac{2x-y}{x^2+2x-y}.$

7. We know that $d = 1-v,$ and directly applying formula (3.8), we have

$$\ddot{a}_{\overline{8|}} = \frac{1-v^8}{d} = \frac{1-(1-d)^8}{d} = \frac{1-(.9)^8}{.1} = 5.695.$$

8. The semiannual interest rate is $j = .06/2 = .03.$ The present value of the payments is

$$100\left(\ddot{a}_{\overline{21|}} + \ddot{a}_{\overline{9|}}\right) = 100(15.87747 + 8.01969) = \$2389.72.$$

9. We will use a comparison date at the point where the interest rate changes. The equation of value at age 65 is

$$3000\ddot{s}_{\overline{25|}.08} = R\ddot{a}_{\overline{15|}.07}$$

so that

$$R = \frac{3000\ddot{s}_{\overline{25|}.08}}{\ddot{a}_{\overline{15|}.07}} = \frac{236,863.25}{9.74547} = \$24,305$$

to the nearest dollar.

10. (*a*) Using formulas (3.1) and (3.7)

$$\ddot{a}_{\overline{n|}} = \left(1+v+v^2+\cdots+v^{n-1}\right) + v^n - v^n$$
$$= \left(v+v^2+\cdots+v^n\right) + 1 - v^n = a_{\overline{n|}} + 1 - v^n.$$

(*b*) Using formulas (3.3) and (3.9)

$$\begin{aligned}\ddot{s}_{\overline{n|}} &= \left[(1+i)^n + (1+i)^{n-1} + \cdots + (1+i)\right] + 1 - 1 \\ &= \left[(1+i)^{n-1} + \cdots + (1+i) + 1\right] + (1+i)^n - 1 \\ &= s_{\overline{n|}} - 1 + (1+i)^n .\end{aligned}$$

(*c*) Each formula can be explained from the above derivations by putting the annuity-immediate payments on a time diagram and adjusting the beginning and end of the series of payments to turn each into an annuity-due.

11. We know that

$$\ddot{a}_{\overline{p|}} = x = \frac{1-v^p}{d} \quad \text{and} \quad s_{\overline{q|}} = y = \frac{(1+i)^q - 1}{i}.$$

Thus, $v^p = 1 - dx = 1 - ivx$ and $(1+i)^q = 1 + iy,$ so that $v^q = (1+iy)^{-1}$.

Finally,

$$\begin{aligned}a_{\overline{p+q|}} &= \frac{1-v^{p+q}}{i} = \frac{1}{i}\left(1 - \frac{1-ivx}{1+iy}\right) \\ &= \frac{(1+iy)-(1-ivx)}{i(1+iy)} = \frac{vx+y}{1+iy}.\end{aligned}$$

12. We will call September 7, $z-1$ $t=0$
so that March 7, $z+8$ is $t=34$
and June 7, $z+12$ is $t=51$
where time t is measured in quarters. Payments are made at $t=3$ through $t=49$, inclusive. The quarterly rate of interest is $j = .06/4 = .015$.

(*a*) $\text{PV} = 100\left(a_{\overline{49|}} - a_{\overline{2|}}\right) = 100(34.5247 - 1.9559) = \$3256.88.$

(*b*) $\text{CV} = 100\left(s_{\overline{32|}} + a_{\overline{15|}}\right) = 100(40.6883 + 13.3432) = \$5403.15.$

(*c*) $\text{AV} = 100\left(s_{\overline{49|}} - s_{\overline{2|}}\right) = 100(71.6087 - 2.0150) = \$6959.37.$

13. One approach is to sum the geometric progression

$$a_{\overline{15|}}\left(1 + v^{15} + v^{30}\right) = a_{\overline{15|}}\frac{1-v^{45}}{1-v^{15}} = a_{\overline{15|}}\frac{a_{\overline{45|}}}{a_{\overline{15|}}} = a_{\overline{45|}}.$$

The formula also can be derived by observing that

$$a_{\overline{15|}}\left(1 + v^{15} + v^{30}\right) = a_{\overline{15|}} + {}_{15|}a_{\overline{15|}} + {}_{30|}a_{\overline{15|}} = a_{\overline{45|}}$$

by splitting the 45 payments into 3 sets of 15 payments each.

14. We multiply numerator and denominator by $(1+i)^4$ to change the comparison date from time $t=0$ to $t=4$ and obtain

$$\frac{a_{\overline{7}|}}{a_{\overline{11}|}}=\frac{a_{\overline{7}|}(1+i)^4}{a_{\overline{11}|}(1+i)^4}=\frac{a_{\overline{3}|}+s_{\overline{4}|}}{a_{\overline{7}|}+s_{\overline{4}|}}.$$

Therefore $x=4,\ y=7,$ and $z=4.$

15. The present value of annuities *X* and *Y* are:

$$\text{PV}_X=a_{\overline{30}|}+v^{10}a_{\overline{10}|} \quad \text{and}$$

$$\text{PV}_Y=K\left(a_{\overline{10}|}+v^{20}a_{\overline{10}|}\right).$$

We are given that $\text{PV}_X=\text{PV}_Y$ and $v^{10}=.5.$ Multiplying through by *i*, we have

$$1-v^{30}+v^{10}\left(1-v^{10}\right)=K\left(1-v^{10}\right)\left(1+v^{20}\right)$$

so that

$$K=\frac{1+v^{10}-v^{20}-v^{30}}{1-v^{10}+v^{20}-v^{30}}=\frac{1+.5-.25-.125}{1-.5+.25-.125}=\frac{1.125}{.625}=1.8.$$

16. We are given ${}_{5|}a_{\overline{10}|}=3\cdot{}_{10|}a_{\overline{5}|}$ or $v^5a_{\overline{10}|}=3v^{10}a_{\overline{5}|}$ and $v^5\frac{1-v^{10}}{i}=3v^{10}\frac{1-v^5}{i}.$

Therefore, we have

$$v^5-v^{15}=3v^{10}-3v^{15} \text{ or } 2v^{15}-3v^{10}+v^5=0 \text{ or } 2-3(1+i)^5+(1+i)^{10}=0$$

which is a quadratic in $(1+i)^5$. Solving the quadratic

$$(1+i)^5=\frac{3\pm\sqrt{(-3)^2-(4)(2)(1)}}{2}=\frac{3\pm 1}{2}=2$$

rejecting the root $i=0.$

17. The semiannual interest rate is $j=.09/2=.045.$ The present value of the annuity on October 1 of the prior year is $2000a_{\overline{10}|}.$ Thus, the present value on January 1 is

$$2000a_{\overline{10}|}(1.045)^{.5}$$

$$=(2000)(7.91272)(1.02225)=\$16{,}178$$

to the nearest dollar.

18. The equation of value at time $t=0$ is

$$1000\ddot{a}_{\overline{20}|}=R\cdot v^{30}\cdot\ddot{a}_{\overline{\infty}|}$$

or

$$1000\frac{1-v^{20}}{d}=R\cdot v^{30}\frac{1}{d}$$

so that

$$R = 1000\frac{1-v^{20}}{v^{30}} = 1000\left(1-v^{20}\right)\left(1+i\right)^{30}$$
$$= 1000\left[\left(1+i\right)^{30} - \left(1+i\right)^{10}\right].$$

19. We are given $i = \frac{1}{9}$ so that $d = \frac{i}{1+i} = \frac{1}{10}$. The equation of value at time $t = 0$ is

$$6561 = 1000 v^n \ddot{a}_{\overline{\infty}|} \quad \text{or} \quad 6.561 = \frac{(1-d)^n}{d} = \frac{(1-.1)^n}{.1}.$$

Therefore, $(.9)^n = (.1)(6.561) = .6561$ and $n = 4$.

20. The equation of value at age 60 is

$$50{,}000 a_{\overline{\infty}|} = R v^5 a_{\overline{20}|}$$

or

$$\frac{50{,}000}{i} = R v^5 \frac{1-v^{20}}{i}$$

so that

$$R = \frac{50{,}000}{v^5 - v^{25}} \quad \text{at} \quad i = .05$$
$$= \frac{50{,}000}{.7835262 - .2953028} = \$102{,}412$$

to the nearest dollar.

21. Per dollar of annuity payment, we have $\text{PV}_A = \text{PV}_D$ which gives

$$\frac{1}{3} a_{\overline{n}|} = v^n \cdot a_{\overline{\infty}|} \quad \text{or} \quad a_{\overline{n}|} = 3 v^n a_{\overline{\infty}|}$$

and $1 - v^n = 3v^n$, so that

$$4v^n = 1 \quad \text{or} \quad v^n = .25 \quad \text{and} \quad \left(1+i\right)^n = 4.$$

22. Per dollar of annuity payment, we have

$$\text{PV}_A = a_{\overline{n}|}, \ \ \text{PV}_B = v^n a_{\overline{n}|}, \ \ \text{PV}_C = v^{2n} a_{\overline{n}|} \quad \text{and} \quad \text{PV}_D = v^{3n} a_{\overline{\infty}|}.$$

We are given

$$\frac{\text{PV}_C}{\text{PV}_A} = v^{2n} = .49 \quad \text{or} \quad v^n = .7.$$

Finally,

$$\frac{\text{PV}_B}{\text{PV}_D} = \frac{v^n a_{\overline{n}|}}{v^{3n} a_{\overline{\infty}|}} = \frac{v^n\left(1-v^n\right)}{v^{3n}}$$

$$= \frac{1-v^n}{v^{2n}} = \frac{1-.7}{(.7)^2} = \frac{.30}{.49} = \frac{30}{49}.$$

23. (*a*) $a_{\overline{5.25}|} = a_{\overline{5}|} + v^{5.25}\left[\frac{(1+i)^{.25}-1}{i}\right]$ at $i = .05$

$$= 4.32946 + (.77402)\left[\frac{(1.05)^{.25}-1}{.05}\right] = 4.5195.$$

(*b*) $a_{\overline{5.25}|} = a_{\overline{5}|} + .25v^{5.25}$

$$= 4.32946 + (.25)(.77402) = 4.5230.$$

(*c*) $a_{\overline{5.25}|} = a_{\overline{5}|} + .25v^{6}$

$$= 4.23946 + (.25)(.74621) = 4.5160.$$

24. At time $t = 0$ we have the equation of value

$$1000 = 100\left(a_{\overline{n}|} - a_{\overline{4}|}\right) \quad \text{or}$$

$$a_{\overline{n}|} = 10 + a_{\overline{4}|} = 13.5875 \quad \text{at} \quad i = .045.$$

Now using a financial calculator, we find that $n = 21$ full payments plus a balloon payment. We now use time $t = 21$ as the comparison date to obtain

$$1000(1.045)^{21} = 100s_{\overline{17}|} + K$$

or

$$K = 1000(1.045)^{21} - 100s_{\overline{17}|}$$

$$= 2520.2412 - 100(24.74171) = 46.07$$

Thus, the balloon payment is

$$100 + 46.07 = \$146.07 \quad \text{at time} \quad t = 21.$$

25. We are given $\text{PV}_1 = \text{PV}_2$ where

$$\text{PV}_1 = 4a_{\overline{36}|} \quad \text{and} \quad \text{PV}_2 = 5a_{\overline{18}|}.$$

We are also given that $(1+i)^n = 2.$ Thus, we have

$$4\cdot\frac{1-v^{36}}{i} = 5\frac{1-v^{18}}{i} \quad \text{or}$$

$$4\left(1-v^{36}\right) = 4\left(1-v^{18}\right)\left(1+v^{18}\right) = 5\left(1-v^{18}\right).$$

Thus, we have

$$4\left(1+v^{18}\right)=5 \quad \text{or} \quad v^{18}=.25.$$

Finally, we have $\left(1+i\right)^{18}=4,$ so that $\left(1+i\right)^{9}=2$ which gives $n=9.$

26. At time $t=20,$ the fund balance would be

$$500\ddot{s}_{\overline{20}|}=24,711.46 \quad \text{at} \quad i=.08.$$

Let n be the number of years full withdrawals of 1000 can be made, so that the equation of value is

$$1000s_{\overline{n}|}=24,711.46 \quad \text{or} \quad s_{\overline{n}|}=24.71146\,.$$

Using a financial calculator we find that only $n=14$ full withdrawals are possible.

27. (*a*) The monthly rate of interest is $j=.12/12=.01.$ The equation of value at time $t=0$ is

$$6000v^{k}=100a_{\overline{60}|}=4495.5038$$

$$v^{k}=.749251 \quad \text{so that} \quad k=\frac{-\ln(.749251)}{\ln(1.01)}=29.$$

(*b*) Applying formula (2.2) we have

$$\bar{t}=\frac{1000(1+2+\cdots+60)}{100(60)}=\frac{(60)(61)}{2(60)}=\frac{61}{2}=30.5.$$

28.(*a*) Set: $\text{N}=48 \quad \text{PV}=12,000 \quad \text{PMT}=-300$ and CPT I to obtain $j=.7701\%.$ The answer is $12j=9.24\%.$

(*b*) We have $300a_{\overline{48}|}=12,000$ or $a_{\overline{48}|}=40.$ Applying formula (3.21) with $n=48$ and $g=40,$ we have

$$j\approx\frac{2(n-g)}{g(n+1)}=\frac{2(48-40)}{40(48+1)}=.8163\%.$$

The answer is $12j=9.80\%.$

29. We have

$$a_{\overline{2}|}=v+v^{2} \quad \text{or} \quad 1.75=(1+i)^{-1}+(1+i)^{-2}\,.$$

Multiplying through $(1+i)^{2}$ gives

$$1.75(1+i)^{2}=(1+i)+1$$

$$1.75\left(1+2i+i^{2}\right)=2+i$$

and $1.75i^{2}+2.5-.25$ or $7i^{2}+10-1=0$ which is a quadratic. Solving for *i*

$$i=\frac{-10\pm\sqrt{(10)^2-(4)(7)(-1)}}{(2)(7)}=\frac{-10\pm\sqrt{128}}{14}$$
$$=\frac{4\sqrt{2}-5}{7} \text{ rejecting the negative root.}$$

30. We have the following equation of value
$$10,000=1538a_{\overline{10}|}=1072a_{\overline{20}|}.$$
Thus $1538(1-v^{10})=1072(1-v^{20})=1072(1-v^{10})(1+v^{10})$, so that $1+v^{10}=\frac{1538}{1072}$ or $v^{10}=.43470.$
Solving for i, we obtain
$$(1+i)^{-10}=.43470 \quad \text{and} \quad i=(.43470)^{-.1}-1=.0869, \quad \text{or} \quad 8.69\%.$$

31. We are given that the following present values are equal
$$a_{\overline{\infty}|7.25\%}=a_{\overline{50}|j}=a_{\overline{n}|j-1}.$$
Using the financial calculator
$$a_{\overline{50}|j}=\frac{1}{.0725}=13.7931$$
and solving we obtain $j=7.00\%$. Since $j-1=6\%$, we use the financial calculator again
$$a_{\overline{n}|6\%}=13.7931 \quad \text{to obtain} \quad n=30.2.$$

32. (*a*) We have $j_1=.08/2=.04$ and $j_2=.07/2=.035$. The present value is
$$a_{\overline{6}|.04}+a_{\overline{4}|.035}(1.04)^{-6}=5.2421+(3.6731)(.79031)$$
$$=8.145.$$

(*b*) The present value is
$$a_{\overline{6}|.04}+a_{\overline{4}|.035}(1.035)^{-6}=5.2421+(3.6731)(.81350)$$
$$=8.230.$$

(*c*) Answer (*b*) is greater than answer (*a*) since the last four payments are discounted over the first three years at a lower interest rate.

33. (*a*) Using formula (3.24)
$$a_{\overline{5}|}=v+v^2+v^3+v^4+v^5$$
$$=\frac{1}{1.06}+\frac{1}{(1.062)^2}+\frac{1}{(1.064)^3}+\frac{1}{(1.066)^4}+\frac{1}{(1.068)^5}$$
$$=4.1543.$$

(*b*) Using formula (3.23)

$$a_{\overline{5}|} = \frac{1}{1.06} + \frac{1}{(1.06)(1.062)} + \frac{1}{(1.06)(1.062)(1.064)} + \frac{1}{(1.06)(1.062)(1.064)(1.066)}$$
$$+ \frac{1}{(1.06)(1.062)(1.064)(1.066)(1.068)} = 4.1831.$$

34. Payments are R at time $t = .5$ and $2R$ at time $t = 1.5,\ 2.5, \ldots,\ 9.5.$ The present value of these payments is equal to P. Thus, we have

$$P = R\left[1 + 2a_{\overline{4}|\, i} + 2a_{\overline{a}|\, j}(1+i)^{-4}\right](1+i)^{-1/2}$$

and

$$R = \frac{P(1+i)^{1/2}}{1 + 2a_{\overline{4}|\, i} + 2(1+i)^{-4}\, a_{\overline{5}|\, j}}.$$

35. The payments occur at $t = 0,\ 1,\ 2, \ldots,\ 19$ and we need the current value at time $t = 2$ using the variable effective rate of interest given. The current value is

$$\left(1+\frac{1}{9}\right)\left(1+\frac{1}{10}\right) + \left(1+\frac{1}{10}\right) + 1 + \left(1+\frac{1}{11}\right)^{-1} + \left(1+\frac{1}{11}\right)^{-1}\left(1+\frac{1}{12}\right)^{-1}$$
$$+ \cdots + \left(1+\frac{1}{11}\right)^{-1}\left(1+\frac{1}{12}\right)^{-1} \cdots \left(1+\frac{1}{27}\right)^{-1}$$
$$= \left(\frac{10}{9}\right)\left(\frac{11}{10}\right) + \frac{11}{10} + 1 + \frac{11}{12} + \left(\frac{11}{12}\right)\left(\frac{12}{13}\right) + \cdots + \left(\frac{11}{12}\cdot\frac{12}{13}\cdots\frac{27}{28}\right)$$
$$= \frac{11}{9} + \frac{11}{10} + \frac{11}{11} + \frac{11}{12} + \frac{11}{13} + \cdots + \frac{11}{28} = \sum_{t=9}^{28} \frac{11}{t}.$$

36. We know that $a^{-1}(t) = 1 - dt$ using simple discount. Therefore, we have

$$a_{\overline{n}|} = \sum_{t=1}^{n} a^{-1}(t) = \sum_{t=1}^{n} (1 - dt) = n - \tfrac{1}{2} n(n+1)d$$

by summing the first n positive integers.

37. We have $a(t) = \dfrac{1}{\log_2(t+2) - \log_2(t+1)} = \dfrac{1}{\log_2 \dfrac{t+2}{t+1}},$ so that $a^{-1}(t) = \log_2 \dfrac{t+2}{t+1}.$

Now

$$\ddot{a}_{\overline{n}|} = \sum_{t=0}^{n-1} a^{-1}(t) = \sum_{t=0}^{n-1} \log_2 \frac{t+2}{t+1}$$
$$= \log_2 \frac{2}{1} + \log_2 \frac{3}{2} + \cdots + \log_2 \frac{n+1}{n}$$
$$= \log_2 \left(\frac{2}{1} \cdot \frac{3}{2} \cdot \ \cdots \ \cdot \frac{n+1}{n} \right) = \log_2 (n+1).$$

38. The accumulated value of 1 paid at time *t* accumulated to time 10 is

$$e^{\int_t^{10} \delta_r dr} = e^{\int_t^{10} \frac{1}{20-r} dr} = e^{\ln(20-r) - \ln 10} = \frac{20-r}{10}.$$

Then

$$s_{\overline{10}|} = \sum_{r=1}^{10} \frac{20-r}{10} = \frac{19}{10} + \frac{18}{10} + \cdots + \frac{10}{10} = 14.5.$$

39. A: $\mathrm{PV}_A = \frac{1}{1.01} + \frac{1}{1.02} + \frac{1}{1.03} + \frac{1}{1.04} + \frac{1}{1.05} = 4.8553$

B: $\mathrm{AV}_B = 1.04 + 1.03 + 1.02 + 1.01 + 1.00 = 5.1000$

and taking the present value

$$\mathrm{PV}_B = \frac{5.1000}{1.05} = 4.8571.$$

The answers differ by 4.8571 - 4.8553=.0018.

40. The present value of the payments in (*ii*) is

$$30a_{\overline{10}|} + 60v^{10}a_{\overline{10}|} + 90v^{20}a_{\overline{10}|} = a_{\overline{10}|}\left(30 + 60v^{10} + 90v^{20}\right).$$

The present value of the payments in (*i*) is

$$55a_{\overline{20}|} = 55a_{\overline{10}|}\left(1 + v^{10}\right).$$

Equating the two values we have the quadratic $90v^{20} + 5v^{10} - 25 = 0$. Solving the quadratic

$$v^{10} = \frac{-5 \pm \sqrt{(5)^2 - (4)(90)(-25)}}{(2)(90)} = \frac{90}{180} = .5$$

rejecting the negative root. Now $v^{10} = .5$ or $(1+i)^{10} = 2$ and $i = .0718$. Finally, $X = 55a_{\overline{20}|.0718} = 574.60.$

41. We have the equation of value at time $t = 3n$

$$98s_{\overline{3n}|} + 98s_{\overline{2n}|} = 8000$$

or

$$\frac{(1+i)^{3n}-1}{i}+\frac{(1+i)^{2n}-1}{i}=\frac{8000}{98}=81.6327.$$

We are given that $(1+i)^n=2.$ Therefore, $\frac{2^3-1}{i}+\frac{2^2-1}{i}=\frac{10}{i}=81.6327$ and $i=.1225,$ or 12.25%.

42. At time $t=0$ we have the equation of value

$$10,000=4ka_{\overline{20|}}-ka_{\overline{15|}}-ka_{\overline{10|}}-ka_{\overline{5|}}$$

so that

$$k=\frac{10,000}{4a_{\overline{20|}}-a_{\overline{15|}}-a_{\overline{10|}}-a_{\overline{5|}}}.$$

43. The present values given are:

(i) $2a_{\overline{2n|}}+a_{\overline{n|}}=36$ or $2(1-v^{2n})+(1-v^n)=36i,$ and

(ii) $2v^n a_{\overline{n|}}=6$ or $2v^n(1-v^n)=6i.$

Thus, $2(1-v^{2n})+(1-v^n)=(6)(2)v^n(1-v^n)$ which simplifies to the quadratic

$$10v^{2n}-13v^n+3=0.$$

Solving,

$$v^n=\frac{13\pm\sqrt{(-13)^2-(4)(10)(3)}}{(2)(10)}=\frac{6}{20}=.3$$

rejecting the root $v^n=1.$ Substituting back into (ii)

$(2)(.3)\frac{1-.3}{i}=6,$ so that $i=\frac{(2)(.3)(.7)}{6}=.07,$ or 7%.

44. An equation of value at time $t=10$ is

$$10,000(1.04)^{10}-K(1.05)(1.04)^6-K(1.05)(1.04)^5$$
$$-K(1.04)^4-K(1.04)^3=10,000.$$

Thus, we have

$$K=\frac{10,000\left[(1.04)^{10}-1\right]}{(1.05)(1.04)^6+(1.05)(1.04)^5+(1.04)^4+(1.04)^3}$$
$$=\$980 \quad \text{to the nearest dollar.}$$

45. $\sum_{n=15}^{40}s_{\overline{n|}}=\frac{1}{i}\sum_{n=15}^{40}\left[(1+i)^n-1\right]=\frac{s_{\overline{41|}}-s_{\overline{15|}}-26}{i}$

using formula (3.3) twice and recognizing that there are 26 terms in the summation.

Chapter 2

1. The quarterly interest rate is

$$j = \frac{i^{(4)}}{4} = \frac{.06}{4} = .015$$

and all time periods are measured in quarters. Using the end of the third year as the comparison date

$$3000(1+j)^{12} + X = 2000v^4 + 5000v^{28}$$
$$X = 2000(.94218) + 5000(.65910) - 3000(1.19562)$$
$$= \$1593.00.$$

2. The monthly interest rate is

$$j = \frac{i^{(12)}}{12} = \frac{.18}{12} = .015.$$

Using the end of the third month as the comparison date

$$X = 1000(1+j)^3 - 200(1+j)^2 - 300(1+j)$$
$$= 1000(1.04568) - 200(1.03023) - 300(1.015)$$
$$= \$535.13.$$

3. We have

$$200v^5 + 500v^{10} = 400.94v^5$$
$$v^{10} = .40188v^5$$
$$v^5 = .40188 \quad \text{or} \quad (1+i)^5 = 2.4883.$$

Now using time $t = 10$ as the comparison date

$$P = 100(1+i)^{10} + 120(1+i)^5$$
$$= 100(2.4883)^2 + 120(2.4883) = \$917.76.$$

4. The quarterly discount rate is 1/41 and the quarterly discount factor is $1 - 1/41 = 40/41$. The three deposits accumulate for 24, 16, and 8 quarters, respectively. Thus,

$$A(28) = 100\left[(1.025)\left(\frac{40}{41}\right)^{-24} + (1.025)^3\left(\frac{40}{41}\right)^{-16} + (1.025)^5\left(\frac{40}{41}\right)^{-8}\right].$$

However,

$$\left(\frac{40}{41}\right)^{-1} = 1.025$$

so that

$$A(28) = 100\left[(1.025)^{25} + (1.025)^{19} + (1.025)^{13}\right] = \$483.11.$$

5. (*a*) At time $t = 10$, we have

$$X = 100(1 + 10i) + 100(1 + 5i) \quad \text{with} \quad i = .05$$
$$= 200 + 1500(.05) = \$275.$$

(*b*) At time $t = 15$, we have

$$X(1 + 5i) = 100(1 + 15i) + 100(1 + 10i) \quad \text{with} \quad i = .05$$
$$X(1.25) = 200 + 2500(.05) = 325$$

and

$$X = \frac{325}{1.25} = \$260.$$

6. The given equation of value is

$$1000(1.06)^n = 2000(1.04)^n$$

so that

$$\left(\frac{1.06}{1.04}\right)^n = 2$$
$$n[\ln 1.06 - \ln 1.04] = \ln 2$$

and

$$n = \frac{.693147}{.058269 - .039221} = 36.4 \text{ years}.$$

7. The given equation of value is

$$3000 + 2000v^2 = 5000v^n + 5000v^{n+5}$$
$$3000 + 2000(1-d)^2 = 5000(1-d)^n\left[1 + (1-d)^5\right]$$
$$\text{and} \quad 3000 + 2000(.94)^2 = 5000(.94)^n\left[1 + (.94)^5\right]$$

since $d = .06$. Simplifying, we have

$$4767.20 = 8669.52(.94)^n$$
$$(.94)^n = \frac{4767.20}{8669.52} = .54988$$
$$n \ln(.94) = \ln(.54988)$$
$$\text{and} \quad n = \frac{\ln(.54988)}{\ln(.94)} = 9.66 \quad \text{years.}$$

8. The given equation of value is

$$100 = 100v^n + 100v^{2n}$$

which is a quadratic in v^n. Solving

$$v^{2n} + v^n - 1 = 0$$

$$v^n = \frac{-1 \pm \sqrt{1-(4)(1)(-1)}}{2} = \frac{-1+\sqrt{5}}{2}$$

$$= .618034 \quad \text{rejecting the negative root.}$$

We are given $i = .08$, so that

$$(1.08)^n = 1/.61803 = 1.618034$$

$$\text{and} \quad n = \frac{\ln 1.618034}{\ln 1.08} = 6.25 \text{ years.}$$

9. Applying formula (2.2)

$$\bar{t} = \frac{n^2 + (2n)^2 + \cdots + (n^2)^2}{n + 2n + \cdots + n^2} = \frac{n^2(1 + 2^2 + \cdots + n^2)}{n(1 + 2 + \cdots + n)}.$$

We now apply the formulas for the sum of the first n positive integers and their squares (see Appendix C) to obtain

$$\frac{n^2\left(\frac{1}{6}\right)(n)(n+1)(2n+1)}{n\left(\frac{1}{2}\right)(n)(n+1)} = \frac{1}{3}(n)(2n+1) = \frac{2n^2+n}{3}.$$

10. We parallel the derivation of formula (2.4)

$$(1+i)^n = 3 \quad \text{or} \quad n = \frac{\ln 3}{\ln(1+i)}$$

and approximating i by .08, we obtain

$$n \approx \frac{\ln 3}{i} \cdot \frac{.08}{\ln(1.08)} = \frac{1.098612}{i} \cdot \frac{.08}{.076961}$$

$$= \frac{1.14}{i} \text{ or a rule of 114, i.e. } n = 114.$$

11. Use time $t = 10$ as the comparison date

$$\text{A: } 10[1+(10)(.11)] + 30[1+(5)(.11)] = 67.5$$

$$\text{B: } 10(1.0915)^{10-n} + 30(1.0915)^{10-2n} = 67.5$$

$$10v^n + 30v^{2n} = 67.5(1.0915)^{-10} = 28.12331$$

which gives the quadratic

$$v^{2n} + .33333v^n - .93744 = 0.$$

Solving

$$v^n = \frac{-.33333 \pm \sqrt{(.33333)^2 - (4)(1)(-.93744)}}{2} = .81579$$

and

$$n = \frac{\ln(.81579)}{-\ln(1.0915)} = 2.33 \text{ years.}$$

12. Let t measure time in years. Then

$$a^A(t) = (1.01)^{12t} \quad \text{and}$$

$$a^B(t) = e^{\int_0^t r/6dr} = e^{t^2/12}.$$

Equate the two expressions and solve for t

$$(1.01)^{12t} = e^{t^2/12} \quad \text{or} \quad (1.01)^{144t} = e^{t^2}$$

$$144t \ln(1.01) = t^2$$

$$\text{and} \quad t = 144 \ln(1.01) = 1.43 \text{ years.}$$

13. Let j be the semiannual interest rate. We have

$$1000(1+j)^{30} = 3000$$

$$\text{and} \quad j = 3^{1/30} - 1 = .0373.$$

The answer is

$$i^{(2)} = 2j = 2(.0373) = .0746, \quad \text{or} \quad 7.46\%.$$

14. The given equation of value is

$$300(1+i)^2 + 200(1+i) + 100 = 700.$$

Simplifying, we get a quadratic

$$3(1+2i+i^2) + 2(1+i) - 6 = 0$$

$$3i^2 + 8i - 1 = 0.$$

Solving the quadratic

$$i = \frac{-8 \pm \sqrt{8^2 - (4)(3)(-1)}}{(2)(3)} = \frac{-8 \pm \sqrt{76}}{6}$$

$$= \frac{-8 + 2\sqrt{19}}{6} = \frac{\sqrt{19} - 4}{3} \quad \text{rejecting the negative root.}$$

15. The given equation of value is

$$100 + 200v^n + 300v^{2n} = 600v^{10}.$$

Substituting the given value of v^n

$$100 + 200(.75941) + 300(.75941)^2 = 600v^{10}$$
$$v^{10} = .708155 \quad \text{or} \quad (1+i)^{10} = 1.41212$$
$$\text{and} \quad i = (1.41212)^{.1} - 1 = .0351, \quad \text{or} \quad 3.51\%.$$

16. The total amount of interest equals
$$1000i(1+2+\cdots+10) = 55,000i.$$
Thus, we have
$$1000 + 55,000i = 1825$$
$$\text{and} \quad i = \frac{1825}{55,000} = .015, \quad \text{or} \quad 1.5\%.$$

17. We have
$$a(10) = e^{\int_0^{10}\delta_t dt} = e^{\int_0^{10} ktdt} = e^{50k} = 2$$
so that
$$50k = \ln 2 \quad \text{and} \quad k = \frac{\ln 2}{50}.$$

18. We will use i to represent both the interest rate and the discount rate, which are not equivalent. We have
$$(1+i)^3 + (1-i)^3 = 2.0096$$
$$(1+3i+3i^2+i^3) + (1-3i+3i^2-i^3) = 2.0096$$
$$2 + 6i^2 = 2.0096 \quad \text{or} \quad 6i^2 = .0096$$
$$i^2 = .0016 \quad \text{and} \quad i = .04, \quad \text{or} \quad 4\%.$$

19. (*a*) Using Appendix A December 7 is Day 341
August 8 is Day 220.

We then have

1941:	24	$= 365 - 341$
1942:	365	
1943:	365	
1944:	366	(leap year)
1945:	220	
Total =	1340	days.

(*b*) Applying formula (2.5)
$$360(Y_2 - Y_1) + 30(M_2 - M_1) + (D_2 - D_1)$$
$$= 360(1945 - 1941) + 30(8 - 12) + (8 - 7) = 1321 \text{ days.}$$

20. (*a*) $I = (10{,}000)(.06)\left(\frac{62}{365}\right) = \$101.92.$

(*b*) $I = (10{,}000)(.06)\left(\frac{60}{360}\right) = \$100.00.$

(*c*) $I = (10{,}000)(.06)\left(\frac{62}{360}\right) = \$103.33.$

21. (*a*) Bankers Rule: $I = Pr\left(\frac{n}{360}\right)$

Exact simple interest: $I = Pr\left(\frac{n}{365}\right)$

where n is the exact number of days in both. Clearly, the Banker's Rule always gives a larger answer since it has the smaller denominator and thus is more favorable to the lender.

(*b*) Ordinary simple interest: $I = Pr\left(\frac{n^*}{360}\right)$

where n^* uses 30-day months. Usually, $n \ge n^*$ giving a larger answer which is more favorable to the lender.

(*c*) Invest for the month of February.

22. (*a*) The quarterly discount rate is

$$(100-96)/100 = .04. \quad \text{Thus,}$$

$$d^{(4)} = 4(.04) = .16, \quad \text{or } 16\%.$$

(*b*) With an effective rate of interest

$$96(1+i)^{.25} = 100$$

$$\text{and} \quad i = \left(\frac{100}{96}\right)^4 - 1 = .1774, \quad \text{or } 17.74\%.$$

23. (*a*) Option A - 7% for six months:

$$(1.07)^{.5} = 1.03441.$$

Option B - 9% for three months:

$$(1.09)^{.25} = 1.02178.$$

The ratio is

$$\frac{1.03441}{1.02178} = 1.0124.$$

(*b*) Option A - 7% for 18 months:

$$(1.07)^{1.5} = 1.10682.$$

Option B - 9% for 15 months:

$$(1.07)^{1.25} = 1.11374.$$

The ratio is

$$\frac{1.10682}{1.11374} = .9938.$$

24. The monthly interest rates are:

$$y_1 = \frac{.054}{12} = .0045 \quad \text{and} \quad y_2 = \frac{.054 - .018}{12} = .003.$$

The 24-month CD is redeemed four months early, so the student will earn 16 months at .0045 and 4 months at .003. The answer is

$$5000(1.0045)^{16}(1.003)^4 = \$5437.17.$$

25. The APR = 5.1% compounded daily. The APY is obtained from

$$1 + i = \left(1 + \frac{.051}{365}\right)^{365} = 1.05232$$

or APY = .05232. The ratio is

$$\frac{\text{APY}}{\text{APR}} = \frac{.05232}{.051} = 1.0259.$$

Note that the term "APR" is used for convenience, but in practice this term is typically used only with consumer loans.

26. (*a*) No bonus is paid, so $i = .0700$, or 7.00%.

(*b*) The accumulated value is $(1.07)^3(1.02) = 1.24954$, so the yield rate is given by

$$(1+i)^3 = 1.24954 \quad \text{or} \quad i = (1.24954)^{1/3} - 1 = .0771, \quad \text{or} \quad 7.71\%.$$

(*c*) The accumulated value is

$$(1.07)^3(1.02)(1.07) = (1.07)^4(1.02) = 1.33701,$$

so the yield rate is given by

$$(1+i)^3 = 1.33701 \quad \text{or} \quad i = (1.33701)^{1/3} - 1 = .0753, \quad \text{or} \quad 7.53\%.$$

27. This exercise is asking for the combination of CD durations that will maximize the accumulated value over six years. All interest rates are convertible semiannually. Various combinations are analyzed below:

4-year/2-year: $1000(1.04)^8(1.03)^4 = 1540.34.$

3-year/3-year: $1000(1.035)^{12} = 1511.08.$

All other accumulations involving shorter-term CD's are obviously inferior. The maximum value is \$1540.34.

28. Let the purchase price be *R*. The customer has two options:

One: Pay $.9R$ in two months.

Two: Pay $(1-.01X)R$ immediately.

The customer will be indifferent if these two present values are equal. We have

$$(1-.01X)R = .9R(1.08)^{-1/6}$$
$$1-.01X = .9(1.08)^{-1/6} = .88853$$

and

$$X = 100(1-.88853) = 11.15\%.$$

29. Let the retail price be *R*. The retailer has two options:

One: Pay $.70R$ immediately.

Two: Pay $.75R$ in six months.

The retailer will be indifferent if these two present values are equal. We have

$$.70R = .75R(1+i)^{-.5}$$
$$.70(1+i)^{.5} = .75$$

and

$$i = \left(\frac{.75}{.70}\right)^2 - 1 = .1480, \quad \text{or} \quad 14.80\%.$$

30. At time 5 years

$$1000(1+i/2)^{10} = X.$$

At time 10.5 years:

$$1000(1+i/2)^{14}(1+2i/4)^{14} = 1980.$$

We then have

$$(1+i/2)^{28} = 1.98$$
$$(1+i/2)^{10} = (1.98)^{10/28} = 1.276$$

and the answer is

$$1000(1.276) = \$1276.$$

31. We are given

$$A(1.06)^{20} + B(1.08)^{20} = 2000$$
$$2A(1.06)^{10} = B(1.08)^{10}$$

which is two linear equations in two unknowns. Solving these simultaneous equations gives:

$$A = 182.82 \quad \text{and} \quad B = 303.30.$$

The answer then is

$$A(1.06)^5 + B(1.08)^5 = (182.82)(1.06)^5 + (303.30)(1.08)^5$$
$$= \$690.30.$$

32. We are given that

$$10,000(1+i)(1+i-.05) = 12,093.75.$$

Solving the quadratic

$$1 + i - .05 + i + i^2 - .05i = 1.209375$$
$$i^2 + 1.95i - .259375 = 0$$
$$i = \frac{-1.95 \pm \sqrt{(1.95)^2 - (4)(1)(-.259375)}}{2}$$
$$= .125 \quad \text{rejecting the negative root.}$$

We then have

$$10,000(1 + .125 + .09)^3 = 10,000(1.215)^3$$
$$= \$17,936.$$

33. The annual discount rate is

$$d = \frac{1000-920}{1000} = \frac{80}{1000} = .08.$$

The early payment reduces the face amount by *X*. We then have

$$X\left[1 - \tfrac{1}{2}(.08)\right] = 288,$$

so that

$$X = \frac{288}{.96} = 300$$

and the face amount has been reduced to

$$1000 - 300 = \$700.$$