Tutorial #5

Question 1:

For the matrices below, obtain (1) A +B, (2) A - B, (3) AC, (4) AB', (5) B'A.

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 6 \\ 3 & 8 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 8 & 1 \\ 5 & 4 & 0 \end{bmatrix}$$

Solution:

$$A + B = \begin{bmatrix} 1 & 4 \\ 2 & 6 \\ 3 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1+1 & 4+3 \\ 2+1 & 6+4 \\ 3+2 & 8+5 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 3 & 10 \\ 5 & 13 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1 & 4 \\ 2 & 6 \\ 3 & 8 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1-1 & 4-3 \\ 2-1 & 6-4 \\ 3-2 & 8-5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 4 \\ 2 & 6 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} 3 & 8 & 1 \\ 5 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1*3+4*5 & 1*8+4*4 & 1*1+4*0 \\ 2*3+6*5 & 2*8+6*4 & 2*1+6*0 \\ 3*3+8*5 & 3*8+8*4 & 3*1+8*0 \end{bmatrix}$$

$$= \begin{bmatrix} 23 & 24 & 1 \\ 36 & 40 & 2 \\ 49 & 56 & 3 \end{bmatrix}$$

$$B' = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

$$(A_{3\times2}B'_{2\times3})_{3\times3} = \begin{bmatrix} 1 & 4 \\ 2 & 6 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1*1+4*3 & 1*1+4*4 & 1*2+4*5 \\ 2*1+6*3 & 2*1+6*4 & 2*2+6*5 \\ 3*1+8*3 & 3*1+8*4 & 3*2+8*5 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 17 & 22 \\ 20 & 26 & 34 \\ 27 & 35 & 46 \end{bmatrix}$$

$$(B'_{2\times3}A_{3\times2})_{2\times2} = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 6 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 1*1+1*2+2*3 & 1*4+1*6+2*8 \\ 3*1+4*2+5*3 & 3*4+4*6+5*8 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 26 \\ 26 & 76 \end{bmatrix}$$

Question 2:

Flavor deterioration. The results shown below were obtained in a small-scale experiment to study the relation between F^0 of storage temperature (X) and number of weeks before flavour deterioration of a food product begins to occur (Y).

i	1	2	3	4	5
X_i	8	4	0	-4	-8
Y_i	7.8	9.0	10.2	11.0	11.7

Assume that first-order regression model (2.1) is applicable. Using matrix methods to Find $(1)Y'Y$, $(2)X'X$, $(3)X'Y$ $(4)B$ $(5)\,V(B)$.

$$Y_{n\times1} = X_{n\times2}\mathbf{B}_{2\times1} + \varepsilon_{n\times1}$$
$$E(Y_{n\times1}) = X_{n\times2}\mathbf{B}_{2\times1}$$
$$\mathbf{B}_{2\times1} = (X'X)^{-1}X'Y$$
$$V(\mathbf{B}) = MSE(X'X)^{-1}$$
$$MSE = \frac{e'e}{n-2}$$

$$X = \begin{bmatrix} 1 & 8 \\ 1 & 4 \\ 1 & 0 \\ 1 & -4 \\ 1 & -8 \end{bmatrix}_{5*2}, \quad Y = \begin{bmatrix} 7.8 \\ 9.0 \\ 10.2 \\ 11.0 \\ 11.7 \end{bmatrix}_{5*1}, \quad X' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 8 & 4 & 0 & -4 & -8 \end{bmatrix}_{2*5}$$

(1) $$Y'_{1\times n}Y_{n\times1} = [\sum y_i^2] = [7.8 \quad 9.0 \quad 10.2 \quad 11.0 \quad 11.7]\begin{bmatrix} 7.8 \\ 9.0 \\ 10.2 \\ 11.0 \\ 11.7 \end{bmatrix} = [\mathbf{503.77}]$$

(2) $$(X'X)_{2\times2} = \begin{bmatrix} n & \sum X_i \\ \sum X_i & \sum X_i^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 8 & 4 & 0 & -4 & -8 \end{bmatrix}\begin{bmatrix} 1 & 8 \\ 1 & 4 \\ 1 & 0 \\ 1 & -4 \\ 1 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1+1+1 & 8+4+0-4-8 \\ 8+4+0-4-8 & 8*8+4*4+0*0+-4*-4+-8*-8 \end{bmatrix} = \begin{bmatrix} \mathbf{5} & \mathbf{0} \\ \mathbf{0} & \mathbf{160} \end{bmatrix}$$

$$(X'X)^{-1} = \frac{1}{\Delta}\begin{bmatrix} \sum X_i^2 & -\sum X_i \\ -\sum X_i & \boldsymbol{n} \end{bmatrix}$$

$$\Delta = \boldsymbol{n}\sum X_i^2 - \sum X_i \sum X_i = \mathbf{5 * 160 - 0 = 800}$$

$$(\boldsymbol{X'X})^{-1} = \frac{\mathbf{1}}{\mathbf{800}}\begin{bmatrix}\mathbf{160} & \mathbf{0}\\ \mathbf{0} & \mathbf{5}\end{bmatrix} = \begin{bmatrix}\mathbf{0.2} & \mathbf{0}\\ \mathbf{0} & \mathbf{0.00625}\end{bmatrix}_{\mathbf{2*2}}$$

(3) $$(\boldsymbol{X'}_{\mathbf{2\times n}}\boldsymbol{Y}_{\boldsymbol{n}\times\mathbf{1}})_{\mathbf{2\times 1}} = \begin{bmatrix}\sum Y_i\\ \sum X_iY_i\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1 & 1\\ 8 & 4 & 0 & -4 & -8\end{bmatrix}\begin{bmatrix}7.8\\ 9.0\\ 10.2\\ 11.0\\ 11.7\end{bmatrix}$$

$$= \begin{bmatrix}7.8 + 9 + 10.2 + 11 + 11.7\\ 8 * 7.8 + 4 * 9 + 0 * 10.2 - 4 * 11 - 8 * 11.7\end{bmatrix} = \begin{bmatrix}\mathbf{49.7}\\ \mathbf{-39.2}\end{bmatrix}_{\mathbf{2*1}}$$

(4) $$\mathbf{B}_{\mathbf{2\times 1}} = (\boldsymbol{X'X})^{-\mathbf{1}}\boldsymbol{X'Y} = \begin{bmatrix}0.2 & 0\\ 0 & 0.00625\end{bmatrix}\begin{bmatrix}49.7\\ -39.2\end{bmatrix}$$

$$= \begin{bmatrix}0.2 * 49.7 + 0 * -39.2\\ 0 * 49.7 + 0.00625 * -39.2\end{bmatrix} = \begin{bmatrix}\mathbf{9.94}\\ \mathbf{-0.245}\end{bmatrix}$$

$$\boldsymbol{\hat{Y} = 9.940 - 0.245X}$$

(5) $$\hat{\boldsymbol{Y}}_{\boldsymbol{n}\times\mathbf{1}} = \boldsymbol{X}_{\boldsymbol{n}\times\mathbf{2}}\mathbf{B}_{\mathbf{2}\times\mathbf{1}} = \begin{bmatrix}1 & 8\\ 1 & 4\\ 1 & 0\\ 1 & -4\\ 1 & -8\end{bmatrix}\begin{bmatrix}9.94\\ -0.245\end{bmatrix} = \begin{bmatrix}9.94 - 0.245 * 8\\ 9.94 - 0.245 * 4\\ 9.94 - 0.245 * 0\\ 9.94 + 0.245 * 4\\ 9.94 + 0.245 * 8\end{bmatrix} = \begin{bmatrix}\mathbf{7.98}\\ \mathbf{8.96}\\ \mathbf{9.94}\\ \mathbf{10.92}\\ \mathbf{11.9}\end{bmatrix}$$

$$\boldsymbol{e}_{\boldsymbol{n}\times\mathbf{1}} = \boldsymbol{Y}_{\boldsymbol{n}\times\mathbf{1}} - \hat{\boldsymbol{Y}}_{\boldsymbol{n}\times\mathbf{1}} = \begin{bmatrix}7.8\\ 9.0\\ 10.2\\ 11.0\\ 11.7\end{bmatrix} - \begin{bmatrix}7.98\\ 8.96\\ 9.94\\ 10.92\\ 11.9\end{bmatrix} = \begin{bmatrix}7.8 - 7.98\\ 9 - 8.96\\ 10.2 - 9.94\\ 11 - 10.92\\ 11.7 - 11.9\end{bmatrix} = \begin{bmatrix}\mathbf{-0.18}\\ \mathbf{0.04}\\ \mathbf{0.26}\\ \mathbf{0.08}\\ \mathbf{-0.2}\end{bmatrix}$$

$$\boldsymbol{e'}_{\mathbf{1}\times\boldsymbol{n}}\boldsymbol{e}_{\boldsymbol{n}\times\mathbf{1}} = \left[\sum e_i^2\right] = [-0.18 \quad 0.04 \quad 0.26 \quad 0.08 \quad -0.2]\begin{bmatrix}-0.18\\ 0.04\\ 0.26\\ 0.08\\ -0.2\end{bmatrix} = [0.148]$$

$$MSE = \frac{\boldsymbol{e'e}}{n-2} = \frac{0.148}{3} = 0.049333$$

$$\boldsymbol{V}(\mathbf{B}) = MSE(\boldsymbol{X'X})^{-\mathbf{1}} = 0.049333\begin{bmatrix}0.2 & 0\\ 0 & 0.00625\end{bmatrix} = \begin{bmatrix}\mathbf{0.009867} & \mathbf{0}\\ \mathbf{0} & \mathbf{0.000308}\end{bmatrix}$$

$$\boldsymbol{V}\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \begin{bmatrix}Var(\widehat{\beta_0}) & cov(\widehat{\beta_0},\widehat{\beta_1})\\ cov(\widehat{\beta_0},\widehat{\beta_1}) & Var(\widehat{\beta_1})\end{bmatrix}$$

By use Minitab:
Input the data in worksheet

i	C1	C2	C3
1	8	7.8	1
2	4	9.0	1
3	0	10.2	1
4	-4	11.0	1
5	-8	11.7	1

To use **Session window**, command language must be enabled.
go to Session window and click "Enable the command prompt (MTB>) to view and generate command language".

Editor >> Enable commands

Then write in project window

```
MTB > Copy C3 C1 m1
MTB > Print m1
```
$X_{n\times 2}$

```
Data Display
 Matrix M1
1    8
1    4
1    0
1   -4
1   -8
```

```
MTB > tran m1 m2      % Transpose matrix
MTB > print m2
```
$X'_{2\times n}$

```
Data Display
 Matrix M2
1   1   1    1    1
8   4   0   -4   -8
```

```
MTB > mult m2 m1 m3
MTB > print m3
```
$(X'X)_{2\times 2}$

```
Data Display
 Matrix M3
5    0
0  160
```

$$(X'X)_{2\times 2} = \begin{bmatrix} n & \sum X_i \\ \sum X_i & \sum X_i^2 \end{bmatrix}$$

```
MTB > inver m3 m4
MTB > print m4
```
$(X'X)^{-1}_{2\times2}$

```
Data Display
 Matrix M4
0.2   0.00000
0.0   0.00625

MTB > copy c2 m5
MTB > Print m5
```
$Y_{n\times1}$

```
Data Display
 Matrix M5
 7.8
 9.0
10.2
11.0
11.7

MTB > mult m2 m5 m6
MTB > print m6
```
$X'_{2\times n}Y_{n\times1} = X'Y_{2\times1}$

```
Data Display
 Matrix M6
 49.7
-39.2

MTB > mult m4 m6 m7
MTB > print m7
```
$(X'X)^{-1}_{2\times2}(X'Y)_{2\times1} = \mathrm{B}_{2\times1}$

```
Data Display
 Matrix M7
 9.940
-0.245
```
Then $\hat{Y} = 9.940 - 0.245X$

```
MTB > tran m5 m13
```
$Y'_{n\times1}$

```
MTB > print m13

Data Display
 Matrix M13
7.8   9   10.2   11   11.7

MTB > mult m13 m5 m14
```
$Y'_{1\times n}Y_{n\times1}$

```
Answer = 503.7700

MTB > mult m1 m7 m8
```
$\hat{Y}_{n\times1} = X_{n\times2}\mathrm{B}_{2\times1}$

```
MTB > print m8

Data Display
 Matrix M8
 7.98
 8.96
 9.94
10.92
11.90

MTB > copy m8 c4            % write m8 in column c4 in worksheet.
MTB > Let c5 = 'y'-C4       % or   Let c5 = c2-C4   or   MTB > Subt m5 m8 m20
                                                         MTB > print m20

MTB > copy c5 m9        e
MTB > tran m9 m10
MTB > print m10         e'

Data Display
 Matrix M10
-0.18  0.04  0.26  0.08  -0.2
MTB > mult m10 m9 m11     e'e

Answer = 0.1480
```

$$MSE = \frac{e'e}{n-2} = \frac{0.1480}{3} = 0.049333$$

```
MTB > mult 0.049333 m4 m12
MTB > print m12
```

$$V(\mathbf{B}) = MSE(X'X)^{-1}$$

```
Data Display
 Matrix M12
0.0098666  0.0000000
0.0000000  0.0003083
```

$$V\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \begin{bmatrix} Var(\widehat{\beta_0}) & cov(\widehat{\beta_0},\widehat{\beta_1}) \\ cov(\widehat{\beta_0},\widehat{\beta_1}) & Var(\widehat{\beta_1}) \end{bmatrix}$$

By use MENU in MINITAB 16 window :

Stat>>Regression >> Regression>> General Regression
Responses: Y
Model : X

Regression Analysis: y versus x

Analysis of Variance

Source	DF	Adj SS	Adj MS	F-Value	P-Value
Regression	1	9.6040	9.60400	194.68	0.001
x	1	9.6040	9.60400	194.68	0.001
Error	3	0.1480	0.04933		
Total	4	9.7520			

Model Summary

S	R-sq	R-sq(adj)	R-sq(pred)
0.222111	98.48%	97.98%	94.11%

Coefficients

Term	Coef	SE Coef	T-Value	P-Value	VIF
Constant	9.9400	0.0993	100.07	0.000	
x	-0.2450	0.0176	-13.95	0.001	1.00

Regression Equation

$$y = 9.9400 - 0.2450 x$$