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Published online: 15 Oct 2012.

To cite this article: A. R. Shafay, N. Balakrishnan & K. S. Sultan (2014) Two-sample Bayesian prediction for sequential order statistics from exponential distribution based on multiply Type-II censored samples, Journal of Statistical Computation and Simulation, 84:3, 526-544, DOI: 10.1080/00949655.2012.718779

To link to this article: http://dx.doi.org/10.1080/00949655.2012.718779

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Two-sample Bayesian prediction for sequential order statistics from exponential distribution based on multiply Type-II censored samples

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(Received 28 April 2012; final version received 2 August 2012)

In this paper, the problem of predicting the future sequential order statistics based on observed multiply Type-II censored samples of sequential order statistics from one- and two-parameter exponential distributions is addressed. Using the Bayesian approach, the predictive and survival functions are derived and then the point and interval predictions are obtained. Finally, two numerical examples are presented for illustration.

Keywords: sequential order statistics; multiple Type-II censoring; Bayesian prediction; one-parameter exponential distribution; two-parameter exponential distribution; one-sample prediction; two-sample prediction

AMS 2000 Subject Classification: Primary: 62G30; Secondary: 62F15

1. Introduction

Sequential order statistics have been introduced as an extension of (ordinary) order statistics to model 'sequential k-out-of-n systems', where the failures of components possibly affect the surviving ones. This can be thought of as a damage caused by failures or as an increased stress put on the successively remaining components. The model of sequential order statistics is flexible in the sense that, upon each failure, the underlying distribution of the residual lifetimes of the surviving components may change. For a more detailed discussion, we refer the readers to Kamps [1] and Cramer and Kamps [2].

Let the ordered random variables $X_{s}^{(1)} \leq X_{s}^{(2)} \leq \cdots \leq X_{s}^{(n)}$ denote the sequential order statistics, and $x^{(1)} \leq x^{(2)} \leq \cdots \leq x^{(n)}$ be the corresponding observations of failure times. Moreover, let $F_{j}$ be the lifetime distribution at the start of the experiment, successively changing to $F_{j}$ at the time of the $(j-1)$th failure, for $2 \leq j \leq n$. The ageing behaviour of the components in the system is determined by the form and shape of the underlying distributions $F_{1}, \ldots, F_{n}$. Here, these
distribution functions are chosen as

\[ F_j = 1 - (1 - F)^{a_j}, \quad 1 \leq j \leq n, \]

with an absolutely continuous distribution function (cdf) \( F \), a corresponding probability density function (pdf) \( f \), and positive real model parameters \( \alpha_1, \ldots, \alpha_n \). These assumptions imply that the failure rate of the components at work after the \( j \)th failure is given by \( \alpha_{j+1}/(1 - F) \). Therefore, the model parameter \( \alpha_{j+1} \) reflects the influence of the \( j \)th failure on the surviving components in a natural way.

The joint density function of the sequential order statistics, \( X_1^{(1)}, X_2^{(2)}, \ldots, X_n^{(n)} \), is given by

\[
f_{X_1^{(1)}, X_2^{(2)}, \ldots, X_n^{(n)}}(x_1, \ldots, x_n) = n! \left( \prod_{j=1}^{n-1} \alpha_j \right) \left( \prod_{i=1}^{n-1} [1 - F(x_i)]^{a_i} f(x_i) \right) [1 - F(x_n)]^{a_n-1} f(x_n),
\]

where \( m_i = (n - i + 1)\alpha_i - (n - i)\alpha_{i+1} - 1 \) for \( i = 1, \ldots, n - 1 \) (see [1]).

Several well-known models of ordered random variables are included in this particular model of sequential order statistics. For instance, the choice \( \alpha_1 = \cdots = \alpha_n = 1 \) yields the usual order statistics, and setting

\[ \alpha_j = \frac{(N - j + 1 - \sum_{i=1}^{j-1} R_i)}{(n - j + 1)}, \quad 1 \leq j \leq n, \]

with \( R_j \) and \( N \) as non-negative integers such that \( N \geq n \) and \( \sum_{i=1}^{n} R_i = N - n \) lead to the model of progressively Type-II censored order statistics (see [3,4]).

Statistical methods based on sequential order statistics may be found in the review article by Cramer and Kamps [3], and in the recent papers by Revathy and Chandrasekar [5], Balakrishnan et al. [6], Beutner [7,8], Beutner and Kamps [9], Balakrishnan et al. [10], Bedbur et al. [11], Burkschat [12], and Burkschat et al. [13].

In many practical problems, one would wish to use previous data to predict a future observation from the same population. One way to do this is to construct an interval that will contain the future observation with a specified probability. Such an interval is called a prediction interval. Bayesian prediction bounds for future observations from the exponential distribution have been discussed by several authors, including Dunsmore [14], Lingappaiah [15], Evans and Nigm [16], Al-Hussaini and Jaheen [17], Abdel-Aty et al. [18], and Mohie El-Din et al. [19].

Recently, Schenk et al. [20] considered the problems of Bayesian estimation and prediction based on observed multiply Type-II censored samples of sequential order statistics from one- and two-parameter exponential distributions. In the case of the one-parameter exponential distribution, they obtained explicit representations for the posterior distribution and the corresponding Bayes estimator under squared error loss as well as the predictor of a future failure time along with its predictive posterior distribution function. In the case of the two-parameter exponential distribution, they obtained explicit representations for the posterior distribution and the corresponding Bayes estimators under squared error loss. But, these representations unfortunately contain some errors. In this paper, the point and interval predictions for the sequential order statistics from a future sample from one-parameter exponential distribution based on observed multiply Type-II censored samples of sequential order statistics from the same distribution are first presented. In the two-parameter case, the correct representations for the posterior distribution and the corresponding Bayes estimators are given. In addition, the prediction of future sequential order statistics from the two-parameter exponential distribution based on multiply Type-II censored samples of sequential order statistics is also discussed in detail.

The rest of this paper is organized as follows. In Section 2, the description of the model of the multiply Type-II censored sample of sequential order statistics is presented. The problem of
predicting the sequential order statistics from a future sample from one-parameter exponential distribution is discussed in Section 3. In Section 4, the prediction of sequential order statistics from a future sample from the two-parameter exponential distribution is discussed. Finally, in Section 5, two numerical examples are presented for the purpose of illustrating all the inferential methods developed here.

2. The model description

In reliability analysis, experiments often get terminated before all units on test have failed due to cost and time considerations. In such cases, failure information is available only on part of the sample, and one has only partial information on all units that had not failed. Such data are said to be censored data. There are several forms of censored data. One of the most common forms of censoring is Type-II censoring. In Type-II censoring scheme, only the first said to be censored data. There are several forms of censored data. One of the most common forms of censoring is Type-II censoring. In Type-II censoring scheme, only the first failure times \( x^{(1)} \leq x^{(2)} \leq \cdots \leq x^{(r)} \) would have been observed and the rest of the data will be known only to be larger than \( x^{(r)} \). A generalization of Type-II censoring scheme is multiple Type-II censoring scheme. Under this scheme, we observe only the \( j_1 \), \( j_2 \), \ldots, \( j_p \)th failure times \( x^{(j_1)} \leq x^{(j_2)} \leq \cdots \leq x^{(j_p)} \), where \( 1 \leq j_1 < j_2 < \cdots < j_q \leq n \), and the rest of data are not available. Particular applications of such censoring are found in reliability theory and survival analysis. Surveys regarding censored data can be found in Nelson [21], Balakrishnan and Cohen [22], Cohen [23], Meeker and Escobar [24], Balakrishnan and Aggarwala [25], and McCool [26]. For a survey on multiple Type-II censoring, one may refer to Kong [27].

Schenk [28] derived the joint density function of multiply Type-II censored sequential order statistics \( X_{s}^{(j)} \leq X_{s}^{(j_2)} \leq \cdots \leq X_{s}^{(j_q)} \) as

\[
f_{\tilde{X},(\tilde{x})} = \left( \prod_{j=p+1}^{q} \prod_{k=j_{p-1}+1}^{j_p} \gamma_k \right) \prod_{p=1}^{q} \sum_{k=j_{p-1}+1}^{j_p} a_{k+p-1}^{(j_p)}(j_p) \left( \frac{1 - F(x^{(j_p)})}{1 - F(x^{(j_{p-1})})} \right)^{\gamma_k} \times \frac{f(x^{(j_p)})}{1 - F(x^{(j_p)})}, \quad x^{(j_1)} < \cdots < x^{(j_q)},
\]

where \( F \) and \( f \) are the parent cdf and pdf, respectively, \( \tilde{X}_s = (X_s^{(j_1)}, \ldots, X_s^{(j_q)}) \) is the vector of observable variables, \( \tilde{x} = (x^{(j_1)}, \ldots, x^{(j_q)}) \) is a vector of realizations, and \( j_0 = 0 \); the constants are given by \( \gamma_k = \alpha_k(n-k+1), 1 \leq k \leq n \), and

\[
a_{k+p-1}^{(j_p)}(j_p) = \prod_{i=j_{p-1}+1}^{j_p} \frac{1}{\gamma_i - \gamma_k}, \quad j_{p-1} + 1 \leq k \leq j_p, \quad 1 \leq p \leq q.
\]

Notice that the above density representation is only valid for pairwise different parameters, i.e. \( \gamma_i \neq \gamma_k \) for all \( i, k \in [j_{p-1} + 1, \ldots, j_p] \), \( 1 \leq p \leq q \), which is assumed throughout this paper. Clearly, this is no restriction for the usual order statistics and progressively Type-II censored order statistics.

We consider here data set consisting of \( s \) independent multiply Type-II censored samples of sequential order statistics. The \( i \)th sample contains \( q_i (\geq 1) \) observations for \( 1 \leq i \leq s \), i.e., the data from the \( i \)th sample are given by

\[
x_{i}^{(j_{1i})} \leq x_{i}^{(j_{2i})} \leq \cdots \leq x_{i}^{(j_{qi})} \quad \text{with} \quad 0 < j_{1i} < j_{2i} < \cdots < j_{qi} \leq n_i.
\]

The corresponding sequential order statistics \( X_{w}^{(j_{i})} \leq \cdots \leq X_{w}^{(j_{qi})} \) are assumed to be independent with respect to index \( i \).
Let \( X_{s+1}^{(1)} \leq X_{s+1}^{(2)} \leq \cdots \leq X_{s+1}^{(n_{s+1})} \) be a future independent sample of sequential order statistics from the same population. In the following sections, the Bayesian prediction of the failure time \( X_{s+1}^{(j)} \) in the future \((s + 1)\)th sample along with its predictive distribution is developed. The marginal density function of the \( j \)th sequential order statistic \( X_{s+1}^{(j)} \) is given by (see [1])

\[
f^{X_{s+1}^{(j)}}(x) = \left( \prod_{k=1}^{j} \gamma_{s+1,k} \right) \sum_{k=1}^{j} a_k(j)f(x)[1 - F(x)]^{\gamma_{s+1,k}-1},
\]

where

\[
a_k(j) = a_k^{(0)}(j) = \prod_{i=1}^{j} \frac{1}{\gamma_{s+1,i} - \gamma_{s+1,k}}, \quad 1 \leq k \leq j, \quad 1 \leq j \leq n_{s+1}.
\]

3. One-parameter exponential distribution

In this section, the underlying distribution is assumed to be a one-parameter exponential distribution with scale parameter \( \sigma > 0 \), and with pdf and cdf as

\[
f(x; \sigma) = \frac{1}{\sigma} \exp \left( -\frac{x}{\sigma} \right), \quad x \geq 0, \quad \sigma > 0,
\]

and

\[
F(x; \sigma) = 1 - \exp \left( -\frac{x}{\sigma} \right), \quad x \geq 0, \quad \sigma > 0,
\]

respectively. For an excellent survey on the exponential distribution, interested readers may refer to the book by Balakrishnan and Basu [29].

The parameter \( \sigma \) is assumed to be a realization of a random variable \( \Sigma \) with prior density function given by

\[
\pi^{\Sigma} (\sigma) \propto \sigma^{-(b+1)} \exp \left( -\frac{a}{\sigma} \right), \quad \sigma > 0,
\]

which is an inverted gamma density, for \( a > 0, b > 0 \).

Schenk et al. [20] derived explicit representations for the posterior distribution of \( \Sigma \) and the corresponding Bayes estimator under squared error loss. These representations, presented in the following lemma without proof, will be used here to derive predictive and survival functions for the sequential order statistics from a future sample and the corresponding Bayesian prediction.
Lemma 3.1 Let us denote
\[ \tilde{X}_{a_l} = (X_{a_l}^{(i_1)}, \ldots, X_{a_l}^{(i_{k_1})}), \quad \tilde{x}_i = (x_i^{(j_1)}, \ldots, x_i^{(j_{q_i})}), \quad i = 1, \ldots, s, \]
\[ \tilde{X} = (\tilde{X}_{a_1}, \ldots, \tilde{X}_{a_s}), \quad \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_s), \]
\[ \tilde{K} = \{\tilde{k} = (k_{i_1}, \ldots, k_{q_i}) | k_{ip} \in \{j_{i,p-1} + 1, \ldots, j_{ip}\}, 1 \leq p \leq q_i, 1 \leq i \leq s\}, \]
\[ \Psi(\tilde{k}) = \prod_{i=1}^{s} \prod_{p=1}^{q_i} a_{k_{ip}}^{(j_{i,p-1})}(j_{ip}), \quad Q = \sum_{i=1}^{s} q_i \]
and
\[ V_\tilde{k} = \sum_{i=1}^{s} \sum_{p=1}^{q_i} \gamma_{k_{ip}}(x_i^{(j_0)} - x_i^{(j_{i,p-1})}) + a. \]

Then, the posterior distribution of \( \Sigma \), given \( \tilde{X} = \tilde{x} \), is given by
\[ \pi^{\Sigma|\tilde{X}}(\sigma | \tilde{x}) = C_1^{-1} \sigma^{-(Q+b+1)} \sum_{\tilde{k} \in \tilde{K}} \Psi(\tilde{k}) \exp \left( -\frac{V_\tilde{k}}{\sigma} \right), \tag{9} \]
with the normalizing constant
\[ C_1 = \Gamma(Q + b) \sum_{\tilde{k} \in \tilde{K}} \Psi(\tilde{k})(V_\tilde{k})^{-(Q+b)}. \tag{10} \]

Therefore, the Bayes estimator of \( \sigma \) under the squared error loss is given by
\[ \hat{\sigma} = \frac{1}{Q + b - 1} \frac{\sum_{\tilde{k} \in \tilde{K}} \Psi(\tilde{k})(V_\tilde{k})^{-(Q+b-1)}}{\sum_{\tilde{k} \in \tilde{K}} \Psi(\tilde{k})(V_\tilde{k})^{-(Q+b)}}, \tag{11} \]
and the posterior second moment of \( \sigma \) is given by
\[ \hat{\sigma}^2 = \frac{1}{(Q + b - 1)(Q + b - 2)} \frac{\sum_{\tilde{k} \in \tilde{K}} \Psi(\tilde{k})(V_\tilde{k})^{-(Q+b-2)}}{\sum_{\tilde{k} \in \tilde{K}} \Psi(\tilde{k})(V_\tilde{k})^{-(Q+b)}}. \tag{12} \]

Theorem 3.2 Using the notation of Lemma 3.1, the predictive density function of \( X_{s+1}^{(j)} \), given \( \tilde{X} = \tilde{x} \), is given by
\[ f_{X_{s+1}^{(j)}|\tilde{X}^{(j)}}(x | \tilde{x}) = C_1^{-1} \Gamma(Q + b + 1) \left( \prod_{k=1}^{j} \gamma_{s+1,k} \right) \sum_{k=1}^{j} a_k(j) \sum_{\tilde{k} \in \tilde{K}} \Psi(\tilde{k})(\nu_{s+1,k}(x))^{-(Q+b+1)}, \tag{13} \]
where \( a_k(j) \) is as in Equation (5) and \( \nu_{s+1,k}(x) = V_{k} + \gamma_{s+1,k}x \). Therefore, the predictive survival function of \( X_{s+1}^{(j)} \), given \( \tilde{X} = \tilde{x} \), is given by
\[ \tilde{F}_{X_{s+1}^{(j)}|\tilde{X}^{(j)}}(t | \tilde{x}) = C_1^{-1} \Gamma(Q + b) \left( \prod_{k=1}^{j} \gamma_{s+1,k} \right) \sum_{k=1}^{j} a_k(j) \sum_{\tilde{k} \in \tilde{K}} \Psi(\tilde{k})(\nu_{s+1,k}^{-1}(t))^{-(Q+b)}. \]
Proof The conditional independence of $X_{s+1}, \ldots, X_{s+1}$ and $X_{s+1} = (X_{s+1}^{(1)}, X_{s+1}^{(2)}, \ldots, X_{s+1}^{(q-1)})$ readily yields

$$f_{X_{s+1}^{(j)}}(x) = f_{X_{s+1}}(x)$$

$$= \left( \prod_{k=1}^{j} \gamma_{s+1,k} \right) \sum_{k=1}^{j} a_k(j) \exp \left( -\frac{\gamma_{s+1,k}}{\sigma} x \right), \; x \geq 0.$$

Thus, the predictive density function of $X_{s+1}^{(j)}$, given $X = \bar{x}$, is given by

$$f_{X_{s+1}^{(j)}}(x|\bar{x}) = \int_{0}^{\infty} f_{X_{s+1}^{(j)}}(x|\bar{x}) \pi \Sigma|X(x|\bar{x}) \, d\sigma$$

$$= C_{-1}^{-1} \left( \prod_{k=1}^{j} \gamma_{s+1,k} \right) \sum_{k=1}^{j} a_k(j) \sum_{k \in K} \Psi(\bar{x}k) \int_{0}^{\infty} \sigma^{-(Q+b+2)} \exp \left( -\frac{V_{s+1,k}}{\sigma} \right) \, d\sigma$$

$$= C_{-1}^{-1} \Gamma(Q + b + 1) \left( \prod_{k=1}^{j} \gamma_{s+1,k} \right) \sum_{k=1}^{j} a_k(j) \sum_{k \in K} \Psi(\bar{x}k)(V_{s+1,k}(x))^{-(Q+b+1)}. \quad (14)$$

From Equation (14), the predictive survival function of $X_{s+1}^{(j)}$, given $X = \bar{x}$, is simply obtained as

$$F_{X_{s+1}^{(j)}}(t|\bar{x}) = \int_{t}^{\infty} f_{X_{s+1}^{(j)}}(x|\bar{x}) \, dx$$

$$= C_{-1}^{-1} \Gamma(Q + b) \left( \prod_{k=1}^{j} \gamma_{s+1,k} \right) \sum_{k=1}^{j} a_k(j) \sum_{k \in K} \Psi(\bar{x}k)(V_{s+1,k}(t))^{-(Q+b)}.$$

This completes the proof of the theorem. \hfill \blacksquare

Lemma 3.3 Let $r \in N, \gamma_l > 0, 1 \leq l \leq r$, and $\gamma_l \neq \gamma_j$ for $l,j \in \{1, \ldots, r\}, l \neq j$, and define

$$a_j(r) = a_j^{(0)}(r) = \prod_{l=1}^{r} \frac{1}{\gamma_l - \gamma_j}, \; r \geq 2 \text{ (with } a_1(1) = 1).$$

Moreover, let $l_0 \in N_0$ and $l_r = 0$. Then,

$$\left( \prod_{j=1}^{r} \frac{1}{\gamma_j} \right) \sum_{j=1}^{r} \frac{1}{\gamma_{l_0+1}} a_j(r) = \sum_{l_0 \geq l_1 \geq \cdots \geq l_{r-1} \geq 0} \prod_{j=1}^{r} \gamma_{l_j-1} - l_{r-j}. \quad (15)$$

In particular, for $l_0 = 0$,

$$\left( \prod_{j=1}^{r} \frac{1}{\gamma_j} \right) \sum_{j=1}^{r} \frac{1}{\gamma_j} a_j(r) = 1; \quad (16)$$

for $l_0 = 1$,

$$\left( \prod_{j=1}^{r} \frac{1}{\gamma_j} \right) \sum_{j=1}^{r} \frac{1}{\gamma_j} a_j(r) = \sum_{j=1}^{r} \frac{1}{\gamma_j}; \quad (17)$$
Thus, we have
\[
\left( \prod_{j=1}^{r} Y_j \right) \sum_{j=1}^{r} \frac{1}{Y_j^3} a_j(r) = \sum_{j=1}^{r} \frac{1}{Y_j^2} + \sum_{l,k=1}^{r} \frac{1}{Y_l Y_k}.
\] (18)

For a proof of this lemma, one may refer to Schenk et al. [20].

**Theorem 3.4** The predictive failure of the jth sequential order statistic in the (s + 1)th sample, given \( \tilde{X} = \tilde{x} \), is given by
\[
\hat{X}_{s+1}^{(j)} = \hat{\sigma} \sum_{k=1}^{j} \frac{1}{\gamma_{s+1,k}},
\] (19)

and the posterior variance of \( \hat{X}_{s+1}^{(j)} \) is given by
\[
\text{Var}(\hat{X}_{s+1}^{(j)}|\tilde{X}) = (\hat{\sigma}^2 - \hat{\sigma}^2) \left( \sum_{k=1}^{j} \frac{1}{\gamma_{s+1,k}} \right)^2 + \hat{\sigma}^2 \sum_{k=1}^{j} \frac{1}{\gamma_{s+1,k}^2}.
\] (20)

**Proof** The predictor of \( X_{s+1}^{(j)} \) is given by the mean of the predictive density function of \( X_{s+1}^{(j)} \). Thus, we have
\[
\hat{X}_{s+1}^{(j)} = E(X_{s+1}^{(j)}|\tilde{X}) = \int_{0}^{\infty} tf_{X_{s+1}^{(j)}}(t|\tilde{x}) \, dt
\]
\[
= C_1^{-1} \Gamma(Q + b + 1) \left( \prod_{k=1}^{j} \gamma_{s+1,k} \right) \sum_{k=1}^{j} a_k(j) \sum_{k \in \tilde{K}} \Psi(\tilde{k}) \int_{0}^{\infty} t(V_{s+1,k}(t))^{-(Q+b+1)} \, dt.
\]

Upon using integration by parts, \( \hat{X}_{s+1}^{(j)} \) becomes
\[
\hat{X}_{s+1}^{(j)} = C_1^{-1} \Gamma(Q + b - 1) \left( \prod_{k=1}^{j} \gamma_{s+1,k} \right) \sum_{k=1}^{j} a_k(j) \left( \sum_{k \in \tilde{K}} \Psi(\tilde{k})(V_{s+1,k})^{-(Q+b-1)} \right).
\] (21)

Upon using Equations (16) and (17), the representation in Equation (21) simplifies to
\[
\hat{X}_{s+1}^{(j)} = \hat{\sigma} \sum_{k=1}^{j} \frac{1}{\gamma_{s+1,k}}.
\]

Similarly, the posterior second moment of \( X_{s+1}^{(j)} \) is given by
\[
E(X_{s+1}^{(j)^2}|\tilde{X}) = 2C_1^{-1} \Gamma(Q + b - 2) \left( \prod_{k=1}^{j} \gamma_{s+1,k} \right) \sum_{k=1}^{j} \frac{a_k(j)}{(\gamma_{s+1,k})^3} \sum_{k \in \tilde{K}} \Psi(\tilde{k})(V_{s+1,k})^{-(Q+b-2)}
\]
\[
= 2\hat{\sigma}^2 \left( \sum_{k=1}^{j} \frac{1}{\gamma_{s+1,k}^2} + \sum_{l,k=1}^{j} \frac{1}{\gamma_{s+1,l} \gamma_{s+1,k}} \right).
\]
Thus, the posterior variance of $\hat{X}_{s+1}^{(j)}$ is given by

$$\text{Var}(\hat{X}_{s+1}^{(j)} | \bar{X}) = (\sigma^2 - \hat{\sigma}^2) \left( \sum_{k=1}^{j} \frac{1}{\gamma_{s+1,k}} \right)^2 + \hat{\sigma}^2 \sum_{k=1}^{j} \frac{1}{\gamma_{s+1,k}},$$

which completes the proof of the theorem. ■

Remark 3.5 The Bayesian predictive bounds of a two-sided equi-tailed 100(1 - \tau)\% interval for the jth sequential order statistic in the (s + 1)th sample, given $\bar{X} = \bar{x}$, can be obtained by solving the following two equations:

$$\tilde{F}_{X_{s+1}^{(j)}}(L|\bar{x}) = 1 - \frac{\tau}{2} \quad \text{and} \quad \tilde{F}_{X_{s+1}^{(j)}}(U|\bar{x}) = \frac{\tau}{2},$$

where $\tilde{F}_{X_{s+1}^{(j)}}(t|\bar{x})$ is the predictive survival function of $X_{s+1}^{(j)}$ given in Theorem 3.2, and $L$ and $U$ denote the lower and upper bounds, respectively.

For the highest posterior density (HPD) method, the following two equations need to be solved:

$$\tilde{F}_{X_{s+1}^{(j)}}(L_{X_{s+1}^{(j)}}|\bar{x}) - \tilde{F}_{X_{s+1}^{(j)}}(U_{X_{s+1}^{(j)}}|\bar{x}) = 1 - \tau$$

and

$$f_{X_{s+1}^{(j)}}(L_{X_{s+1}^{(j)}}|\bar{x}) = f_{X_{s+1}^{(j)}}(U_{X_{s+1}^{(j)}}|\bar{x}),$$

where $f_{X_{s+1}^{(j)}}(x|\bar{x})$ is the predictive density function of $X_{s+1}^{(j)}$ given in Theorem 3.2, and $L_{X_{s+1}^{(j)}}$ and $U_{X_{s+1}^{(j)}}$ denote the HPD lower and upper bounds, respectively.

4. Two-parameter exponential distribution

In this section, the underlying distribution is assumed to be a two-parameter exponential distribution with location parameter $\mu \in \mathbb{R}$ and scale parameter $\sigma > 0$, and with pdf and cdf as

$$f(x; \sigma, \mu) = \frac{1}{\sigma} \exp \left( -\frac{x - \mu}{\sigma} \right), \quad x \geq \mu, \quad \sigma > 0, \quad \mu \in \mathbb{R}, \quad (22)$$

and

$$F(x; \sigma, \mu) = 1 - \exp \left( -\frac{x - \mu}{\sigma} \right), \quad x \geq \mu, \quad \sigma > 0, \quad \mu \in \mathbb{R}, \quad (23)$$

respectively. The parameters $\sigma$ and $\mu$ are assumed to be realizations of two random variables $\Sigma$ and $\Delta$ with joint prior density function given by

$$\pi_{\Sigma,\Delta}(\sigma, \mu) \propto \sigma^{-(b+1)} \exp \left( -\frac{a - c\mu}{\sigma} \right) 1_{[N,M]}(\mu), \quad \sigma > 0, \quad \mu \in \mathbb{R}, \quad (24)$$

which is a proper density for $a > cM$, $b > 1$, $c > 0$ and $M > N$ with $N, M \in \mathbb{R}$ (see [16,30] for related results with a similar prior). Notice that the choice $a = 0$, $b \in \mathbb{R}$, $c = 0$ and $M \to \infty, N \to -\infty$ leads to the improper prior $\pi_{\Sigma,\Delta}(\sigma, \mu) \propto \sigma^{-(b+1)}$, $\sigma > 0$, $\mu \in \mathbb{R}$.

Schenk et al. [20] derived explicit representations for the joint posterior distribution of $\Sigma$ and $\Delta$ and the corresponding Bayes estimators under the squared error loss. But, these representations
Unfortunately contain some errors. Here, the correct representations are presented in the following two lemmas without proof for the sake of brevity.

**Lemma 4.1** Under the notation in Lemma 3.1, let \( Q + b - 1 > 0 \) and

\[
z = \min\{x_1^{(j_1)}, \ldots, x_s^{(j_1)}\}, \quad M_0 = \min\{z, M\},
\]

\[
H_k(t) = V_k - t\left(c + \sum_{i=1}^s \gamma_{ki}\right)
\]

with \( N < z \) and \( H_k(M_0) > 0 \). Then, the joint posterior distribution of \( \Sigma \) and \( \Delta \), given \( \tilde{X} = \tilde{x} \), is given by

\[
\pi(\Sigma, \Delta | \tilde{x}, \mu) = C_2^{-1} \sigma^{-(Q+b+1)} \sum_{k \in K} \Psi(\tilde{k}) \exp\left(-\frac{H_k(\mu)}{\sigma}\right) 1_{[N,M_0]}(\mu),
\]

with the normalizing constant

\[
C_2 = \Gamma(Q + b - 1) \sum_{k \in K} \frac{\Psi(\tilde{k})}{(c + \sum_{i=1}^s \gamma_{ki})((H_k(M_0))^{-(Q+b-1)} - [H_k(N)]^{-(Q+b-1)})}.
\]

**Lemma 4.2** Under the notation in Lemma 4.1, if \( Q + b - 3 > 0 \), then the Bayes estimators of \( \sigma^2, \mu^2 \) and \( \sigma \mu \) under the squared error loss are given, respectively, by

\[
\hat{\sigma} = \frac{1}{Q + b - 2} \frac{1}{(Q + b - 2)} \sum_{k \in K} \frac{(\Psi(\tilde{k}))(H_k(M_0))^{-(Q+b-3)} - [H_k(N)]^{-(Q+b-3)}}{(c + \sum_{i=1}^s \gamma_{ki})(H_k(M_0))^{-(Q+b-3)} - [H_k(N)]^{-(Q+b-3)}},
\]

\[
\hat{\sigma}^2 = \frac{1}{(Q + b - 2)} \sum_{k \in K} \frac{(\Psi(\tilde{k}))(H_k(M_0))^{-(Q+b-3)} - [H_k(N)]^{-(Q+b-3)}}{(c + \sum_{i=1}^s \gamma_{ki})(H_k(M_0))^{-(Q+b-3)} - [H_k(N)]^{-(Q+b-3)}},
\]

\[
\hat{\mu} = \frac{1}{Q + b - 2} \sum_{k \in K} \frac{(\Psi(\tilde{k}))(H_k(M_0))^{-(Q+b-3)} - [H_k(N)]^{-(Q+b-3)}}{(c + \sum_{i=1}^s \gamma_{ki})(H_k(M_0))^{-(Q+b-3)} - [H_k(N)]^{-(Q+b-3)}},
\]

\[
\hat{\mu}^2 = \frac{1}{Q + b - 2} \sum_{k \in K} \frac{(\Psi(\tilde{k}))(H_k(M_0))^{-(Q+b-3)} - [H_k(N)]^{-(Q+b-3)}}{(c + \sum_{i=1}^s \gamma_{ki})(H_k(M_0))^{-(Q+b-3)} - [H_k(N)]^{-(Q+b-3)}}.
\]
and
\[
\sigma^2 = \frac{1}{Q + b - 2} \left( \frac{1}{Q + b - 2} \right) \left( \sum_{\tilde{k} \in \tilde{K}} (\Psi(\tilde{k})/(c + \sum_{i=1}^{s} Y_{i\tilde{k}_i}))((M_0[H^*_{k}(M_0)]^{-((Q+b-2)} - N[H^*_{k}(N)]^{-((Q+b-2)})
\right)
\]
\[
- \frac{1}{(Q + b - 2)(Q + b - 3)}
\times \left( \sum_{\tilde{k} \in \tilde{K}} (\Psi(\tilde{k})/(c + \sum_{i=1}^{s} Y_{i\tilde{k}_i}))((H^*_{k}(M_0)]^{-((Q+b-3)} - [H^*_{k}(N)]^{-((Q+b-3)})
\right)
\]
\[
\times \left( \sum_{\tilde{k} \in \tilde{K}} (\Psi(\tilde{k})/(c + \sum_{i=1}^{s} Y_{i\tilde{k}_i}))((H^*_{k}(M_0)]^{-((Q+b-1)} - [H^*_{k}(N)]^{-((Q+b-1)})
\right).
\]

**Theorem 4.3** Under the notation in Lemma 4.1, the predictive density function of \(X_{s+1}^{(j)}\), given \(\tilde{X} = \tilde{x}\), is given by
\[
f^{X_{s+1}^{(j)}|\tilde{x}}(x|\tilde{x}) = \begin{cases} 
    f_1^{X_{s+1}^{(j)}|\tilde{x}}(x|\tilde{x}), & N < x < M_0, \\
    f_2^{X_{s+1}^{(j)}|\tilde{x}}(x|\tilde{x}), & x \geq M_0,
\end{cases}
\]

where
\[
f_1^{X_{s+1}^{(j)}|\tilde{x}}(x|\tilde{x}) = C^{-1}_{2} \Gamma(Q + b) \left( \prod_{k=1}^{j} \gamma_{s+1,k} \right) \sum_{k \in \tilde{K}} a_k(j) \sum_{c = Y_{s+1,k} + \sum_{i=1}^{s} Y_{i\tilde{k}_i}} \Psi(\tilde{k})
\]
\[
\times ((H^*_{k}(x)]^{-((Q+b)} - [G^k_{s+1,\tilde{k}}(N,x)]^{-((Q+b)})
\]

and
\[
f_2^{X_{s+1}^{(j)}|\tilde{x}}(x|\tilde{x}) = C^{-1}_{2} \Gamma(Q + b) \left( \prod_{k=1}^{j} \gamma_{s+1,k} \right) \sum_{k \in \tilde{K}} a_k(j) \sum_{c = Y_{s+1,k} + \sum_{i=1}^{s} Y_{i\tilde{k}_i}} \Psi(\tilde{k})
\]
\[
\times ((G^k_{s+1,\tilde{k}}(M_0,x)]^{-((Q+b)} - [G^k_{s+1,\tilde{k}}(N,x)]^{-((Q+b)})
\]

with \(G^k_{s+1,\tilde{k}}(t,x) = H^*_{k}(t) + \gamma_{s+1,k}(x-t)\). Therefore, the predictive survival function of \(X_{s+1}^{(j)}\), given \(\tilde{X} = \tilde{x}\), is given by
\[
\tilde{F}^{X_{s+1}^{(j)}|\tilde{x}}(t|\tilde{x}) = \begin{cases} 
    \tilde{F}_1^{X_{s+1}^{(j)}|\tilde{x}}(t|\tilde{x}), & N < t < M_0, \\
    \tilde{F}_2^{X_{s+1}^{(j)}|\tilde{x}}(t|\tilde{x}), & t \geq M_0,
\end{cases}
\]

where
\[
\tilde{F}_1^{X_{s+1}^{(j)}|\tilde{x}}(t|\tilde{x}) = C^{-1}_{2} \Gamma(Q + b - 1) \left( \prod_{k=1}^{j} \gamma_{s+1,k} \right) \sum_{k \in \tilde{K}} a_k(j) \sum_{c = Y_{s+1,k} + \sum_{i=1}^{s} Y_{i\tilde{k}_i}} \Psi(\tilde{k})
\]
\[
\times \left[ \frac{1}{c + \sum_{i=1}^{s} Y_{i\tilde{k}_i}} \left( [H^*_{k}(M_0)]^{-((Q+b-1)} - [H^*_{k}(t)]^{-((Q+b-1)}) \right)
\right.
\]
\[
+ \frac{1}{Y_{s+1,k}} \left( [H^*_{k}(M_0)]^{-((Q+b-1)} - [G^k_{s+1,\tilde{k}}(N,t)]^{-((Q+b-1)}) \right]
\]

and
\[
\tilde{F}_2^{X_{s+1}^{(j)}|\tilde{x}}(t|\tilde{x}) = C^{-1}_{2} \Gamma(Q + b - 1) \left( \prod_{k=1}^{j} \gamma_{s+1,k} \right) \sum_{k \in \tilde{K}} a_k(j) \sum_{c = Y_{s+1,k} + \sum_{i=1}^{s} Y_{i\tilde{k}_i}} \Psi(\tilde{k})
\]
\[
\times \left( [G^k_{s+1,\tilde{k}}(M_0,t)]^{-((Q+b-1)} - [G^k_{s+1,\tilde{k}}(N,t)]^{-((Q+b-1)}) \right),
\]
Proof The predictive density function of $X_{s+1}^{(j)}$, given $\bar{X} = \bar{x}$, is given by

$$f^{X_{s+1}^{(j)}|\bar{X}}(x|\bar{x}) = \begin{cases} f^{X_{s+1}^{(j)}|\bar{X}}(x|\bar{x}), & N < x < M_0, \\ f^{X_{s+1}^{(j)}|\bar{X}}(x|\bar{x}), & x \geq M_0, \end{cases} \tag{34}$$

where

$$f^{X_{s+1}^{(j)}|\bar{X}}(x|\bar{x}) = \int_N^x \int_0^\infty f^{X_{s+1}^{(j)}|\bar{X}}(x|\bar{x}) \pi(\Sigma, \Delta|x) \, d\sigma \, d\mu$$

$$= C_2^{-1} \left( \prod_{k=1}^j \gamma_{s+1,k} \right) \sum_{k=1}^j a_k(j) \sum_{\tilde{k} \in K} \Psi(\tilde{k})$$

$$\times \int_N^x \int_0^\infty \sigma^{-(Q+b+2)} \exp \left( -\frac{G_{s+1,k}^{k}(\mu, x)}{\sigma} \right) \, d\sigma \, d\mu$$

$$= C_2^{-1} \Gamma(Q + b) \left( \prod_{k=1}^j \gamma_{s+1,k} \right) \sum_{k=1}^j a_k(j) \sum_{\tilde{k} \in K} \Psi(\tilde{k})$$

$$\times \int_N^{M_0} \int_0^\infty \sigma^{-(Q+b+2)} \exp \left( -\frac{G_{s+1,k}^{k}(\mu, x)}{\sigma} \right) \, d\sigma \, d\mu$$

$$= C_2^{-1} \Gamma(Q + b) \left( \prod_{k=1}^j \gamma_{s+1,k} \right) \sum_{k=1}^j a_k(j) \sum_{\tilde{k} \in K} \frac{\Psi(\tilde{k})}{c + \gamma_{s+1,k} + \sum_{i=1}^{s} \gamma_{ik}}$$

$$\times ([G_{s+1,k}^{k}(M_0, x)]^{-(Q+b)} - [G_{s+1,k}^{k}(N, x)]^{-(Q+b)}).$$

From Equation (34), the predictive survival function of $X_{s+1}^{(j)}$, given $\bar{X} = \bar{x}$, is simply obtained as

$$\bar{F}^{X_{s+1}^{(j)}|\bar{X}}(t|\bar{x}) = \begin{cases} \bar{F}^{X_{s+1}^{(j)}|\bar{X}}(t|\bar{x}), & N < t < M_0, \\ \bar{F}^{X_{s+1}^{(j)}|\bar{X}}(t|\bar{x}), & t \geq M_0, \end{cases}$$
where

\[
\tilde{F}_1^{X^{(j)}|\bar{x}}(t|\bar{x}) = \int_M f_1^{X^{(j)}|\bar{x}}(x|\bar{x}) \, dx + \int_{M_0}^\infty f_2^{X^{(j)}|\bar{x}}(x|\bar{x}) \, dx
\]

\[
= C_2^{-1} \Gamma(Q + b - 1) \left( \prod_{k=1}^j \gamma_{s+1,k} \right) \sum_{k \in K} a_k(j) \sum_{k \in K} \frac{\Psi(\tilde{k})}{c + \gamma_{s+1,k} + \sum_{i=1}^s \gamma_{k_i}} \times \left[ \frac{1}{\gamma_{s+1,k}} (\left[ H_k(M_0) \right]^{-(Q+b-1)} - \left[ H_k(t) \right]^{-(Q+b-1)}) + \frac{1}{\gamma_{s+1,k}} (\left[ G^k_{s+1,k}(M_0, t) \right]^{-(Q+b-1)} - \left[ G^k_{s+1,k}(N, t) \right]^{-(Q+b-1)}) \right]
\]

and

\[
\tilde{F}_2^{X^{(j)}|\bar{x}}(t|\bar{x}) = \int_M f_2^{X^{(j)}|\bar{x}}(x|\bar{x}) \, dx
\]

\[
= C_2^{-1} \Gamma(Q + b - 1) \left( \prod_{k=1}^j \gamma_{s+1,k} \right) \sum_{k \in K} a_k(j) \sum_{k \in K} \frac{\Psi(\tilde{k})}{c + \gamma_{s+1,k} + \sum_{i=1}^s \gamma_{k_i}} \times (\left[ G^k_{s+1,k}(M_0, t) \right]^{-(Q+b-1)} - \left[ G^k_{s+1,k}(N, t) \right]^{-(Q+b-1)})
\]

This completes the proof of the theorem. \[\blacksquare\]

**Theorem 4.4** The predictive failure of the jth sequential order statistic in the \((s + 1)\)th sample, given \(\bar{X} = \bar{x}\), is given by

\[
\hat{X}^{(j)}_{s+1} = \hat{\mu} + \hat{\sigma} \sum_{k=1}^j \frac{1}{\gamma_{s+1,k}},
\]

(35)

where \(\hat{\sigma}\) and \(\hat{\mu}\) are the Bayes estimators of \(\sigma\) and \(\mu\) as presented in Equations (27) and (29), respectively. The posterior variance of \(\hat{X}^{(j)}_{s+1}\) is given by

\[
\text{Var}(\hat{X}^{(j)}_{s+1}|\bar{X}) = (\hat{\mu}^2 - \hat{\mu}^2) + 2(\hat{\sigma}\hat{\mu} - \hat{\sigma}\hat{\mu}) \sum_{k=1}^j \frac{1}{\gamma_{s+1,k}} + (\hat{\sigma}^2 - \hat{\sigma}^2) \left( \sum_{k=1}^j \frac{1}{\gamma_{s+1,k}} \right)^2
\]

\[+ \hat{\sigma}^2 \sum_{k=1}^j \frac{1}{\gamma_{s+1,k}^2} m_i \]

(36)

where \(\hat{\sigma}^2\), \(\hat{\mu}^2\) and \(\hat{\sigma}\hat{\mu}\) are as presented in Equations (28), (30), and (31), respectively.
Proof  The predictor of \( X^{(j)}_{s+1} \) is given by the mean of \( X^{(j)}_{s+1} \), given \( \bar{X} = \bar{x} \). Thus, we have

\[
\hat{X}^{(j)}_{s+1} = E(X^{(j)}_{s+1} | \bar{X}) = \int_{-\infty}^{\infty} tf^{X^{(j)}_{s+1} | \bar{X}}(t | \bar{x}) \, dt
\]

\[
= \int_{-\infty}^{\infty} tf_{1}^{X^{(j)}_{s+1}}(t | \bar{x}) \, dt + \int_{-\infty}^{\infty} tf_{2}^{X^{(j)}_{s+1}}(t | \bar{x}) \, dt
\]

\[
= C^{-1} \Gamma(Q + b) \left( \prod_{k=1}^{j} \gamma_{s+1,k} \right) \sum_{k=1}^{j} a_{k}(j) \sum_{k \in K} \gamma_{s+1,k} + \sum_{i=1}^{s} \Psi(\bar{k})
\]

\[
\times \left\{ \int_{-\infty}^{M_{0}} t[\bar{H}_{k}(t)]^{-{(Q+b)}} - [G_{s+1,k}^{k}(N,t)]^{-{(Q+b)}} \, dt + \int_{M_{0}}^{\infty} t[\bar{H}_{k}(t)]^{-{(Q+b)}} - [G_{s+1,k}^{k}(N,t)]^{-{(Q+b)}} \, dt \right\}.
\]

Upon using integration by parts, \( \hat{X}^{(j)}_{s+1} \) is obtained as

\[
\hat{X}^{(j)}_{s+1} = C^{-1} \Gamma(Q + b - 1) \left( \prod_{k=1}^{j} \gamma_{s+1,k} \right) \sum_{k=1}^{j} a_{k}(j) \sum_{k \in K} \gamma_{s+1,k} + \sum_{i=1}^{s} \Psi(\bar{k})
\]

\[
\times \left\{ \frac{c + \gamma_{s+1,k} + \sum_{i=1}^{s} \gamma_{k,i}}{\gamma_{s+1,k}(c + \sum_{i=1}^{s} \gamma_{k,i})} (M_{0}[H_{k}(M_{0})]^{-{(Q+b-1)}} - N[H_{k}(N)]^{-{(Q+b-1)}})
\]

\[
- \frac{1}{Q + b - 2} \frac{\gamma_{s+1,k}^{2} - (c + \sum_{i=1}^{s} \gamma_{k,i})^{2}}{\gamma_{s+1,k}^{2}(c + \sum_{i=1}^{s} \gamma_{k,i})^{2}} ([H_{k}(M_{0})]^{-{(Q+b-2)}} - [H_{k}(N)]^{-{(Q+b-2)}})
\]

\[
\right\}.
\]

Now by using Equations (16) and (17), the expression in Equation (37) simplifies to

\[
\hat{X}^{(j)}_{s+1} = \hat{\mu} + \hat{\sigma} \sum_{k=1}^{j} \frac{1}{\gamma_{s+1,k}}.
\]

Similarly, the posterior second moment of \( X^{(j)}_{s+1} \) is given by

\[
E(X^{(j)}_{s+1}^{2} | \bar{X}) = \hat{\mu}^{2} + 2\hat{\sigma} \hat{\mu} \sum_{k=1}^{j} \frac{1}{\gamma_{s+1,k}} + 2\hat{\sigma}^{2} \left( \sum_{k=1}^{j} \frac{1}{\gamma_{s+1,k}^{2}} + \sum_{l,k=1}^{j} \frac{1}{\gamma_{s+1,l} \gamma_{s+1,k}} \right).
\]
Thus, the posterior variance of $\hat{X}_{s+1}^{(j)}$ is given by

$$\text{Var}(\hat{X}_{s+1}^{(j)}|\tilde{X}) = (\hat{\mu}^2 - \tilde{\mu}^2) + 2(\sigma \hat{\mu} - \tilde{\sigma} \hat{\mu}) \sum_{k=1}^{j} \frac{1}{\gamma_{s+1,k}} + (\tilde{\sigma}^2 - \hat{\sigma}^2) \left( \sum_{k=1}^{j} \frac{1}{\gamma_{s+1,k}} \right)^2$$

$$+ \tilde{\sigma}^2 \sum_{k=1}^{j} \frac{1}{\gamma_{s+1,k}^2},$$

which completes the proof of the theorem.

Remark 4.5 The Bayesian predictive bounds of a two-sided equi-tailed 100$(1 - \tau)$% interval for the $j$th sequential order statistic in the $(s + 1)$th sample, given $\tilde{X} = \tilde{x}$, can be obtained by solving the following two equations:

$$\bar{F}_{X_{s+1}^{(j)}}(\tilde{X} \mid \tilde{x}) = 1 - \frac{\tau}{2} \quad \text{and} \quad \bar{F}_{X_{s+1}^{(j)}}(U \mid \tilde{x}) = \frac{\tau}{2},$$

where $\bar{F}_{X_{s+1}^{(j)}}(t \mid \tilde{x})$ is the predictive survival function of $X_{s+1}^{(j)}$ given in Theorem 4.3, and $L$ and $U$ denote the lower and upper bounds, respectively.

For the HPD method, the following two equations need to be solved:

$$\bar{F}_{X_{s+1}^{(j)}}(L_{X_{s+1}^{(j)}} \mid \tilde{x}) - \bar{F}_{X_{s+1}^{(j)}}(U_{X_{s+1}^{(j)}} \mid \tilde{x}) = 1 - \tau$$

and

$$f_{X_{s+1}^{(j)}}(L_{X_{s+1}^{(j)}} \mid \tilde{x}) = f_{X_{s+1}^{(j)}}(U_{X_{s+1}^{(j)}} \mid \tilde{x}),$$

where $f_{X_{s+1}^{(j)}}(x \mid \tilde{x})$ is the predictive density function of $X_{s+1}^{(j)}$ given in Theorem 4.3, and $L_{X_{s+1}^{(j)}}$ and $U_{X_{s+1}^{(j)}}$ denote the HPD lower and upper bounds, respectively.

Remark 4.6 By choosing $\alpha_{ij} = 1$ for $i = 1, \ldots, s + 1$ and $j = 1, \ldots, n_i$, the Bayesian predictor $\hat{X}_{s+1}^{(j)}$ and its predictive posterior distribution under the model of the usual order statistics from one- and two-parameter exponential distributions are readily obtained. Similarly, the Bayesian predictor $\hat{X}_{s+1}^{(r)}$ and its predictive posterior distribution under the model of progressively Type-II censored order statistics from one- and two-parameter exponential distributions are readily obtained by setting $\alpha_{ij} = (N_i - j + 1 - \sum_{q=1}^{i} R_{i,q})/(n_i - j + 1)$ for $i = 1, \ldots, s + 1$ and $j = 1, \ldots, n_i$.

5. Numerical examples

5.1. Numerical example 1

To illustrate the inferential procedures developed in Section 3, we consider $s = 4$ simulated samples from a sequential 2-out-of-5 system based on components following the one-parameter
The results so obtained are presented in Tables 1 and 2, and from the results in these tables, the exponential distribution with \( \sigma = 20 \):

\[
\begin{align*}
&x_1^{(1)} = 1.940, \quad x_2^{(2)} = - -, \quad x_3^{(3)} = 6.216, \quad x_4^{(4)} = 12.060, \quad x_5^{(5)} = - - , \\
&x_2^{(1)} = 3.815, \quad x_2^{(2)} = 5.125, \quad x_2^{(3)} = 6.280, \quad x_2^{(4)} = 17.345, \quad x_2^{(5)} = - - , \\
&x_3^{(1)} = 5.566, \quad x_3^{(2)} = 5.692, \quad x_3^{(3)} = 20.247, \quad x_3^{(4)} = - - , \quad x_3^{(5)} = - - , \\
&x_4^{(1)} = - -, \quad x_4^{(2)} = 3.354, \quad x_4^{(3)} = 19.134, \quad x_4^{(4)} = 47.88, \quad x_4^{(5)} = - - .
\end{align*}
\]

The influence of failures on the remaining components in the system is described by the increasing sequence of parameters \( \alpha_1 = 1, \alpha_2 = 1.2, \alpha_3 = 1.4, \alpha_4 = 1.6, \alpha_5 = 1.8 \). Notice that in the first and fourth samples, some failure times are missing. Moreover, the failure of the fifth unit in the second sample is not observed, while in the third sample we only observed the first three failures.

These samples are now assumed to have come from the one-parameter exponential distribution, with \( \sigma \) being unknown. Based on the above multiply Type-II censored samples, the results in Section 3 have been used to calculate the Bayes estimates \( \hat{\sigma} \) and the posterior standard errors \( SE(\hat{\sigma}) \). In addition, suppose we want to predict the failure times from a future sample. In this case, the predictors \( \hat{X}_s^{(j)} \), for \( j = 1, \ldots, 5 \), along with the posterior standard errors \( SE(\hat{X}_s^{(j)}) \), and equi-tailed intervals and HPD intervals for \( X_s^{(j)} \) are obtained for five different choices of the hyperparameters \( a \) and \( b \), namely,

1. Jeffreys’ prior: \( \pi(\Sigma)(\sigma) \propto \sigma^{-1} (a = 0, b = 0) \).
2. Improper prior 1: \( \pi(\Sigma)(\sigma) \propto 1 (a = 0, b = -1) \).
3. Improper prior 2: \( \pi(\Sigma)(\sigma) \propto \sigma^{-2} (a = 0, b = 1) \).
4. Informative prior 1: \( E(\Sigma) = 20 \) and \( \text{Var}(\Sigma) = 5 (a = 1620, b = 82) \).
5. Informative prior 2: \( E(\Sigma) = 20 \) and \( \text{Var}(\Sigma) = 3 (a = 8060/3, b = 406/3) \).

The results so obtained are presented in Tables 1 and 2, and from the results in these tables, the following points can be observed:

1. A comparison of the results for the informative priors with the corresponding ones for the non-informative priors reveals that the former produces more precise results, as we would expect.
2. The results obtained based on Improper prior 2 are the best among those obtained based on non-informative priors.
3. The results obtained based on Informative prior 2 (with \( \text{Var}(\Sigma) = 3 \)) are more precise than the corresponding ones based on Informative prior 1 (with \( \text{Var}(\Sigma) = 5 \)), as we would expect.
4. Table 2 shows that the HPD prediction interval is more precise than the corresponding equi-tailed interval for all cases considered.
5. The predictive density functions of \( X_s^{(j)} \) are highly right skewed and consequently the upper bounds of equi-tailed prediction intervals become quite large as compared to those of HPD intervals; thus, equi-tailed prediction intervals become less precise.
Table 2. 95% Bayesian prediction bounds for $X^{(j)}_{s5}$ ($j = 1, \ldots, 5$) from the one-parameter exponential distribution.

<table>
<thead>
<tr>
<th>Prior</th>
<th>$\hat{X}^{(j)}_{s5}$</th>
<th>SE($\hat{X}^{(j)}_{s5}$)</th>
<th>$L_{X^{(j)}_{s5}}$</th>
<th>$U_{X^{(j)}_{s5}}$</th>
<th>Width</th>
<th>$L_{X^{(j)}_{s5}}$</th>
<th>$U_{X^{(j)}_{s5}}$</th>
<th>Width</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jeffreys’ prior</td>
<td>5.393</td>
<td>5.793</td>
<td>0.128</td>
<td>21.051</td>
<td>20.923</td>
<td>0.000</td>
<td>16.691</td>
<td>16.691</td>
</tr>
<tr>
<td>Improper prior 1</td>
<td>5.808</td>
<td>6.273</td>
<td>0.137</td>
<td>22.762</td>
<td>22.625</td>
<td>0.000</td>
<td>18.015</td>
<td>18.015</td>
</tr>
<tr>
<td>Improper prior 2</td>
<td>5.034</td>
<td>5.381</td>
<td>0.120</td>
<td>19.578</td>
<td>19.458</td>
<td>0.000</td>
<td>15.547</td>
<td>15.547</td>
</tr>
<tr>
<td>Informative prior 1</td>
<td>4.161</td>
<td>4.205</td>
<td>0.104</td>
<td>15.486</td>
<td>15.382</td>
<td>0.000</td>
<td>13.354</td>
<td>13.354</td>
</tr>
<tr>
<td>Informative prior 2</td>
<td>4.104</td>
<td>4.131</td>
<td>0.103</td>
<td>15.224</td>
<td>15.121</td>
<td>0.000</td>
<td>12.335</td>
<td>12.335</td>
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</table>

5.2. Numerical example 2

To illustrate the inferential procedures developed in Section 4, we consider $s = 5$ simulated samples from a sequential 2-out-of-5 system based on components following the two-parameter exponential distribution with $\sigma = 20$ and $\mu = 10$:

\[
\begin{align*}
    x_1^{(1)} &= -\ldots, & x_1^{(2)} &= 16.601, & x_1^{(3)} &= 20.426, & x_1^{(4)} &= 25.408, & x_1^{(5)} &= -\ldots, \\
    x_2^{(1)} &= 10.980, & x_2^{(2)} &= 11.150, & x_2^{(3)} &= 22.861, & x_2^{(4)} &= 23.902, & x_2^{(5)} &= -\ldots, \\
    x_3^{(1)} &= -\ldots, & x_3^{(2)} &= 12.378, & x_3^{(3)} &= -\ldots, & x_3^{(4)} &= 18.140, & x_3^{(5)} &= -\ldots, \\
    x_4^{(1)} &= 10.527, & x_4^{(2)} &= 20.866, & x_4^{(3)} &= 35.856, & x_4^{(4)} &= 37.913, & x_4^{(5)} &= -\ldots, \\
    x_5^{(1)} &= 10.714, & x_5^{(2)} &= 13.822, & x_5^{(3)} &= 20.945, & x_5^{(4)} &= -\ldots, & x_5^{(5)} &= -\ldots.
\end{align*}
\]

The influence of failures on the remaining components in the system is described by the increasing sequence of parameters $\alpha_1 = 1, \alpha_2 = 1.2, \alpha_3 = 1.4, \alpha_4 = 1.6, \alpha_5 = 1.8$. Notice that in the first and third samples, some failure times are missing. Moreover, the failure of the fifth unit in the second and fourth samples is not observed, while in the last sample, we only observed the first three failures.

These samples are now assumed to have come from the two-parameter exponential distribution, with both parameters $\sigma$ and $\mu$ being unknown. Based on the above multiply Type-II censored samples, the Bayes estimates and the posterior standard errors for the unknown parameters $\sigma$ and $\mu$ are determined. In addition, suppose we want to predict the failure times from a future
The results so obtained are presented in Tables 3 and 4, and from the results in these tables, the sample. In this case, the predictors \( \hat{X}_{46}^{(j)} \) along with posterior standard errors \( \text{SE}(\hat{X}_{46}^{(j)}) \), and equitailed intervals and the HPD intervals for \( X_{46}^{(j)} \) are all obtained for five different choices of the hyperparameters \( a, b, c, M \) and \( N \), namely,

1. Jeffreys’ prior: \( \pi(\Sigma, \mu) \propto \sigma^{-1} \) (\( a = 0, b = 0, c = 0, M \to \infty, N \to -\infty \)).
2. Improper prior 1: \( \pi(\Sigma, \mu) \propto 1 \) (\( a = 0, b = -1, c = 0, M \to \infty, N \to -\infty \)).
3. Improper prior 2: \( \pi(\Sigma, \mu) \propto \sigma^{-2} \) (\( a = 0, b = 1, c = 0, M \to \infty, N \to -\infty \)).
4. Informative prior 1: \( E(\Sigma) = 20, \quad \text{Var}(\Sigma) = 22, \quad E(\Delta) = 10 \) and \( \text{Var}(\Delta) = 1.2 \) (\( a = 595.990, b = 21.182, c = 19.235, M = 11.040, N \to -\infty \)).
5. Informative prior 2: \( E(\Sigma) = 20, \quad \text{Var}(\Sigma) = 24, \quad E(\Delta) = 10 \) and \( \text{Var}(\Delta) = 1.5 \) (\( a = 546.153, b = 19.667, c = 17.282, M = 11.157, N \to -\infty \)).

The results so obtained are presented in Tables 3 and 4, and from the results in these tables, the following points can be observed:

<table>
<thead>
<tr>
<th>Prior</th>
<th>( \hat{X}_{46}^{(j)} )</th>
<th>\text{SE}(\hat{X}_{46}^{(j)})</th>
<th>L_{X_{46}^{(j)}}</th>
<th>U_{X_{46}^{(j)}}</th>
<th>\text{Width}</th>
<th>\text{Width}</th>
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(1) A comparison of the results for the informative priors with the corresponding ones for the
non-informative priors reveals that the former produces more precise results, as we would
expect.
(2) The results obtained based on improper prior 2 are the best among those obtained based on
non-informative priors.
(3) The results obtained based on the informative prior 1 (with \( \text{Var}(\Sigma) = 22 \) and \( \text{Var}(\Delta) = 1.2 \))
are more precise than the corresponding ones based on the informative prior 2 (with \( \text{Var}(\Sigma) = 22 \) and \( \text{Var}(\Delta) = 1.5 \)), as we would expect.
(4) Table 4 shows that the HPD prediction interval is more precise than the corresponding equi-
tailed interval for all cases considered.
(5) Here again, the predictive density functions of \( X_{x0}^{(j)} \) are highly right skewed and consequently
the upper bounds of equi-tailed prediction intervals become quite large as compared to those
of HPD intervals; thus, equi-tailed prediction intervals become less precise.

6. Conclusions and discussion

In this paper, Bayesian prediction of future sequential order statistics has been discussed based
on multiply Type-II censored samples of sequential order statistics observed from one- and two-
parameter exponential distributions. Both point and interval predictions have been developed, and
two examples have been presented to illustrate the results. The computational results show that
predictions based on an informative prior with less variability are more precise than those based
on an informative prior with more variability. Moreover, the HPD prediction intervals are more
precise than the equi-tailed prediction intervals.

The results developed here may be extended to other forms of censored samples of sequential
order statistics from one- and two-parameter exponential distributions. It may also be of interest to
develop corresponding results when the lifetimes are from a Pareto distribution. We are currently
working on these problems and hope to report these findings in a future paper.

Acknowledgements

The authors express their sincere thanks to the Associate Editor and the anonymous referees for their constructive comments
and suggestions on the original version of this manuscript, which led to this improved version.

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