Question7(7). (a) Study the uniform convergence of the sequence $f_n(x) = e^{-nx^2}(x^2+1) + \dfrac{n^2x+1}{n^2+1}$

(1) on $\Re$ (2) on the interval [1,2].

(b) Evaluate the limit: $\lim\limits_{n\to\infty} \displaystyle\int_1^2 \left(e^{-nx^2}(x^2+1) + \frac{n^2x+1}{n^2+1} \right) dx$

$\ldots, \forall n \in \mathbb{N}.$

15. افــرض أن $f_n \xrightarrow{u} f$ على D حيث كل f_n متصلة عند $c \in D$. إذا كانت (x_n) متتالية في D نهايتها c، فأثبت أن $f_n(x_n) \to f(c)$.

16. ليكن D مجموعة متراصة وافرض أن (f) متتالية من ...

لا يكتب في هذا الهامش

__P.W. limit__: $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \left[\frac{1+x^2}{e^{nx^2}} + \frac{n^2x+1}{n^2+1} \right]$
$= x = f(x)$

(1) on $\mathbb{R}$, we use the last excercise:
Take $x_n = \frac{1}{\sqrt{n}} \to 0 = c \in \mathbb{R}$

$f_n(x_n) = f_n\left(\frac{1}{\sqrt{n}}\right) = \frac{1+\frac{1}{n}}{e} + \frac{n^{\frac{3}{2}}+1}{n^2+1} \to \frac{1}{e}$

But $f(c) = f(0) = 0 \neq \frac{1}{e}$, and $f_n(x)$ is continuous on $\mathbb{R}$ $\forall n$

$\Rightarrow f_n \overset{u}{\not\to} f(x) = x$

(2) on $[1,2]$:

$|f_n(x) - f(x)| = \left| \frac{1+x^2}{e^{nx^2}} + \frac{n^2x+1}{n^2+1} - x \right|$

$= \left| \frac{1+x^2}{e^{nx^2}} - \frac{x}{1+n^2} + \frac{1}{n^2+1} \right|$

By triangle inequality

$\leq \left[\frac{2}{e^n} + \frac{2+1}{1+n^2} \right] \to 0$ as $n \to \infty$

$\Rightarrow \sup |f_n(x) - f(x)| \to 0 \Rightarrow f_n \xrightarrow{u} f$ on $[1,2]$

Proof

$\left[\frac{1+x^2}{e^{nx^2}} \right]' = \frac{e^{nx^2} \cdot 2x - (x^2+1) e^{nx^2} \cdot 2nx}{(e^{nx^2})^2} = 0 \Rightarrow x = 0, \notin [1,2]$

no critical number

At $x = 1 \to \frac{1+1}{e} = \frac{2}{e}$ and at $x = 2 \to \frac{5}{e^{4n}}$

$\frac{2}{e}$ is bigger.

(3) Since $f_n \xrightarrow{u} f \Rightarrow \int_1^2 f_n(x)\,dx \xrightarrow[n\to\infty]{} \int_1^2 f(x)\,dx$

$= \int_1^2 x\,dx = \frac{x^2}{2}\Big]_1^2$

$= 2 - \frac{1}{2} = \frac{3}{2}$

$\Rightarrow \lim\limits_{n\to\infty} \int_1^2 f_n(x)\,dx = \frac{3}{2}$