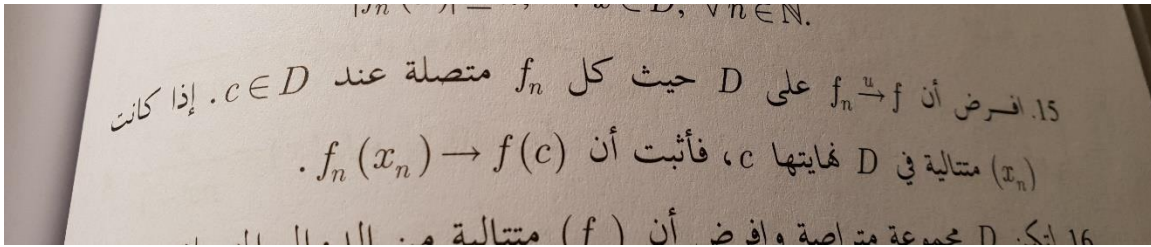


Question 7 (7°). (a) Study the uniform convergence of the sequence $f_n(x) = e^{-nx^2}(x^2+1) + \frac{n^2x+1}{n^2+1}$

(1) on \mathbb{R} (2) on the interval $[1,2]$.

(b) Evaluate the limit: $\lim_{n \rightarrow \infty} \int_1^2 \left(e^{-nx^2}(x^2+1) + \frac{n^2x+1}{n^2+1} \right) dx$



لا يكتب في هذا الهامش

P.W. limit: $\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \left[\frac{1+x^2}{e^{nx^2}} + \frac{n^2x+1}{n^2+1} \right] = x = f(x)$

(1) on \mathbb{R} , we use the last exercise:
 Take $x_n = \frac{1}{\sqrt{n}} \rightarrow 0 = c \in \mathbb{R}$
 $f_n(x_n) = f_n\left(\frac{1}{\sqrt{n}}\right) = \frac{1+\frac{1}{n}}{e} + \frac{n^2 \cdot \frac{1}{\sqrt{n}} + 1}{n^2+1} \rightarrow \frac{1}{e}$
 But $f(c) = f(0) = 0 \neq \frac{1}{e}$, and $f_n(x)$ is continuous on \mathbb{R} $\forall n$
 $\Rightarrow f_n \not\rightarrow f$ on \mathbb{R}

(2) on $[1,2]$:
 $|f_n(x) - f(x)| = \left| \frac{1+x^2}{e^{nx^2}} + \frac{n^2x+1}{n^2+1} - x \right|$
 $= \left| \frac{1+x^2}{e^{nx^2}} - \frac{x}{1+n^2} + \frac{1}{n^2+1} \right|$
 By triangle inequality $\leq \left[\frac{2}{e^n} + \frac{2+1}{1+n^2} \right] \rightarrow 0$ as $n \rightarrow \infty$
 $\Rightarrow \sup |f_n(x) - f(x)| \rightarrow 0 \Rightarrow f_n \xrightarrow{u} f$ on $[1,2]$

Proof
 $\left[\frac{1+x^2}{e^{nx^2}} \right]' = \frac{e^{-nx^2} \cdot 2x - (x^2+1) \cdot e^{-nx^2} \cdot 2nx}{(e^{nx^2})^2} = 0 \Rightarrow x=0, \notin [1,2]$
 no critical number
 At $x=1 \rightarrow \frac{1+1}{e} = \frac{2}{e}$ and at $x=2 \rightarrow \frac{5}{e^{4n}}$
 $\frac{2}{e}$ is bigger.

(3) Since $f_n \xrightarrow{u} f \Rightarrow \int_1^2 f_n(x) dx \xrightarrow{n \rightarrow \infty} \int_1^2 f(x) dx$
 $= \int_1^2 x dx = \left[\frac{x^2}{2} \right]_1^2 = 2 - \frac{1}{2} = \frac{3}{2}$
 $\Rightarrow \lim_{n \rightarrow \infty} \int_1^2 f_n(x) dx = \frac{3}{2}$