Part I

Inner Product Space

1 Vector Space

We are going to learn

- Vector space.
- Linearly independent vectors, Linearly dependent vectors.
- Spanning set of a vector space
- Basis of a vector space.
- Subspace of a vector space.
- Dimension of a vector space.

Definition 1

A vector space over $\mathbf{F}$ is a set X on which two operations

$$\begin{aligned} + &: X \times X \to X, \\ \cdot &: \mathbf{F} \times X \to X, \end{aligned}$$

are defined such that:

1. X is commutative group under addition.
2. Scalar multiplication between the elements of $\mathbf{F}$ and X satisfies two conditions.
3. The two distributive properties hold.

The elements of X are called *vectors* and those of $\mathbf{F}$ are called *scalars.*

Remark 2 *(Real and complex vector space)*

1. $\mathbf{F} = \mathbb{R} \Longrightarrow X$ is a *real vector space.*
2. $\mathbf{F} = \mathbb{C} \Longrightarrow X$ is a *complex vector space.*
3. We often write ax instead of $a \cdot x$.

Example 3 *(Vector spaces or not! Addition? scalar multiplication?)*

1. The set of $n-$tuples of real numbers, i.e.
$$\mathbb{R}^n = \{(x_1, x_2, ..., x_n) : x_i \in \mathbb{R}\}.$$
2. The set of $n-$tuples of complex numbers, i.e.
$$\mathbb{C}^n = \{(x_1, x_2, ..., x_n) : x_i \in \mathbb{C}\}$$
3. The set $\mathbb{C}^n$ over $\mathbb{R}$.
4. The set $\mathbb{R}^n$ over $\mathbb{C}$.
5. The set $\mathcal{P}_m(I)$ of polynomials on an interval I with real coefficients and degree $\leq m$, i.e.
$$\mathcal{P}_m(I) = \{a_n x^n + ... + a_1 x + a_0,\ a_i \in \mathbb{R}, n \leq m\},$$
over $\mathbb{R}$. When $I = \mathbb{R}$, we write $\mathcal{P}_m(\mathbb{R}) = \mathcal{P}_m$.
6. The set $\mathcal{P}(I)$ of polynomials on an interval I with real coefficients, i.e.
$$\mathcal{P}(I) = \{a_n x^n + ... + a_1 x + a_0,\ a_i \in \mathbb{R}, n \in \mathbb{N}_0\},$$
where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. When $I = \mathbb{R}$, we write $\mathcal{P}(\mathbb{R}) = \mathcal{P}$.

7. The set $\mathcal{C}^m(I)$ of real functions on an interval I with whose first m derivatives are continuous, i.e.

$$\mathcal{C}^m(I) = \{f : f^{(n)} \in \mathcal{C}(I) : n \in \{0, 1, ..., m\}\}$$

where $\mathcal{C}(I)$ is $\mathcal{C}^{(0)}(I)$. When $I = \mathbb{R}$, we write $\mathcal{C}^m(\mathbb{R}) = \mathcal{C}^m$.

8. The set $\mathcal{C}^\infty(I)$ of real functions on an interval I with whose all derivatives are continuous, i.e.

$$\mathcal{C}^\infty(I) = \cap_{i=1}^\infty \mathcal{C}^i(I)$$

When $I = \mathbb{R}$, we write $\mathcal{C}^\infty(\mathbb{R}) = \mathcal{C}^\infty$.

Definition 4

Let $\{x_1, ..., x_n\}$ be any finite set of vectors in a vector space X. The sum

$$a_1 x_1 + ... + a_n x_n, \ a_i \in \mathbf{F}$$

or in a short notation

$$\sum_{i=1}^n a_i x_i$$

is called a *linear combination* of the vectors in the set and the scalars a_i are called *coefficients*.

Definition 5

1. A finite set of vectors $\{x_1, ..., x_n\}$ is said to be *linearly independent* if

$$\sum_{i=1}^n a_i x_i = 0 \Rightarrow a_i = 0 \ \ \forall i \in \{1, ..., n\}.$$

2. A infinite set of vectors $\{x_1, x_2, ...\}$ is said to be *linearly independent* if every subset of it is linearly independent.

Definition 6

The set $\{x_1, ..., x_n\}$ is said to be *linearly dependent* if it is not *linearly independent.* A finite set of vectors is *linearly dependent* iff one of the vectors can be represented as a linear combination of the others.

Definition 7

A set $\mathcal{A}$ of vectors in a vector space X is said to *span* X if every vector in X can be written as a linear combination of the vectors in $\mathcal{A}$.

Definition 8

If a set $\mathcal{A}$ spans a vector space $\mathcal{A}$ and is linearly independent, then it is called a *basis* of X.

Definition 9

1. If a vector space X has a finite basis, then every other basis of X is finite with the same number of vectors. This number is called the dimension of X, and is denoted by $\dim X$.

2. If a vector space X has a infinite basis, then we write $\dim X = \infty$.

Definition 10

A subset Y of a vector space X is called a *subspace* of X if every linear combination of vectors in Y lies in Y. In other words, the set Y is closed under addition and scalar multiplication.

Example 11 *(Basis and dimension)*

1. The real vector space $\mathbb{R}^n$.

2. The complex vector space $\mathbb{C}^n$.

3. The real vector space $\mathbb{C}^n$.

4. The real vector space $\mathcal{P}_m(I)$.

Exercise 12 ***(Homework)***

1. Prove that $\mathcal{C}^m$ is a vector space for each $m \in \mathbb{N}_0$.

2. Prove that $\mathcal{C}^\infty$, $\mathcal{P}$ is a vector space (Hint: use the concept of subspace).

3. Find a basis for $\mathcal{P}$.

4. What is the dimension of $\mathcal{P}$?

5. What is the dimension of $\mathcal{C}^\infty$? (use the relation between $\mathcal{P}$ and $\mathcal{C}^\infty$).

2 Inner Product Space

We are going to learn

- Inner product space.
- Norm of a vector.
- Chauchy-Bunyakowsky-Schwarz Inequality.
- Triangle Inequality.
- Orthogonality of vectors.
- Orthogonal and orthonormal set.
- Relation between linear independence and orthogonality.

Definition 13

Let X be a vector space over $\mathbf{F}$. A function from $X \times X$ to $\mathbf{F}$ is called an *inner product* in X if, for any pair of vectors $x, y \in X$, the inner product

$$(x, y) \to \langle x, y \rangle \in \mathbf{F}$$

satisfies the following conditions:

1. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in X$.
2. $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle$ for all $a, b \in \mathbf{F}; x, y, z \in X$.
3. $\langle x, x \rangle \geq 0$ for all $x \in X$.
4. $\langle x, x \rangle = 0 \Leftrightarrow x = 0$.

A vector space on which an inner product is defined is called an *inner product space.*

Example 14 *(Inner product space)*

1. The $n-$dimensional Euclidean space is the vector space $\mathbb{R}^n$ together with the inner product of the vectors $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ defined by
$$\langle x, y \rangle = x_1 y_1 + ... + x_n y_n,$$
2. The $n-$dimensional Euclidean space is the vector space $\mathbb{R}^n$ together with the inner product of the vectors $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ defined by
$$\langle x, y \rangle = c x_1 y_1 + ... + c x_n y_n,$$
where $c > 0$.
3. In $\mathbb{C}^n$ we define the inner product of $z = (z_1, ..., z_n)$ and $w = (w_1, ..., w_n)$ by
$$\langle z, w \rangle = z_1 \overline{w}_1 + ... + z_n \overline{w}_n,$$
4. For $f, g \in C([a, b])$ we define their inner product by
$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx,$$

Theorem 15

If X is an inner product space, then

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle \text{ for all } x, y \in X. \qquad \text{(CBS)}$$

Proof:

- CASE 1: If $x = 0$ or $y = 0$, the CBS inequality clearly holds.
- CASE 2: If $x \neq 0$ and $y \neq 0$, If $\langle x, y \rangle \in \mathbb{C}$, then we can write

  $$\langle x, y \rangle = |\langle x, y \rangle| \, e^{i\theta}$$

  $\Rightarrow$

  $$e^{-i\theta} \langle x, y \rangle = e^{-i\theta} |\langle x, y \rangle| \, e^{i\theta}$$

  $\Rightarrow$

  $$\left\langle e^{-i\theta} x, y \right\rangle = |\langle x, y \rangle| \in \mathbb{R}$$

  and

  $$\left| \left\langle e^{-i\theta} x, y \right\rangle \right|^2 = |\langle x, y \rangle|^2$$

  So, by taking $\left\langle e^{-i\theta} x, y \right\rangle \in \mathbb{R}$ instead of $\langle x, y \rangle \in \mathbb{C}$, the CBS inequality remain the same. Therefore, we assume without loss of generality that $\langle x, y \rangle \in \mathbb{R}$.

  For any $t \in \mathbb{R}$, the real quadratic expression

  $$0 \leq \langle x + ty, x + ty \rangle = \langle x, x \rangle + 2 \langle x, y \rangle t + \langle y, y \rangle t^2 \tag{1.1}$$

  have a minimum at $t = \frac{-\langle x, y \rangle}{\langle y, y \rangle}$.

  Substituting $t = \frac{-\langle x, y \rangle}{\langle y, y \rangle}$ in (1.1) gives

  $$0 \leq \langle x, x \rangle - \frac{\langle x, y \rangle^2}{\langle y, y \rangle},$$

  or

  $$\langle x, y \rangle^2 \leq \langle x, x \rangle \langle y, y \rangle$$

  which is the desired inequality.

Definition 16

Let X be an inner product space over $\mathbf{F}$. We define the *norm* on X

$$\|\| : X \to [0, \infty)$$

by

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Example 17 *(norm of an inner product space)*

1. The $n-$dimensional Euclidean space is the vector space $\mathbb{R}^n$ together with the inner product of the vectors $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ defined by

   $$\langle x, y \rangle = x_1 y_1 + ... + x_n y_n,$$

   This gives

   $$\|x\| = \sqrt{x_1^2 + ... + x_n^2}.$$

2. The $n-$dimensional Euclidean space is the vector space $\mathbb{R}^n$ together with the inner product of the vectors $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ defined by
$$\langle x, y \rangle = cx_1 y_1 + ... + cx_n y_n,$$
where $c > 0$. This gives
$$\|x\| = \sqrt{cx_1^2 + ... + cx_n^2}.$$

3. In $\mathbb{C}^n$ we define the inner product of $z = (z_1, ..., z_n)$ and $w = (w_1, ..., w_n)$ by
$$\langle z, w \rangle = z_1 \overline{w}_1 + ... + z_n \overline{w}_n$$
This gives
$$\|z\| = \sqrt{|z_1|^2 + ... + |z_n|^2}.$$

4. For $f, g \in C([a, b])$ we define their inner product by
$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx$$
This gives
$$\|f\| = \left[\int_a^b |f(x)|^2 dx \right]^{\frac{1}{2}}$$

Properties of the norm (**Homework:** prove properties 1, 2, 3)

1. $\|x\| \geq 0 \quad \forall x \in X.$
2. $\|x\| = 0 \Leftrightarrow x = 0.$
3. $\|ax\| = |a| \|x\| \quad \forall a \in \mathbf{F}, \forall x \in X.$
4. $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in X.$ [Triangle inequality].

Remark 18 *(Another form of the CBS inequality)*

If X is an inner product space, then
$$|\langle x, y \rangle| \leq \|x\| \|y\| \text{ for all } x, y \in X.$$

Proof. (of triangle inequality)

$$\begin{aligned}
\|x + y\|^2 &= \langle x + y, x + y \rangle \\
&= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\
&= \|x\|^2 + \langle x, y \rangle + \overline{\langle x, y \rangle} + \|y\|^2 \\
&= \|x\|^2 + 2 \operatorname{Re} \langle x, y \rangle + \|y\|^2 \\
&\leq \|x\|^2 + 2 |\langle x, y \rangle| + \|y\|^2 \\
&\leq \|x\|^2 + 2 \|x\| \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2
\end{aligned}$$

Taking the square root of both sides gives the triangle inequality. ■

Definition 19

Let X be an inner product space. The distance between the vectors $x \in X$ and $y \in X$ is given by $\|x - y\|$.

Remark 20 *Geometrical interpretation of the triangle inequality*

Consider a triangle with vertices x, y, z, then

$$\begin{aligned} \|x - y\| &= \|x - z + z - y\| \\ &\leq \|x - z\| + \|z - y\|. \end{aligned}$$

Remark 21 *(Derivation of CBS and Triangle inequality in $\mathbb{R}^n$)*

See book p(10).

Concept of orthogonality in $\mathbb{R}^n$

In $\mathbb{R}^n$, the angle $\theta \in [0, \pi]$ between the vectors x and y is defined by

$$\langle x, y \rangle = \|x\| \|y\| \cos\theta$$

In $\mathbb{R}^2$ and $\mathbb{R}^3$, the vectors x and y are orthogonal if

$$\theta = 90^\circ \Leftrightarrow \cos\theta = 0 \Leftrightarrow \langle x, y \rangle = 0$$

In general, two nonzero vectors in $\mathbb{R}^n$ are said to be orthogonal if

$$\cos\theta = 0 \Leftrightarrow \langle x, y \rangle = 0$$

Definition 22 *(Orthogonal vectors in an inner product space)*

1. A pair of nonzero vectors x and y in the inner product space X is said to be *orthogonal* if $\langle x, y \rangle = 0$, symbolically written as $x \perp y$.
2. A set of nonzero vectors $\mathcal{V}$ in X is *orthogonal* if every pair in $\mathcal{V}$ is orthogonal.
3. An orthogonal set $\mathcal{V} \subseteq X$ is said to be orthonormal if $\|x\| = 1$ for all $x \in \mathcal{V}$.

Example 23 *(An orthonormal set)*

In the Euclidean space $\mathbb{R}^n$, the set $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ is orthonormal.

Relation between linear independence and orthogonality

1. Let $\{x_1, x_2, ..., x_n\}$ be an orthogonal set of vectors in the inner product space X, then they are necessarily linearly independent.
2. Let $\{x_1, x_2, ..., x_n\}$ be a linearly independent set of vectors in the inner product space X, then we can always form an orthogonal set of vectors from it. [Give an example of a linearly independent set that is not orthogonal!].

To show that 1 is true, let

$$a_1x_1 + a_2x_2 + ... + a_nx_n = 0,$$

then by taking the inner product of both sides in the above equation with $x_k, k \in \{1, ..., n\}$ we have

$$\langle a_1x_1 + a_2x_2 + ... + a_nx_n, x_k\rangle = \langle 0, x_k\rangle$$

or

$$\sum_{i=1}^{n} a_i \langle x_i, x_k\rangle = 0$$

but $\langle x_i, x_k\rangle = 0 \ \forall i \neq k$, hence

$$a_k \|x_k\|^2 = 0$$

which gives

$$a_k = 0$$

This is true for all $k \in \{1, ..., n\}$.

The second part is established using the Gram-Schmidt method. We need the following definition:

Definition 24

If x and $y \neq 0$ are any vectors in the inner product space X, then the projection of x on y is given by

$$\left\langle x, \frac{y}{\|y\|} \right\rangle$$

and the projection vector of x along y is given by

$$\left\langle x, \frac{y}{\|y\|} \right\rangle \frac{y}{\|y\|}$$

or

$$\frac{\langle x, y\rangle}{\|y\|^2} y$$

Gram-Schmidt method

Let $\{x_1, x_2, ..., x_n\}$ be a linearly independent set of vectors. Construct $\{y_1, y_2, ..., y_n\}$ as follows

$$\begin{aligned}
y_1 &= x_1, \\
y_2 &= x_2 - \frac{\langle x_2, y_1\rangle}{\|y_1\|^2} y_1, \\
y_3 &= x_3 - \frac{\langle x_3, y_1\rangle}{\|y_1\|^2} y_1 - \frac{\langle x_3, y_2\rangle}{\|y_2\|^2} y_2, \\
& \quad ... \\
y_n &= x_n - \frac{\langle x_n, y_1\rangle}{\|y_1\|^2} y_1 - ... - \frac{\langle x_n, y_{n-1}\rangle}{\|y_{n-1}\|^2} y_{n-1},
\end{aligned}$$

then the set $\{y_1, y_2, ..., y_n\}$ is orthogonal [Homework: Justify!].

3 The Space $\mathcal{L}^2$

We are going to learn

- Why the $\mathcal{L}^2$ space?
- What is the $\mathcal{L}^2$ space?
- Properties of the $\mathcal{L}^2$ space.
- Inner product with respect to a weight function.

Recall that we defined an inner product on the vector space $C([a,b])$ by

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx$$

The associated norm is given by

$$\|f\| = \left[\int_a^b |f(x)|^2 \, dx\right]^{\frac{1}{2}}$$

1. In the vector space $C([a,b])$, a sequence of function might converges (in a sense to be defined latter) to a limit that is not $C([a,b])$. This is problematic and thus we need to enlarge the space to avoid this problem.

2. The larger space, which we will denote by $X([a,b])$, must be chosen such that the inner product

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx$$

is defined for all $f, g \in X([a,b])$. That is, we need to make sure that for all $f, g \in X([a,b])$,

$$\left|\int_a^b f(x)\overline{g(x)}dx\right| < \infty,$$

Using the CBS inequality we have

$$\left|\int_a^b f(x)\overline{g(x)}dx\right| = |\langle f, g\rangle|$$

$$\leq \|f\|\,\|g\| = \left[\int_a^b |f(x)|^2 \, dx\right]^{\frac{1}{2}} \left[\int_a^b |g(x)|^2 \, dx\right]^{\frac{1}{2}}$$

Therefore, $\langle f, g\rangle$ is defined if $|f|^2$ and $|g|^2$ are both integrable.

3. Define $\mathcal{L}^2(a,b)$ to be the set of function $f : [a,b] \to \mathbb{C}$ such that $|f|^2$ is integrable, i.e.

$$\int_a^b |f(x)|^2 \, dx < \infty$$

then

(a) $\mathcal{L}^2(a,b)$ is a vector space. For all $f, g \in \mathcal{L}^2(a,b)$ we have

$$\begin{aligned} \|\alpha f + \beta g\| &\leq \|\alpha f\| + \|\beta g\| \\ &= |\alpha|\,\|f\| + |\beta|\,\|g\| \end{aligned}$$

i.e., $\alpha f + \beta g \in \mathcal{L}^2(a,b)$.

(b) $C([a,b])$ is a proper subset of $\mathcal{L}^2(a,b)$.For example, take

$$h(x) = \begin{cases} 1, & x = 1 \\ 0, & x \in (1,2] \end{cases}.$$

(c) In $\mathcal{L}^2(a,b)$,

$$f = 0 \Leftrightarrow \|f\| = 0$$

which is not equivalent to $f(x) = 0\ \forall x \in [a,b]$. Note that $f = 0$ in $\mathcal{L}^2(a,b)$ if it is zero on all but a finite number of points in I.

(d) We say that f and g are equal in $\mathcal{L}^2(a,b)$ if $\|f - g\| = 0$.

(e) In $\mathcal{L}^2(a,b)$ the integral is not affected by the end points of the interval (a,b). Therefore, $\mathcal{L}^2(a,b)$, $\mathcal{L}^2([a,b))$, $\mathcal{L}^2((a,b])$, $\mathcal{L}^2([a,b])$ are all the same. Also, the interval can be unbounded and we have $\mathcal{L}^2(-\infty,b)$, $\mathcal{L}^2(a,\infty)$, $\mathcal{L}^2(-\infty,\infty)$.

Example 25 *1.10*

Determine which of the following functions belong to $\mathcal{L}^2$ and compute its norm.

(i) $f(x) = \left\{ \begin{array}{ll} 1, & 0 \le x \le \frac{1}{2} \\ 0, & \frac{1}{2} \le x \le 1. \end{array} \right\}.$

(ii) $f(x) = \frac{1}{\sqrt{x}}$, $0 \le x \le 1$.

(iii) $f(x) = \frac{1}{\sqrt[3]{x}}$, $0 \le x \le 1$.

(iv) $f(x) = \frac{1}{x}$, $1 \le x \le \infty$.

Solution

(i)

$$\|f\|^2 = \int_0^1 f^2(x)\,dx = \int_0^{\frac{1}{2}} 1dx = \frac{1}{2},$$

Hence $f \in \mathcal{L}^2(0,1)$ and $\|f\| = \frac{1}{\sqrt{2}}$.

(ii)

$$\|f\|^2 = \int_0^1 \frac{1}{x}dx = \lim_{r\to 0^+} \int_r^1 \frac{1}{x}dx = \lim_{r\to 0^+} \ln x|_r^1 = -\lim_{r\to 0^+} \ln r = \infty,$$

Hence $f \notin \mathcal{L}^2(0,1)$.

(iii)

$$\|f\|^2 = \int_0^1 \frac{1}{x^{\frac{2}{3}}}dx = \lim_{r\to 0^+} \int_r^1 \frac{1}{x^{\frac{2}{3}}}dx = \lim_{r\to 0^+} 3x^{\frac{1}{3}}\Big|_r^1 = \lim_{r\to 0^+} 3\left(1 - r^{\frac{1}{3}}\right) = 3$$

Hence $f \in \mathcal{L}^2(0,1)$ and $\|f\| = \sqrt{3}$.

(iv)

$$\|f\|^2 = \int_1^\infty \frac{1}{x^2}dx = \lim_{r\to\infty}\int_1^r \frac{1}{x^2}dx = \lim_{r\to\infty} -x^{-1}\Big|_1^r = -\lim_{r\to\infty}\left(\frac{1}{r}-1\right) = 1$$

Hence $f \in \mathcal{L}^2(1,\infty)$ and $\|f\| = 1$.

Example 26 *1.11*

Consider the infinite set of functions $\mathcal{V} = \{1, \cos x, \sin x, \cos 2x, \sin 2x, ...\}$.

(i) Prove that $\mathcal{V}$ is orthogonal in the real inner product space $\mathcal{L}^2(-\pi,\pi)$.

(ii) Construct an orthonormal set using $\mathcal{V}$.

Solution
First note that $\mathcal{V} \subset \mathcal{L}^2(-\pi,\pi)$

(i) We need to show that each pair of vectors in $\mathcal{V}$ is orthogonal, i.e.

$$\langle f, g\rangle = 0 \;\; \forall f, g \in \mathcal{V}, f \neq g.$$

1. For all $n \in \mathbb{N}$,

$$\langle 1, \cos nx\rangle = \int_{-\pi}^{\pi} \cos nx dx = \frac{\sin nx}{n}\Big|_{-\pi}^{\pi} = 0.$$

2. For all $n \in \mathbb{N}$,

$$\langle 1, \sin nx\rangle = \int_{-\pi}^{\pi} \sin nx dx = \frac{-\cos nx}{n}\Big|_{-\pi}^{\pi} = 0.$$

3. For all $n, m \in \mathbb{N}$, $n \neq m$

$$\begin{aligned}\langle \cos nx, \cos mx\rangle &= \int_{-\pi}^{\pi} \cos nx \cos mx dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(n-m)x + \cos(n+m)x]\, dx \\ &= \frac{1}{2}\left[\frac{1}{n-m}\sin(n-m)x + \frac{1}{n+m}\sin(n+m)x\right]\Big|_{-\pi}^{\pi} dx \\ &= 0,\end{aligned}$$

4. For all $n, m \in \mathbb{N}$, $n \neq m$

$$\begin{aligned}\langle \sin nx, \sin mx\rangle &= \int_{-\pi}^{\pi} \sin nx \sin mx dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(n-m)x - \cos(n+m)x]\, dx \\ &= \frac{1}{2}\left[\frac{1}{n-m}\sin(n-m)x - \frac{1}{n+m}\sin(n+m)x\right]\Big|_{-\pi}^{\pi} dx \\ &= 0,\end{aligned}$$

5. For all $n, m \in \mathbb{N}$,

$$\langle \cos nx, \sin mx \rangle = \int_{-\pi}^{\pi} \cos nx \sin mx dx = 0.$$

(ii) We normalize all the vector in $\mathcal{V}$ by dividing each vector by its norm.

$$\begin{aligned} \|1\| &= \sqrt{\int_{-\pi}^{\pi} dx} = \sqrt{2\pi}, \\ \|\cos nx\| &= \sqrt{\int_{-\pi}^{\pi} \cos^2 nx dx} = \sqrt{\int_{-\pi}^{\pi} \frac{1 + \cos 2nx}{2} dx} = \sqrt{\pi}, \\ \|\sin nx\| &= \sqrt{\int_{-\pi}^{\pi} \sin^2 nx dx} = \sqrt{\int_{-\pi}^{\pi} \frac{1 - \cos 2nx}{2} dx} = \sqrt{\pi}. \end{aligned}$$

Therefore, we have the orthonormal set

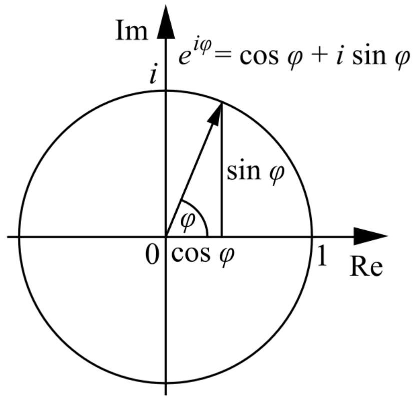
$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, ... \right\}.$$

Example 27 *1.12*

Show that the set of functions

$$\left\{ e^{inx} : n \in z \right\} = \left\{ ..., e^{-i2x}, e^{-ix}, 1, e^{ix}, e^{i2x}, ... \right\},$$

is orthogonal in the complex space $\mathcal{L}^2(-\pi, \pi)$.



For $n \neq m$ we have

$$\begin{aligned}
\left\langle e^{inx}, \overline{e^{imx}} \right\rangle &= \int_{-\pi}^{\pi} e^{inx}\overline{e^{imx}}dx \\
&= \int_{-\pi}^{\pi} e^{inx}e^{-imx}dx \\
&= \int_{-\pi}^{\pi} e^{i(n-m)x}dx \\
&= \frac{1}{i(n-m)}e^{i(n-m)x}\Big|_{-\pi}^{\pi} \\
&= \frac{1}{i(n-m)}\left[\cos(n-m)x + i\sin(n-m)x\right]\Big|_{-\pi}^{\pi} \\
&= 0
\end{aligned}$$

We can construct an orthonormal set by defining each function by its norm

$$\begin{aligned}
\left\| e^{inx} \right\| &= \sqrt{\int_{-\pi}^{\pi} e^{inx}\overline{e^{inx}}dx} \\
&= \sqrt{\int_{-\pi}^{\pi} e^{inx}e^{-inx}dx} \\
&= \sqrt{\int_{-\pi}^{\pi} 1dx} \\
&= \sqrt{2\pi}, n \in \mathbb{Z}
\end{aligned}$$

The orthonormal set is thus given by,

$$\left\{ \frac{e^{inx}}{\sqrt{2\pi}} : n \in z \right\}$$

Definition 28 *Inner product with respect to a weight function*

Let $\rho \in C(a,b)$ and $\rho(x) > 0 \; \forall x \in (a,b)$, then for $f, g \in C(a,b)$ we define the inner product with respect to the weight function ρ by

$$\langle f, g \rangle_{\rho} = \int_{a}^{b} f(x)\,\overline{g}(x)\,\rho(x)\,dx$$

[verify that the above definition satisfies the inner product conditions].
The norm is therefore defined by

$$\|f\|_{\rho} = \sqrt{\int_{a}^{b} |f(x)|^{2}\,\rho(x)\,dx}$$

The set of functions

$$f : (a, b) \to \mathbb{C}$$

that satisfy

$$\|f\|_\rho < \infty$$

is denoted by $\mathcal{L}^2_\rho (a, b)$. Note that for $\rho = 1$, this set is nothing but $\mathcal{L}^2 (a, b)$.

If $\langle f, g\rangle_\rho = 0$,then f is said to be orthogonal to g with respect to the weight function ρ.

4 Sequences of Functions

We are going to learn

- Pointwise convergence of sequences of functions.
- Uniform convergence of sequences of functions.
- Pointwise convergence of series of functions.
- Absolute convergence of series of functions.
- Uniform convergence of series of functions.

Definition 29

A sequence of functions (real or complex)

$$f_n : I \to \mathbf{F}$$

is said to *converge pointwise* to a function

$$f : I \to \mathbf{F}$$

i.e.,

$$\lim_{n\to\infty} f_n = f, \quad \lim f_n = f, \text{ or } f_n \to f,$$

if for every $x \in I$,

$$lim_{n\to\infty} f_n(x) = f(x).$$

That is, $\forall \epsilon > 0$, $\exists N(\epsilon, x)$ such that

$$n \geq N \Rightarrow |f_n(x) - f(x)| < \epsilon. \tag{1.15}$$

Example 30 *1.14*

Find the pointwise limit of the following functions

(i) $f_n(x) = \frac{1}{n} \sin nx$, $x \in \mathbb{R}$.

(ii) $f_n(x) = x^n$, $x \in [0, 1]$.

(iii) $f_n(x) = \frac{nx}{1+nx}$, $x \in [0, \infty)$.

Solution

(i) Let $f_n(x) = \frac{1}{n} \sin nx$, $x \in \mathbb{R}$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{n} \sin nx = 0.$$

Hence, $f_n \to f$ where $f(x) = 0\ \forall x \in \mathbb{R}$.

(ii) Let $f_n(x) = x^n$, $x \in [0, 1]$.

$$\lim_{n\to\infty} x^n = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

Thus, $f_n \to f$ where

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

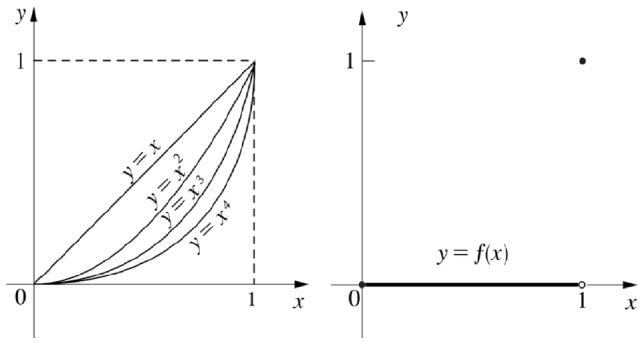


Figure 1.1 The sequence $f_n(x) = x^n$.

(iii) Let $f_n(x) = \frac{nx}{1+nx}$, $x \in [0, \infty)$.

$$\lim_{n\to\infty} \frac{nx}{1+nx} = \begin{cases} 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Example 31 *1.15*

For each $n \in \mathbb{N}$, define the sequence $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0, & x = 0 \\ n, & 0 < x \le \frac{1}{n} \\ 0, & \frac{1}{n} < x \le 1 \end{cases}$$

Find the pointwise limit of f_n.
Solution
First note that

$$f_n(0) \to 0.$$

For $x > 0$, we will show that there is always an $N \in \mathbb{N}$ such that $f_n(x) = 0$ $\forall n \ge N$. Now, since $x > 0$ and $\frac{1}{n} \to 0$ then $\exists N(x) \in \mathbb{N}$ such that

$$n \ge N \Rightarrow \frac{1}{n} \le \frac{1}{N} < x$$

Therefore,

$$\lim_{n\to\infty} f_n(x) = 0.$$

Definition 32

A sequence of functions (real or complex)

$$f_n : I \to \mathbf{F}$$

is said to *converge uniformly* to a function

$$f : I \to \mathbf{F}$$

i.e.,

$$f_n \xrightarrow{u} f,$$

if $\forall \epsilon > 0$, $\exists N(\epsilon)$ such that

$$n \geq N \Rightarrow |f_n(x) - f(x)| < \epsilon \quad \forall x \in \mathbb{R}.$$

Remark 33 *(Relation between pointwise and uniform convergence)*

Uniform convergence$\Rightarrow$Pointwise convergence.

Example 34 *1.14 (revisited)*

Which of the following sequences converges uniformly to its pointwise limit

(i) $f_n(x) = \frac{1}{n} \sin nx$, $x \in \mathbb{R}$.

(ii) $f_n(x) = x^n$, $x \in [0, 1]$.

(iii) $f_n(x) = \frac{nx}{1+nx}$, $x \in [0, \infty)$.

Solution

(i) Let $\epsilon > 0$, then for $n \geq N > \frac{1}{\epsilon}$ and for all $x \in \mathbb{R}$

$$|f_n(x) - f(x)| = \left|\frac{1}{n} \sin nx - 0\right| = \left|\frac{1}{n} \sin nx\right| \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

i.e. $f_n \xrightarrow{u} f$

(ii) Let $0 < \epsilon < 1$

$$|f_n(x) - f(x)| = |x^n - 0| = x^n$$

So for any $N \in \mathbb{N}$,

$$x^N < \epsilon \Leftrightarrow x < \sqrt[N]{\epsilon}$$

that is for all $x \in [\sqrt[N]{\epsilon}, 1)$,

$$\left|x^N - 0\right| > \epsilon$$

i.e. $f_n \overset{u}{\nrightarrow} f$

(iii) Let $0 < \epsilon < 1$

$$|f_n(x) - f(x)| = \left|\frac{nx}{1+nx} - 1\right| = \frac{1}{1+nx}$$

So for any $N \in \mathbb{N}$,

$$\frac{1}{1+Nx} < \epsilon \Leftrightarrow x > \frac{1-\epsilon}{N\epsilon}$$

that is for all $x \in \left[0, \frac{1-\epsilon}{N\epsilon}\right]$,

$$\left|\frac{Nx}{1+Nx} - 1\right| > \epsilon$$

i.e. $f_n \overset{u}{\nrightarrow} f$

Remark 35 *1.16*

1. In the definition of pointwise and uniform convergence, we can replace the "$<$" relation by "$\leq$" and the "ϵ" by "$c\epsilon$", where $c > 0$.

2. The statement
$$|f_n(x) - f(x)| \leq \epsilon \qquad \forall x \in I$$
is equivalent to
$$\sup_{x \in I} |f_n(x) - f(x)| \leq \epsilon.$$
Therefore, $f_n \overset{u}{\to} f \Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}$ such that
$$n \geq N \Rightarrow \sup_{x \in I} |f_n(x) - f(x)| \leq \epsilon$$

Definition 36

A sequence of functions (real or complex)
$$f_n : I \to \mathbf{F}$$
is said to *converge uniformly* to a function
$$f : I \to \mathbf{F}$$
i.e.,
$$f_n \overset{u}{\to} f,$$
if
$$\lim_{n \to \infty} \sup_{x \in I} |f_n(x) - f(x)| = 0$$

Example 37 *1.14 (revisited)*

Use the above criteria to decide which of the following sequences converges uniformly to its pointwise limit

(i) $f_n(x) = \frac{1}{n} \sin nx$, $x \in \mathbb{R}$.

(ii) $f_n(x) = x^n$, $x \in [0, 1]$.

(iii) $f_n(x) = \frac{nx}{1+nx}$, $x \in [0, \infty)$.

Solution

(i)
$$|f_n(x) - f(x)| = \left| \frac{1}{n} \sin nx - 0 \right| = \left| \frac{1}{n} \sin nx \right|$$

but

$$\begin{aligned} 0 \quad &\leq \quad \left|\frac{1}{n}\sin nx\right| \leq \frac{1}{n}, \quad \forall x \in \mathbb{R} \\ &\Rightarrow \quad 0 \leq \sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sin nx\right| \leq \frac{1}{n} \\ &\Rightarrow \quad \lim_{n\to\infty}\sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sin nx\right| = 0 \quad \text{[Why?]} \end{aligned}$$

Thus, $f_n \xrightarrow{u} f$.

(ii)

$$|f_n(x) - f(x)| = \begin{cases} x^n, & x \in [0,1) \\ 0, & x = 1 \end{cases}$$

Hence

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1)} x^n = 1$$

Therefore,

$$\lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| = 1 \neq 0$$

i.e., $f_n \overset{u}{\nrightarrow} f$.

(iii)

$$|f_n(x) - f(x)| = \begin{cases} 0, & x = 0 \\ \frac{1}{1+nx}, & x > 0 \end{cases}$$

and hence

$$\sup_{x\in[0,\infty)} |f_n(x) - f(x)| = \sup_{x\in(0,\infty)} |f_n(x) - f(x)| = 1$$

Therefore,

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} |f_n(x) - f(x)| = 1 \neq 0$$

i.e., $f_n \overset{u}{\nrightarrow} f$.

Theorem 38 *1.17*

Let (f_n) be a sequence of functions defined on the interval I, then

1. If f_n is continuous on I $\forall n$ and $f_n \xrightarrow{u} f$, then f is continuous on I.
2. If f_n is integrable on I $\forall n$, $f_n \xrightarrow{u} f$ and I is bounded, then f is integrable on I and
$$\int_I f(x)\,dx = \lim \int_I f_n(x)\,dx$$

3. If f_n is differentiable on $I\ \forall n,\ f_n \to f,\ f_n^{'} \xrightarrow{u} g$ and I is bounded, then $f_n \xrightarrow{u} f$, f is differentiable and

$$g = f'$$

Remark 39 *1.18*

In part 3, the pointwise convergence $f_n \to f$ can be replaced by a weaker condition in part 3, namely the convergence $f_n(x_0) \to f(x_0)$ for some point $x_0 \in I$.

Example 40 *1.14 (revisited)*

Check if the conditions of theorem 1.17 are satisfied for the following sequences

(i) $f_n(x) = \frac{1}{n}\sin nx,\ x \in I \subset \mathbb{R}.$

(ii) $f_n(x) = x^n,\ x \in [0,1].$

(iii) $f_n(x) = \frac{nx}{1+nx},\ x \in I \subset [0,\infty).$[Homework!]

Example 41 *1.15 (revisited)*

For each $n \in \mathbb{N}$, define the sequence $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0, & x = 0 \\ n, & 0 < x \leq \frac{1}{n} \\ 0, & \frac{1}{n} < x \leq 1 \end{cases}$$

We know that $f_n \to 0$.

1. $\lim_{n\to\infty} f_n$ is a continuous function.

2. $$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim_{n\to\infty}\int_0^{\frac{1}{n}} n dx = 1 \neq 0 = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx$$

   which implies that $\lim_{n\to\infty} f_n$ is not uniform.

3. f_n is not differentiable but $\lim_{n\to\infty} f_n = 0$ is differentiable.

Definition 42

Given a sequence of (real or complex) functions (f_n) defined on a real interval I, we define its nth partial sum by

$$S_n(x) = f_1(x) + ... + f_n(x) = \sum_{k=1}^{n} f_k(x), \quad x \in I.$$

The sequence of functions (S_n) is called an *infinite series* of functions and is denoted $\sum f_k$.

Definition 43

The series $\sum f_k$ is said to *converge pointwise* (or simply, converge) on I if the sequence (S_n) converges pointwise on I. The sum of the series is given by the limit

$$\sum_{k=1}^{\infty} f_k(x) = \lim_{n\to\infty} S_n(x).$$

If the series $\sum_{k=1}^{\infty} f_k(x)$ does not converge, then it is said to *diverge* at the point x.

Definition 44

The series $\sum f_k$ is said to be *absolutely convergent* on I if the positive series $\sum |f_k|$ is pointwise convergent on I.

Definition 45

The series $\sum f_k$ is said to *converge uniformly* on I if the sequence (S_n) converges uniformly on I.

Corollary 46 *1.19*

Suppose the series $\sum f_n$ converges pointwise on the interval I.

1. If f_n is continuous on I $\forall n$ and $\sum f_n$ converges uniformly on I, then its sum $\sum_{n=1}^{\infty} f_n$ is continuous.
2. If f_n is integrable on I $\forall n$, I is bounded, and $\sum f_n$ converges uniformly on I, then $\sum_{n=1}^{\infty} f_n$ is integrable on I and

$$\int_I \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty} \int_I f_n(x)\,dx$$

3. If f_n is differentiable on I $\forall n$, I is bounded, and $\sum f_n'$ converges uniformly on I, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on I and its limit is differentiable on I and satisfies

$$\left(\sum_{n=1}^{\infty} f_n\right)' = \sum_{n=1}^{\infty} f_n'$$

The definition of pointwise, uniform and absolute convergence of a series require that we test the convergence of the sequence of partial sums. The following test provides an easier way to do that!

Theorem 47 *1.20 (Weierstrass M-Test)*

Let (f_n) be a sequence of functions on I, and suppose that there is a sequence of nonnegative numbers M_n such that

$$|f_n(x)| \leq M_n \text{ for all } x \in I,\ n \in \mathbb{N}.$$

If $\sum M_n$ converges, then $\sum f_n$ converges uniformly and absolutely on I.

Proof

We want to prove that $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that

$$n \geq N \Rightarrow |S(x) - S_n(x)| < \epsilon, \quad \forall x \in I$$

or equivalently

$$n \geq N \Rightarrow \left|\sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^{n} f_k(x)\right| < \epsilon, \quad \forall x \in I$$

But for all $x \in I$

$$\begin{aligned} &\left|\sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^{n} f_k(x)\right| \\ = &\left|\sum_{k=n+1}^{\infty} f_k(x)\right| \\ \leq &\sum_{k=n+1}^{\infty} |f_k(x)| \\ \leq &\sum_{k=n+1}^{\infty} M_k \end{aligned}$$

Now, if $\sum_{k=1}^{\infty} M_k$ is convergent, then so is $\sum_{k=n+1}^{\infty} M_k$. Therefore, for a chosen $\epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that

$$n \geq N \Rightarrow \sum_{k=n+1}^{\infty} M_k = \left|\sum_{k=1}^{\infty} M_k - \sum_{k=1}^{n} M_k\right| < \epsilon$$

Thus, we have

$$n \geq N \Rightarrow \left|\sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^{n} f_k(x)\right| < \epsilon \qquad \forall x \in I$$

which proves that $\sum_{k=1}^{\infty} f_k(x)$ is uniformly convergent.

To prove that $\sum_{k=1}^{\infty} f_k(x)$ is absolutely convergent we need to prove that $\sum_{k=1}^{\infty} |f_k(x)|$ is pointwise convergent.

Now, for each $x \in I$, both $\sum_{k=1}^{\infty} |f_k(x)|$ and $\sum_{k=1}^{\infty} M_k$ is a series of nonnegative numbers that satisfy

$$|f_k(x)| \leq M_k \quad \forall k \in N$$

Therefore, since $\sum_{k=1}^{\infty} M_k$ is convergent, then so is $\sum_{k=1}^{\infty} |f_k(x)|$ according to the comparison test.

Example 48 *1.21*

In the following, determine if the series is uniformly convergent and check the properties listed in Corollary 1.19:

(i) $\sum \frac{1}{n^2} \sin nx,\ x \in \mathbb{R}$.

(ii) $\sum \frac{1}{n^3} \sin nx,\ x \in \mathbb{R}$.

Solution:

(i) Since

$$\left| \frac{1}{n^2} \sin nx \right| \leq \frac{1}{n^2} \quad \forall x \in \mathbb{R}, n \in \mathbb{N}$$

and the nonnegative series of numbers $\sum \frac{1}{n^2}$ is convergent, we can use the M-Test to conclude that $\sum \frac{1}{n^2} \sin nx$ is uniformly convergent on $\mathbb{R}$.

1. Since (a) $\frac{1}{n^2} \sin nx$ is continuous on $\mathbb{R}$ for all n and (b) $\sum \frac{1}{n^2} \sin nx$ is uniformly convergent on $\mathbb{R}$, then $\sum \frac{1}{n^2} \sin nx$ is continuous on $\mathbb{R}$.

2. Since (a) $\frac{1}{n^2} \sin nx$ is integrable on $[a, b] \subseteq \mathbb{R}$ and (b) $\sum \frac{1}{n^2} \sin nx$ is uniformly convergent on $[a, b] \subseteq \mathbb{R}$, then $\sum \frac{1}{n^2} \sin nx$ is integrable on $[a, b] \subseteq \mathbb{R}$ and

$$\begin{aligned} \int_a^b \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \sin nx \right) dx &= \sum_{n=1}^{\infty} \frac{1}{n^2} \int_a^b \sin nx dx \\ &= \sum_{n=1}^{\infty} \frac{1}{n^3} (\cos na - \cos nb) \\ &\leq 2 \sum_{n=1}^{\infty} \frac{1}{n^3} \end{aligned}$$

which is a convergent $p-$series.

3. $f_n = \frac{1}{n^2} \sin nx$ is differentiable on $\mathbb{R}$ for all n with $f_n' = \frac{1}{n} \cos nx$, but $\sum \frac{1}{n} \cos nx$ is not convergent at some points of $x \in \mathbb{R}$. Therefore, we **cannot** deduce that

$$\frac{d}{dx} \sum_{n=1}^{\infty} \left(\frac{1}{n^2} \sin nx \right) = \sum_{n=1}^{\infty} \frac{1}{n} \cos nx$$

(ii) Since

$$\left| \frac{1}{n^3} \sin nx \right| \leq \frac{1}{n^3} \quad \forall x \in \mathbb{R}, n \in \mathbb{N}$$

and the nonnegative series of numbers $\sum \frac{1}{n^3}$ is convergent, we can use the M-Test to conclude that $\sum \frac{1}{n^3} \sin nx$ is uniformly convergent on $\mathbb{R}$.

1. Since (a) $\frac{1}{n^3}\sin nx$ is continuous on $\mathbb{R}$ for all n and (b) $\sum \frac{1}{n^3}\sin nx$ is uniformly convergent on $\mathbb{R}$, then $\sum \frac{1}{n^3}\sin nx$ is continuous on $\mathbb{R}$.

2. Since (a) $\frac{1}{n^3}\sin nx$ is integrable on $[a,b] \subseteq \mathbb{R}$ and (b) $\sum \frac{1}{n^3}\sin nx$ is uniformly convergent on $[a,b] \subseteq \mathbb{R}$, then $\sum \frac{1}{n^3}\sin nx$ is integrable on $[a,b] \subseteq \mathbb{R}$ and

$$\int_a^b \sum_{n=1}^{\infty} \left(\frac{1}{n^3}\sin nx\right) dx = \sum_{n=1}^{\infty} \int_a^b \left(\frac{1}{n^3}\sin nx\right) dx$$

3. $f_n = \frac{1}{n^3}\sin nx$ is differentiable on $[a,b] \subseteq \mathbb{R}$ for all n with $f_n' = \frac{1}{n^2}\cos nx$. Here, $\sum \frac{1}{n^2}\cos nx$ is uniformly convergent on $[a,b] \subseteq \mathbb{R}$. Therefore, we deduce that $\sum \frac{1}{n^3}\sin nx$ is uniformly convergent on $\mathbb{R}$ and

$$\frac{d}{dx}\sum_{n=1}^{\infty} \frac{1}{n^3}\sin nx = \sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx$$

holds for all $x \in [a,b]$.

5 Convergence in $\mathcal{L}^2$

We are going to learn

- Convergence of sequences of functions in $\mathcal{L}^2$.
- Relation between pointwise convergence and convergence in $\mathcal{L}^2$.
- Relation between uniform convergence and convergence in $\mathcal{L}^2$.
- The Cauchy sequence and its relation to convergence.
- Completeness of $\mathcal{L}^2$.
- Density of C in $\mathcal{L}^2$.

Definition 49 *1.22*

A sequence of functions (f_n) in $\mathcal{L}^2(a,b)$ is said to converge in $\mathcal{L}^2$ if there is a function $f \in \mathcal{L}^2(a,b)$ such that

$$\lim_{n\to\infty} \|f_n - f\| = 0, \qquad ((1.19))$$

that is, if for every $\epsilon > 0$ there is an integer N such that

$$n \geq N \Rightarrow \|f_n - f\| < \epsilon.$$

Equation (1.19) is equivalent to writing

$$f_n \overset{\mathcal{L}^2}{\to} f,$$

and f is called the limit in $\mathcal{L}^2$ of the sequence (f_n).

Example 50 *1.23*

In the following determine if the sequence converge in $\mathcal{L}^2$

(i) $f_n(x) = x^n$, $x \in [0,1]$.

(ii) $f_n(x) = \begin{cases} 0, & \text{if} \quad x = 0 \\ n, & \text{if} \quad 0 < x \leq \frac{1}{n} \\ 0, & \text{if} \quad \frac{1}{n} < x \leq 1 \end{cases}$

Solution:

(i) We already know that the pointwise limit of x^n on $[0,1]$ is given by

$$f(x) = \begin{cases} 0, & \text{if} \quad 0 \leq x < 1 \\ 1, & \text{if} \quad x = 1 \end{cases},$$

Note that $\mathcal{L}^2([0,1]) = \mathcal{L}^2([0,1))$ and $f = 0$ in $\mathcal{L}^2$. Moreover,

$$\begin{aligned} \|x^n - 0\| &= \left[\int_0^1 (x^n)^2\, dx\right]^{\frac{1}{2}} \\ &= \left[\left.\frac{x^{2n+1}}{2n+1}\right|_0^1\right]^{\frac{1}{2}} \\ &= \left[\frac{1}{2n+1}\right]^{\frac{1}{2}} \end{aligned}$$

i.e.,

$$\lim_{n\to\infty} \|x^n - 0\| = \lim_{n\to\infty} \left[\frac{1}{2n+1}\right]^{\frac{1}{2}} = 0$$

Therefore,

$$x^n \overset{\mathcal{L}^2}{\to} 0$$

(ii) We know that the pointwise limit of

$$f_n(x) = \begin{cases} 0, & \text{if} \quad x = 0 \\ n, & \text{if} \quad 0 < x \le \frac{1}{n} \\ 0, & \text{if} \quad \frac{1}{n} < x \le 1 \end{cases}$$

is

$$f(x) = 0, \quad x \in [0,1].$$

Now,

$$\begin{aligned} \|f_n(x) - 0\| &= \left[\int_0^{\frac{1}{n}} (n)^2 \, dx \right]^{\frac{1}{2}} \\ &= \left[n^2 x\Big|_0^{\frac{1}{n}} \right]^{\frac{1}{2}} \\ &= n^{\frac{1}{2}} \end{aligned}$$

i.e.,

$$\lim_{n\to\infty} \|f_n(x) - 0\| = \lim_{n\to\infty} n^{\frac{1}{2}} = \infty$$

Therefore, f_n does not converge in $\mathcal{L}^2$.

Remark

1. pointwise convergence $\nrightarrow$ convergence in $\mathcal{L}^2$.
2. If a sequence converges both pointwise and in $\mathcal{L}^2$, then the limit functions are equal in $\mathcal{L}^2$.

Lemma 51 *(Exercise 1.41, page 29)*

If $f_n \xrightarrow{u} f$ on $[a,b]$, then $|f_n - f| \xrightarrow{u} 0$ on $[a,b]$ and hence $|f_n - f|^2 \xrightarrow{u} 0$ on $[a,b]$.

Proof:

If $f_n \xrightarrow{u} f$ on $[a,b]$, then

$$\lim_{n\to\infty} \sup_{x\in[a,b]} |f_n - f| = 0,$$

which is equivalent to

$$\lim_{n\to\infty} \sup_{x\in[a,b]} ||f_n - f| - 0| = 0$$

That is, $|f_n - f| \xrightarrow{u} 0$ on $[a,b]$.

Next, we prove that $|f_n - f|^2 \xrightarrow{u} 0$ on $[a,b]$. Let $\epsilon > 0$ and take $\epsilon_1 = \min(\epsilon, 1)$, then since $|f_n - f| \xrightarrow{u} 0$, $\exists N(\epsilon) \in \mathbb{N}$ such that

$$n \ge N \Rightarrow \sup_{x\in[a,b]} |f_n - f| \le \epsilon_1 \le \epsilon$$

That is,

$$n \geq N \Rightarrow |f_n - f| \leq \epsilon_1 \quad \forall x \in [a,b]$$

or

$$n \geq N \Rightarrow |f_n - f|^2 \leq \epsilon_1^2 < \epsilon_1 \leq \epsilon \quad \forall x \in [a,b]$$

Hence,

$$n \geq N \Rightarrow \sup_{x\in[a,b]} |f_n - f|^2 \leq \epsilon$$

or in other words

$$\lim_{n\to\infty} \sup_{x\in[a,b]} \left| |f_n - f|^2 - 0 \right| = 0$$

i.e., $|f_n - f|^2 \overset{u}{\to} 0$.

Theorem 52 *(Relation between uniform convergence and convergence in $\mathcal{L}^2$)*

Let f_n be a sequence of functions in $\mathcal{L}^2(I)$ where I is bounded. If $f_n \overset{u}{\to} f$,where $f \in \mathcal{L}^2(I)$, then $f_n \overset{\mathcal{L}^2}{\to} f$.

Proof:

We want to prove that

$$\lim_{n\to\infty} \|f_n - f\| = 0$$

Now,

$$\lim_{n\to\infty} \|f_n - f\|^2 = \lim_{n\to\infty} \int_I |f_n - f|^2 \, dx$$

But $f_n \overset{u}{\to} f$. Hence, using the above lemma we have $|f_n - f|^2 \overset{u}{\to} 0$. Moreover, using Theorem 1.17 gives

$$\lim_{n\to\infty} \int_I |f_n - f|^2 \, dx = \int_I \lim_{n\to\infty} |f_n - f|^2 \, dx = 0$$

from which we get

$$\lim_{n\to\infty} \|f_n - f\|^2 = 0$$

which gives

$$\lim_{n\to\infty} \|f_n - f\| = 0.$$

Example 53 *1.24*

Show that the series $\sum_{k=1}^{\infty} \frac{1}{k^2} \sin kx$ is convergent in $\mathcal{L}^2(-\pi,\pi)$.

Solution:

We say that $\sum_{k=1}^{\infty} \frac{1}{k^2} \sin kx$ is convergent in $\mathcal{L}^2(-\pi,\pi)$ if the sequence of partial sums $S_n = \sum_{k=1}^{n} \frac{1}{k^2} \sin kx$ is convergent in $\mathcal{L}^2(-\pi,\pi)$. Using the above theorem it is sufficient to prove that the sequence (S_n) is in $\mathcal{L}^2(-\pi,\pi)$ and $S_n \overset{u}{\to} S$ in $[-\pi,\pi]$ for some $S \in \mathcal{L}^2(-\pi,\pi)$.

We know from example 1.21 that $S_n \xrightarrow{u} S$ in $[-\pi, \pi]$, where $S = \sum_{k=1}^{\infty} \frac{1}{k^2} \sin kx$. Now, since $\frac{1}{k^2} \sin kx$ is continuous $\forall k$ on $[-\pi, \pi]$, then so is S_n, which proves that (S_n) is in $\mathcal{L}^2(-\pi, \pi)$. Next, since $S_n \xrightarrow{u} S$, then S is also continuous on $[-\pi, \pi]$. Thus, $S \in \mathcal{L}^2(-\pi, \pi)$.

Remark 54 *(Testing the convergence of $\sum_{k=1}^{\infty} \frac{1}{k} \cos kx$)*

We cannot use the above argument to prove the convergence of the series $\sum_{k=1}^{\infty} \frac{1}{k} \cos kx$ in $[-\pi, \pi]$. In fact, the series $\sum_{k=1}^{\infty} \frac{1}{k} \cos kx$ is divergent at all $x = 2z\pi, z \in \mathbb{Z}$.

Definition 55 *1.25 ()*

A sequence in $\mathcal{L}^2(-\pi, \pi)$ is called a Cauchy sequence if, $\forall\ \epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n, m \geq N \Rightarrow \|f_n - f_m\| < \epsilon$$

Theorem 56 *1.26 (Convergence and Cauchy sequence)*

1. Every convergent sequence (f_n) in $\mathcal{L}^2$ is a Cauchy sequence [Homework!].
2. For every Cauchy sequence (f_n) in $\mathcal{L}^2$ there is a function $f \in \mathcal{L}^2$ such that $f_n \xrightarrow{\mathcal{L}^2} f$. That is to say $\mathcal{L}^2$ is "complete".

Example 57 *1.28*

Show that the series $\sum_{k=1}^{\infty} \frac{1}{k} \cos kx$ is convergent in $\mathcal{L}^2(-\pi, \pi)$.

Solution:

Clearly, the sequence (S_n), where $S_n = \sum_{k=1}^{n} \frac{1}{k} \cos kx$, is in $\mathcal{L}^2(-\pi, \pi)$. So, we only need to prove that (S_n) is a Cauchy sequence.

Let $\epsilon > 0$,

$$\|S_n(x) - S_m(x)\|^2 = \left\| \sum_{k=m+1}^{n} \frac{1}{k} \cos kx \right\|^2, m < n,$$

Since the set $\{\cos kx : k \in \mathbb{N}\}$ is orthogonal in $\mathcal{L}^2(-\pi, \pi)$ (Example 1.11), we have

$$\left\| \sum_{k=m+1}^{n} \frac{1}{k} \cos kx \right\|^2 = \sum_{k=m+1}^{n} \frac{1}{k^2} \|\cos kx\|^2 = \pi \sum_{k=m+1}^{n} \frac{1}{k^2}$$

but $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent series, therefore the sequence of partial sums $\left(\sum_{k=1}^{n} \frac{1}{k^2}\right)$ is a Cauchy sequence in $[-\pi, \pi]$. In other words for the given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n, m \geq N \Rightarrow \left| \sum_{k=1}^{n} \frac{1}{k^2} - \sum_{k=1}^{m} \frac{1}{k^2} \right| < \frac{\epsilon^2}{\pi}$$

or

$$n, m \geq N \Rightarrow \sum_{k=m+1}^{n} \frac{1}{k^2} < \frac{\epsilon^2}{\pi}$$

Form which we get

$$n, m \geq N \Rightarrow \left\| \sum_{k=m+1}^{n} \frac{1}{k} \cos kx \right\|^2 = \pi \sum_{k=m+1}^{n} \frac{1}{k^2} < \epsilon^2$$

Taking the square root, we get

$$n, m \geq N \Rightarrow \left\| \sum_{k=m+1}^{n} \frac{1}{k} \cos kx \right\| < \epsilon$$

Remark 58 *convergence in $\mathcal{L}^2 \nrightarrow$pointwise convergence.*

Remark 59 *(A series that is divergent pointwise but is convergent in $\mathcal{L}^2(-\pi, \pi)$)*

Using the above steps, we can prove that the series $\sum_{k=1}^{\infty} \frac{1}{k} \cos kx$ is convergent in $\mathcal{L}^2(-\pi, \pi)$. The series $\sum_{k=1}^{\infty} \frac{1}{k} \cos kx$ is divergent at all $x = 2z\pi, z \in \mathbb{Z}$. Thus, convergence in $\mathcal{L}^2 \nrightarrow$pointwise convergence.

Theorem 60 *1.27 (Density of C in $\mathcal{L}^2$)*

For any $f \in \mathcal{L}^2(a, b)$ and any $\epsilon > 0$, there is a continuous function g on $[a, b]$ such that $\|f - g\| < \epsilon$.

The above theorem shows that the set of continuous function on $[a, b]$ is dense in $\mathcal{L}^2(a, b)$ in the same way the rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$. In fact, for every $f \in \mathcal{L}^2(a, b)$, there is a sequence of continuous functions (f_n) such that $f_n \stackrel{\mathcal{L}^2}{\to} f$.

Example 61 *(Density of C in $\mathcal{L}^2$)*

Show that the following sequence of functions in $C[-1, 1]$

$$f_n(x) = \begin{cases} 0, & -1 \leq x \leq \frac{-1}{n} \\ nx + 1, & \frac{-1}{n} < x < 0 \\ 1, & 0 \leq x \leq 1 \end{cases}$$

converges to the function

$$f(x) = \begin{cases} 0, & -1 \leq x < 0 \\ 1, & 0 \leq x \leq 1 \end{cases}$$

in $\mathcal{L}^2(-1, 1)$.

Solution:

We need to show that

$$\lim_{n \to \infty} \|f_n - f\| = 0$$

Now

$$\begin{aligned}
\lim_{n\to\infty}\|f_n - f\| &= \lim_{n\to\infty}\left[\int_{-1}^{1}|f_n - f|^2\,dx\right]^{\frac{1}{2}} \\
&= \lim_{n\to\infty}\left[\int_{-\frac{1}{n}}^{0}(nx+1)^2\,dx\right]^{\frac{1}{2}} \\
&= \lim_{n\to\infty}\left[\left.\frac{(nx+1)^3}{3n}\right|_{x=\frac{-1}{n}}^{0}\right]^{\frac{1}{2}} \\
&= \lim_{n\to\infty}\frac{1}{\sqrt{3n}} = 0
\end{aligned}$$

6 Orthogonal Functions

We are going to learn

- Given an orthogonal set S in $\mathcal{L}^2$ and a function $f \in \mathcal{L}^2$, we answer the following questions:
  - If f is a linear combination of a finite number of elements in S, what are the coefficients in this linear combination?
  - How to find the linear combination in S that best approximates f?
- Completeness of an orthogonal set.
- Bessel's inequality.
- Parseval's relation.

Consider an orthogonal set of functions $S = \{\varphi_1, \varphi_2, \varphi_3, ...\}$ in the complex space $\mathcal{L}^2$.

Question 1

Let $f \in \mathcal{L}^2$ be a finite linear combination of the functions in S, that is,

$$f = \sum_{i=1}^{n} \alpha_i \varphi_i, \qquad \alpha_i \in \mathbb{C}.$$

Find the coefficients "α_i's"!

Answer

If we take the inner product of f with φ_k, $k = 1, 2, ..., n$.

$$\begin{aligned}
\langle f, \varphi_k \rangle &= \sum_{i=1}^{n} \langle \alpha_i \varphi_i, \varphi_k \rangle \\
&\Rightarrow \langle f, \varphi_k \rangle = \sum_{i=1}^{n} \alpha_i \langle \varphi_i, \varphi_k \rangle \\
&\Rightarrow \langle f, \varphi_k \rangle = \alpha_k \|\varphi_k\|^2 \\
&\Rightarrow \alpha_k = \frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2}.
\end{aligned}$$

Therefore, we have

$$f = \sum_{k=1}^{n} \frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2} \varphi_k,$$

Remark 62 *1*

1. f is the sum of the projection vectors of f along φ_k,namely $\frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2} \varphi_k$, $k = 1, 2, ..., n$.

2. In terms of the orthonormal set $\{\psi_k = \frac{\varphi_k}{\|\varphi_k\|}\}$, we have

$$\begin{aligned}
f &= \sum_{k=1}^{n} \frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2} \varphi_k \\
&= \sum_{k=1}^{n} \left\langle f, \frac{\varphi_k}{\|\varphi_k\|} \right\rangle \frac{\varphi_k}{\|\varphi_k\|} \\
&= \sum_{k=1}^{n} \langle f, \psi_k \rangle \psi_k \\
&= \sum_{k=1}^{n} \beta_k \psi_k
\end{aligned}$$

In this case, the coefficient "β_k" is equal to $\frac{\langle f, \psi_k \rangle}{\|\psi_k\|}$, which is the projection of f on ψ_k.

Question 2

Let f be any function in $\mathcal{L}^2$. Find the finite linear combination of the functions in S that best approximates f!

Answer

The best approximation of f is the function $\sum_{k=1}^{n} \alpha_k \varphi_k$ that minimizes the quantity

$$\left\| f - \sum_{k=1}^{n} \alpha_k \varphi_k \right\|$$

but

$$\begin{aligned}
\left\| f - \sum_{k=1}^{n} \alpha_k \varphi_k \right\|^2 &= \left\langle f - \sum_{k=1}^{n} \alpha_k \varphi_k, f - \sum_{k=1}^{n} \alpha_k \varphi_k \right\rangle \\
&= \left\langle f, f - \sum_{k=1}^{n} \alpha_k \varphi_k \right\rangle - \left\langle \sum_{k=1}^{n} \alpha_k \varphi_k, f - \sum_{k=1}^{n} \alpha_k \varphi_k \right\rangle \\
&= \|f\|^2 - \sum_{k=1}^{n} \overline{\alpha_k} \langle f, \varphi_k \rangle - \sum_{k=1}^{n} \alpha_k \langle \varphi_k, f \rangle + \left\langle \sum_{k=1}^{n} \alpha_k \varphi_k, \sum_{j=1}^{n} \alpha_j \varphi_j \right\rangle \\
&= \|f\|^2 - \sum_{k=1}^{n} \overline{\alpha_k} \langle f, \varphi_k \rangle - \sum_{k=1}^{n} \alpha_k \overline{\langle f, \varphi_k \rangle} + \sum_{k=1}^{n} \sum_{j=1}^{n} \alpha_k \overline{\alpha_j} \langle \varphi_k, \varphi_j \rangle \\
&= \|f\|^2 - \sum_{k=1}^{n} \overline{\alpha_k} \langle f, \varphi_k \rangle - \sum_{k=1}^{n} \overline{\overline{\alpha_k} \langle f, \varphi_k \rangle} + \sum_{k=1}^{n} \alpha_k \overline{\alpha_k} \langle \varphi_k, \varphi_k \rangle \\
&= \|f\|^2 - \sum_{k=1}^{n} \left[\overline{\alpha_k} \langle f, \varphi_k \rangle + \overline{\overline{\alpha_k} \langle f, \varphi_k \rangle} \right] + \sum_{k=1}^{n} |\alpha_k|^2 \|\varphi_k\|^2 \\
&= \|f\|^2 - 2 \sum_{k=1}^{n} \operatorname{Re} \overline{\alpha_k} \langle f, \varphi_k \rangle + \sum_{k=1}^{n} |\alpha_k|^2 \|\varphi_k\|^2 \\
&= \|f\|^2 - \sum_{k=1}^{n} \frac{|\langle f, \varphi_k \rangle|^2}{\|\varphi_k\|^2} \\
&\quad + \sum_{k=1}^{n} \|\varphi_k\|^2 \left[|\alpha_k|^2 - 2 \operatorname{Re} \overline{\alpha_k} \frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2} + \frac{|\langle f, \varphi_k \rangle|^2}{\|\varphi_k\|^4} \right] \\
&= \|f\|^2 - \sum_{k=1}^{n} \frac{|\langle f, \varphi_k \rangle|^2}{\|\varphi_k\|^2} \\
&\quad + \sum_{k=1}^{n} \|\varphi_k\|^2 \left[\left(\alpha_k - \frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2} \right) \left(\overline{\alpha_k} - \frac{\overline{\langle f, \varphi_k \rangle}}{\|\varphi_k\|^2} \right) \right] \\
&= \|f\|^2 - \sum_{k=1}^{n} \frac{|\langle f, \varphi_k \rangle|^2}{\|\varphi_k\|^2} + \sum_{k=1}^{n} \|\varphi_k\|^2 \left| \alpha_k - \frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2} \right|^2
\end{aligned}$$

Note that only the last term involves α_k and this term is always ≥ 0. By choosing

$$\alpha_k = \frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2},\ k = 1, 2, ...n.$$

the term attains its minimum, namely 0.As a result, the best approximation of f is given by

$$\sum_{k=1}^{n} \frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2} \varphi_k$$

and the minimum of the quantity $\|f - \sum_{k=1}^{n} \alpha_k \varphi_k\|^2$ is given by

$$\left\| f - \sum_{k=1}^{n} \alpha_k \varphi_k \right\|^2 = \|f\|^2 - \sum_{k=1}^{n} \frac{|\langle f, \varphi_k \rangle|^2}{\|\varphi_k\|^2}.$$

Remark 63 *2*

1. Since
$$\left\| f - \sum_{k=1}^{n} \alpha_k \varphi_k \right\|^2 \geq 0$$
we have,
$$\sum_{k=1}^{n} \frac{|\langle f, \varphi_k \rangle|^2}{\|\varphi_k\|^2} \leq \|f\|^2$$
for all n.Therefore, it is also true as $n \to \infty$, which gives the *Bessel's inequality*, namely
$$\sum_{k=1}^{\infty} \frac{|\langle f, \varphi_k \rangle|^2}{\|\varphi_k\|^2} \leq \|f\|^2$$
for any orthogonal set $\{\varphi_k : k \in \mathbb{N}\}$ and $f \in \mathcal{L}^2$.

2. Bessel's inequality becomes equality if and only if
$$\left\| f - \sum_{k=1}^{\infty} \frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2} \varphi_k \right\|^2 = 0$$
[why? Homework]. That is,
$$f = \sum_{k=1}^{\infty} \frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2} \varphi_k$$
are equal in $\mathcal{L}^2$.

Definition 64 *1.29*

An orthogonal set $\{\varphi_n : n \in \mathbb{N}\}$ in $\mathcal{L}^2$ is said to be complete if, for any $f \in \mathcal{L}^2$,

$$\sum_{k=1}^{n} \frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2} \varphi_k \xrightarrow{\mathcal{L}^2} f$$

Remark 65 *3*

The above definition states that an orthogonal set $\{\varphi_n : n \in \mathbb{N}\}$ in $\mathcal{L}^2$ is complete if for each $f \in \mathcal{L}^2$, we have

$$f = \sum_{k=1}^{\infty} \frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2} \varphi_k$$

in $\mathcal{L}^2$. In other words, $\{\varphi_n : n \in \mathbb{N}\}$ is a basis for $\mathcal{L}^2$.

Theorem 66 *1.30*

An orthogonal set $\{\varphi_n : n \in \mathbb{N}\}$ in $\mathcal{L}^2$ is complete if and only if

$$\sum_{k=1}^{\infty} \frac{|\langle f, \varphi_k \rangle|^2}{\|\varphi_k\|^2} = \|f\|^2$$

Remark 67 *1.31*

1. The equality in theorem 1.31 is called *Parseval's relation* or the *Completeness relation.*
2. We have shown that for any orthogonal set $\{\varphi_n : n \in \mathbb{N}\}$ in $\mathcal{L}^2$, the best approximation for $f \in \mathcal{L}^2$ is $\sum_{k=1}^{n} \frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2} \varphi_k$, and this choice is independent of n. Moreover, if the orthogonal set $\{\varphi_n : n \in \mathbb{N}\}$ is complete, then

$$f = \sum_{k=1}^{\infty} \frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2} \varphi_k$$

   in $\mathcal{L}^2$.
3. When the orthogonal set $\{\varphi_n : n \in \mathbb{N}\}$ is normalized to $\{\psi_n : n \in \mathbb{N}\}$, then
   (a) The Bessel's inequality becomes

$$\sum_{k=1}^{\infty} |\langle f, \psi_k \rangle|^2 \leq \|f\|^2$$

   (b) The Parseval's equality becomes

$$\sum_{k=1}^{\infty} |\langle f, \psi_k \rangle|^2 = \|f\|^2$$

4. Since $\|f\| < \infty$ for all $f \in \mathcal{L}^2$, Bessel's inequality implies that $\langle f, \psi_n \rangle \to 0$ whether the set $\{\psi_n : n \in \mathbb{N}\}$ is complete or not.

5. Parseval's relation can be seen as a generalization of the Pythagoras theorem from $\mathbb{R}^n$ to $\mathcal{L}^2$, where

$$\begin{aligned} \|f\|^2 &\equiv \text{ the square of the length of the vector} \\ \sum_{k=1}^{\infty} |\langle f, \psi_k \rangle|^2 &\equiv \text{ the sum of the squares of the projections of } f \\ & \text{ on the orthonormal basis.} \end{aligned}$$

Part II
The Sturm-Liouville Theory

7 Linear Second-Order Equations

We are going to learn

- Some Terminology related to second-order ordinary differential equations.
- Properties of the solution of a second-order ordinary differential equation.
- Initial and boundary conditions.
- The Wronskian and its relation to the solution of a second-order ordinary differential equation.

Consider the second-order ordinary differential equation on the real interval I

$$a_0(x)\,y'' + a_1(x)\,y' + a_2(x)\,y = f(x)\,, \tag{2.1}$$

where $a_0, a_1 a_2$ and f are given complex functions on I.

1. If $f = 0$ on I, the equation is called *homogeneous*, otherwise it is *nonhomogeneous*.

2. A function $\varphi \in C^2(I)$ is a solution of the above equation if the substituting $y = \varphi$ gives an identity, i.e.

$$a_0(x)\,\varphi'' + a_1(x)\,\varphi' + a_2(x)\,\varphi = f(x) \qquad \forall x \in I.$$

3. Let
$$L = a_0(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_2(x)\,,$$
then in terms of the *differential operator* L, we can write
$$Ly = f(x)$$

   (a) Note that for any functions $\varphi, \psi \in C^2(I)$ and any constants $c_1, c_2 \in \mathbb{C}$, we have
   $$L(c_1\varphi + c_2\psi) = c_1 L\varphi + c_2 L\psi$$
   thus, L is a *linear differential operator.*

   (b) If φ and ψ are solutions of a linear homogeneous equation, then
   $$L\varphi = 0, L\psi = 0$$
   and thus
   $$L(c_1\varphi + c_2\psi) = c_1 L\varphi + c_2 L\psi = 0$$
   This property of *linear homogeneous equations* is the *superposition principle.* That is, if φ and ψ are solutions of a linear homogeneous equation, then so is any linear combination "$c_1\varphi + c_2\psi$" of them.

4. If $a_0(x) \neq 0 \; \forall x \in I$, then equation (2.1) is said to be regular on I, and can be written in the equivalent form (i.e. the two equations have the same set of solutions)
$$y'' + q(x)\,y' + r(x)\,y = g(x)$$
where
$$q = \frac{a_1}{a_0}, r = \frac{a_2}{a_0}, g = \frac{f}{a_0}.$$

5. If $a_0(x_0) = 0$ at some $x_0 \in I$, then equation (2.1) is said to be *singular* and x_0 is called a *singular point* of the equation.

Consider the second-order ordinary differential equation on the real interval I

$$y'' + q(x) y' + r(x) y = g(x) \tag{2.2}$$

1. If q, r and g are continuous functions on I and x_0 is any point in I,then for any numbers ξ and η, there is a unique solution φ of equation (2.2) on I such that

$$\varphi(x_0) = \xi, \quad \varphi'(x_0) = \eta \tag{2.3}$$

Equation (2.2) with the *initial conditions* (2.3) is called an *initial-value problem.*

2. If $g(x) = 0 \ \forall x \in I$, then (2.2) becomes the homogeneous equation

$$y'' + q(x) \acute{y}' + r(x) y = 0 \tag{2.4}$$

and

(a) Equation (2.2) has two linearly independent solutions $y_1(x)$ and $y_2(x)$ on I.

(b) Any solution of (2.2) can be written in the form

$$c_1 y_1 + c_2 y_2 \tag{2.5}$$

for some constants c_1 and c_2. Thus, (2.5) is called the *general solution* of (2.4).

(c) Using $c_1 = c_2 = 0$ in (2.5) shows that 0 is always a solution of (2.4). This solution is called the *trivial solution.*

(d) According to the existence and uniqueness theorem, the trivial solution is the only solution of (2.4) if $\xi = \eta = 0$ in the initial conditions (2.3).

(e) If the coefficients q and r are constants, the general solution of (2.4) is found by solving the second degree equation

$$m^2 + qm + r = 0$$

i. If the roots are distinct, then the general solution is given by

$$c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

ii. If $m_1 = m_2 = m$, then the general solution is given by

$$c_1 e^{mx} + c_2 x e^{mx}$$

3. If the coefficients q and r are *analytical functions* at some point x_0 in the interior of I, i.e. each of them can be represented in an open interval centered at x_0 by a power series in $(x - x_0)$, then

(a) the general solution of (2.4) is also analytic at x_0 and is given by

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

The above series converges in the intersection of I and the two intervals of convergence of q and r.

(b) The coefficients c_n, $n = 2, 3, 4...$ in the above power series can be written in terms of c_0 and c_1 by substituting the series into equation (2.4).

4. If $g(x_0) \neq 0$ for some $x_0 \in I$, then (2.2) is nonhomogeneous and

   (a) If $y_p(x)$ is a particular solution of equation (2.2), then the general solution is given by
   $$y_p + c_1 y_1 + c_2 y_2$$
   (b) A unique solution is obtained by using the initial conditions (2.3) to determine the values of c_1 and c_2.

5. The special case of equation (2.1)
   $$x^2 y'' + axy' + by = 0,$$
   where a and b are constants is called the *Cauchy-Euler equation.* The general solution is found by solving the second degree equation
   $$m^2 + (a-1)m + b = 0$$
   (a) i. If the roots are distinct, then the general solution is given by
   $$c_1 x^{m_1} + c_2 x^{m_2}$$
   ii. If $m_1 = m_2 = m$, then the general solution is given by
   $$c_1 x^m + c_2 x^{mx} \log x$$

In the existence and uniqueness theorem of equation (2.2), we mentioned initial conditions at a point x_0 in I. In most physical application, the differential equation is subject to boundary conditions. That is, if we let $I = [a, b]$ then the conditions are imposed at the end points of the interval, namely a and b.

The general form of the boundary conditions is given by

$$\begin{aligned} \alpha_1 y(a) + \alpha_2 y'(a) + \alpha_3 y(b) + \alpha_4 y'(b) &= \xi, \\ \beta_1 y(b) + \beta_2 y'(b) + \beta_3 y(a) + \beta_4 y'(a) &= \eta \end{aligned}$$

where α_i and β_i are constants and satisfy

$$\sum_{i=1}^{4} |\alpha_i| > 0 \text{ and } \sum_{i=1}^{4} |\beta_i| > 0.$$

Equation (2.1) with the boundary conditions is called a *boundary-value problem.*

1. If $\xi = \eta = 0$, then we have *homogenous boundary conditions*
2. If $\alpha_3 = \alpha_4 = \beta_3 = \beta_4 = 0$, then we have *separated boundary conditions.*
3. Unseparated boundary conditions are called *coupled boundary conditions.*
4. If $y(a) = y(b)$ and $y'(a) = y'(b)$, then we have *periodic boundary conditions.*

Definition 68 *2.1*

For any two functions $f, g \in C^1$ the determinant

$$W(f,g)(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)g'(x) - g(x)f'(x)$$

is called the Wronskian of f and g. The symbol $W(f,g)(x)$ is sometimes abbreviated to $W(x)$.

Lemma 69 *2.2*

If y_1 and y_2 are solutions of the homogeneous equation

$$y'' + q(x)y' + r(x)y = 0, \qquad x \in I$$

where $q \in C(I)$, then either $W(y_1, y_2) = 0$ for all $x \in I$, or $W(y_1, y_2)(x) \neq 0$ for any $x \in I$.

Proof:

Using the definition of the Wronskian, we have

$$W(x) = y_1 y_2' - y_1' y_2$$

Therefore,

$$\begin{aligned} W'(x) &= y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2' \\ &= y_1 y_2'' - y_1'' y_2 \end{aligned}$$

But since y_1 and y_2 are solution of the differential equation, we have

$$\begin{aligned} y_1'' + q y_1' + r y_1 &= 0, \\ y_2'' + q y_2' + r y_2 &= 0 \end{aligned}$$

Multiplying the first equation by $-y_2$ and the second by y_1 gives

$$\begin{aligned} -y_1'' y_2 - q y_1' y_2 - r y_1 y_2 &= 0, \\ y_2'' y_1 + q y_2' y_1 + r y_2 y_1 &= 0 \end{aligned}$$

Adding the above equations gives

$$y_2'' y_1 - y_1'' y_2 + q(y_2' y_1 - y_1' y_2) = 0$$

or

$$W' + qW = 0$$

This is a linear first order ordinary differential equation. Thus, multiplying the above equation by the integrating factor

$$e^{\int q(x)dx}$$

gives

$$e^{\int q(x)dx} W' + q e^{\int q(x)dx} W = 0$$

or

$$\frac{d}{dx}\left[e^{\int q(x)dx} W\right] = 0$$

or

$$e^{\int q(x)dx} W = c$$

or

$$W(x) = c e^{-\int q(x)dx}$$

where c is a constant. If $c = 0$, then $W(x) = 0$ for all x. Otherwise, $W(x) \neq 0$ for all x.

Lemma 70 *2.4*

Any two solutions y_1 and y_2 of equation (2.10) are linearly independent if, and only if, $W(y_1, y_2)(x) \neq 0$ on I.

Proof:

We will prove that: any two solutions y_1 and y_2 of (2.10) are linearly dependent $\Leftrightarrow W(y_1, y_2) = 0$ for some point x_0 in I.

($\Rightarrow$) If y_1 and y_2 are linearly dependent, then $\exists\, k$ constant such that $y_1 = ky_2$. Therefore,

$$\begin{aligned} W &= y_1 y_2' - y_1' y_2 \\ &= k y_2 y_2' - k y_2' y_2 \\ &= 0. \end{aligned}$$

($\Leftarrow$) Let $W(y_1, y_2) = 0$ for some point x_0 in I, then by Lemma 2.2 $W(y_1, y_2) = 0$ for all $x \in I$. This implies that

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0$$

Which means that the system

$$k_1 (y_1, y_1') + k_2 (y_2, y_2') = (0, 0)$$

has a nontrivial solution. Consequently

$$k_1 y_1 + k_2 y_2 = 0$$

has a nontrivial solution, i.e. y_1 and y_2 are linearly dependent functions.

Remark 71 *2.5*

Note that we used the fact that y_1 and y_2 are solutions of (2.10) only to prove that if the Wronskian vanishes on some point in I, then these solutions are linearly dependent. This means we can find two linearly independent functions with a Wronskian that vanishes at some points of their domain. Take for example x, x^2on $[-1, 1]$.

Example 72 *2.6*

Find the solution of the equation

$$y'' + y = 0 \tag{2.12}$$

on the interval $[0, \pi]$, with the following set of conditions

1. $$y(0) = 0, \quad y'(0) = 1.$$
2. $$y(0) = 0, \quad y'(0) = 0.$$
3. $$y(0) = 0, \quad y(\pi) = 0.$$

then show that any choice of the initial conditions

$$y(x_0) = \xi, \quad y'(x_0) = \eta, \quad x_0 \in [0, \pi]$$

with (2.12) gives a unique solution.

Solution
The general solution of (2.12) is

$$y(x) = c_1 \cos x + c_2 \sin x$$

1. The derivative of the solution

$$y'(x) = -c_1 \sin x + c_2 \cos x$$

Using the first choice of initial condition gives

$$\begin{aligned} 0 &= y(0) = c_1 \cos 0 + c_2 \sin 0 = c_1, \\ 1 &= y'(0) = -c_1 \sin 0 + c_2 \cos 0 = c_2. \end{aligned}$$

That is, these initial conditions gives the solution

$$y(x) = \sin x$$

2. Using the second choice of initial condition gives

$$\begin{aligned} 0 &= y(0) = c_1 \cos 0 + c_2 \sin 0 = c_1, \\ 0 &= y'(0) = -c_1 \sin 0 + c_2 \cos 0 = c_2. \end{aligned}$$

   That is, these initial conditions gives the trivial solution.

3. Using the third choice of boundary condition gives

$$\begin{aligned} 0 &= y(0) = c_1 \cos 0 + c_2 \sin 0 = c_1, \\ 0 &= y(\pi) = c_1 \cos \pi + c_2 \sin \pi = c_1. \end{aligned}$$

   which gives the solution

$$y(x) = c_2 \sin x,$$

   where c_2 is any constant. That is, this choice of boundary conditions does not give a unique solution for the problem.

Using the general initial conditions

$$y(x_0) = \xi, \quad y'(x_0) = \eta$$

we have

$$\begin{aligned} \xi &= y(0) = c_1 \cos x_0 + c_2 \sin x_0, \\ \eta &= y'(0) = -c_1 \sin x_0 + c_2 \cos x_0. \end{aligned}$$

which gives a unique solution for $[c_1, c_2]^t$ if and only if

$$\begin{vmatrix} \cos x_0 & \sin x_0 \\ -\sin x_0 & \cos x_0 \end{vmatrix} \neq 0$$

or equivalently

$$\cos^2 x_0 + \sin^2 x_0 \neq 0$$

which is always true.

8 Self-Adjoint Differential Operator

We are going to learn

- The adjoint of a linear operator in a finite-dimensional inner product space.
- Properties of the eigenvalue problem $Lu+\lambda u = 0$ for a self-adjoint operator L in a finite-dimensional inner product space.
- The extension of the above concepts to a infinite-dimensional inner product space, namely $\mathcal{L}^2$.

A *linear operator* in a vector space X is a mapping

$$A : X \to X$$

which satisfies

$$A(ax + by) = aAx + bAy$$

for all $a, b \in \mathbf{F}$ and all $x, y \in X$.

The *adjoint* of a linear operator A, if it exists, is the mapping

$$A' : X \to X$$

that satisfies

$$\langle Ax, y \rangle = \langle x, A'y \rangle$$

for all $x, y \in X$. If $A' = A$, then A is said to *self-adjoint*.

Example 73 *(Adjoint of a linear operator in a finite-dimensional inner product space)*

Consider a finite-dimensional inner product space X (e.g. $\mathbb{C}^n$ over $\mathbb{C}$). If $B = \{\mathbf{e}_i : 1 \le i \le n\}$ is an orthonormal basis for X, then any linear operator T can be represented by a matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & . & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

that is,

$$A[x]_B = [T(x)]_B$$

for all $x \in X$. Here, $[y]_B$ is the coordinate vector of y with respect to the basis B.

The columns of A are given by

$$[T(\mathbf{e}_i)]_B = \begin{bmatrix} a_{1i} & \cdots & a_{ni} \end{bmatrix}^t, i = 1, 2, ..., n.$$

In this case, the adjoint of A is given by

$$A' = \begin{bmatrix} \overline{a}_{11} & \cdots & \overline{a}_{n1} \\ \vdots & . & \vdots \\ \overline{a}_{1n} & \cdots & \overline{a}_{nn} \end{bmatrix} = \overline{A}^T$$

Definition 74 *(Eigenvalues and eigenvectors)*

Consider a linear operator A on a inner product space X. If there exists an nonzero vector x in X such that

$$Ax = ax$$

for $a \in \mathbb{C}$, then x is called an eigenvector of A corresponding to the eigenvalue a.

Remark 75 *(Properties of a self-adjoint matrix)*

If A is a self-adjoint (*Hermitian*) matrix, then

1. The eigenvalues of A are all real numbers.
2. The eigenvectors of A corresponding to distinct eigenvalues are orthogonal.
3. The eigenvectors of A form a basis of X.

Adjoints of operators generalize conjugate transposes of square matrices to infinite-dimensional situations.

8.1 Generalization to The Space $\mathcal{L}^2$

Consider the second-order homogenous differential equation

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = 0,$$

which can be written in terms in of the linear operator

$$\begin{aligned} L &: \mathcal{L}^2(I) \cap C^2(I) \to \mathcal{L}^2(I), \\ L &= p(x)\frac{d^2}{dx^2} + q(x)\frac{d}{dx} + r(x) \end{aligned}$$

as follows

$$Ly = 0.$$

Remark 76 *1*

1. We have assumed that p, q, r are all in $C^2(I)$.
2. If I is a closed bounded interval, then

$$C^2(I) \subset C(I) \subset \mathcal{L}^2(I)$$

and thus

$$\mathcal{L}^2(I) \cap C^2(I) = C^2(I)$$

[Give an example for a function in $f \in C^2(I)$, but $f \notin \mathcal{L}^2(I)$]

By definition, the adjoint of L, denoted by L', must satisfy

$$\langle Lf, g\rangle = \langle f, L'g\rangle$$

for all $f, g \in \mathcal{L}^2(I) \cap C^2(I)$.

Let $I = (a, b)$ where I can be infinite. Now,

$$
\begin{aligned}
\langle Lf, g\rangle &= \int_a^b \left(pf^{''} + qf^{'} + rf\right)\overline{g}dx \\
&= \int_a^b pf^{''}\overline{g}dx + \int_a^b qf^{'}\overline{g}dx + \int_a^b rf\overline{g}dx \\
&= \int_a^b pf^{''}\overline{g}dx + \left(\int_a^b f^{'}\left(p\overline{g}\right)^{'} dx - \int_a^b f^{'}\left(p\overline{g}\right)^{'} dx\right) + \\
&\quad \int_a^b qf^{'}\overline{g}dx + \left(\int_a^b f\left(q\overline{g}\right)^{'} dx - \int_a^b f\left(q\overline{g}\right)^{'} dx\right) + \int_a^b rf\overline{g}dx \\
&= \int_a^b \left(f^{'}\left(p\overline{g}\right)\right)^{'} dx - \int_a^b f^{'}\left(p\overline{g}\right)^{'} dx + \\
&\quad + \int_a^b \left(f\left(q\overline{g}\right)\right)' dx - \int_a^b f\left(q\overline{g}\right)^{'} dx + \int_a^b rf\overline{g}dx \\
&= f^{'}p\overline{g}\Big|_a^b - \int_a^b f^{'}\left(p\overline{g}\right)^{'} dx + \left(-\int_a^b f\left(p\overline{g}\right)^{''} dx + \int_a^b f\left(p\overline{g}\right)^{''} dx\right) \\
&\quad + fq\overline{g}|_a^b - \int_a^b f\left(q\overline{g}\right)^{'} dx + \int_a^b rf\overline{g}dx \\
&= f^{'}p\overline{g}\Big|_a^b - \int_a^b \left(f\left(p\overline{g}\right)^{'}\right)^{'} dx + \int_a^b f\left(p\overline{g}\right)^{''} dx \\
&\quad + fq\overline{g}|_a^b - \int_a^b f\left(q\overline{g}\right)^{'} dx + \int_a^b rf\overline{g}dx \\
&= f^{'}p\overline{g}\Big|_a^b - f\left(p\overline{g}\right)^{'}\Big|_a^b + fq\overline{g}|_a^b \\
&\quad + \int_a^b f\left(p\overline{g}\right)^{''} dx - \int_a^b f\left(q\overline{g}\right)^{'} dx + \int_a^b rf\overline{g}dx \\
&= f^{'}p\overline{g} - fp^{'}\overline{g} - fp\overline{g}^{'} + fq\overline{g}\Big|_a^b \\
&\quad + \int_a^b f\left[\overline{\left(\overline{p}g\right)^{''} - \left(\overline{q}g\right)^{'} + \overline{r}g}\right] dx \\
&= p\left(f^{'}\overline{g} - f\overline{g}^{'}\right) + \left(q - p^{'}\right)f\overline{g}\Big|_a^b + \left\langle f, \left(\overline{p}g\right)^{''} - \left(\overline{q}g\right)^{'} + \overline{r}g\right\rangle
\end{aligned}
$$

Thus,

$$
\langle Lf, g\rangle = p\left(f^{'}\overline{g} - f\overline{g}^{'}\right) + \left(q - p^{'}\right)f\overline{g}\Big|_a^b + \left\langle f, \left(\overline{p}g\right)^{''} - \left(\overline{q}g\right)^{'} + \overline{r}g\right\rangle
$$

Remark 77 *2*

1. If (a, b) is infinite or the any of integrands are unbounded at a or b, then the integrals are improper.

2. The right hand side is well defined if $p \in C^2(a,b), q \in C^1(a,b)$ and $r \in C(a,b)$.

We can write

$$\langle Lf, g\rangle = p\left(f^{'}\overline{g} - f\overline{g}^{'}\right) + \left(q - p^{'}\right) f\overline{g}\Big|_a^b + \langle f, L^* g\rangle \tag{2.24}$$

where

$$\begin{aligned} L^* g &= (\overline{p}g)^{''} - (\overline{q}g)^{'} + \overline{r}g \\ &= (\overline{p}'g + \overline{p}g')' - \left(\overline{q}^{'}g + \overline{q}g^{'}\right) + \overline{r}g \\ &= \overline{p}^{''}g + 2\overline{p}'g' + \overline{p}g^{''} - \overline{q}^{'}g - \overline{q}g^{'} + \overline{r}g \\ &= \overline{p}g^{''} + (2\overline{p}' - \overline{q})g' + \left(\overline{p}^{''} - \overline{q}^{'} + \overline{r}\right) g \end{aligned}$$

The operator

$$L^* = \overline{p}\frac{d^2}{dx^2} + (2\overline{p}' - \overline{q})\frac{d}{dx} + \left(\overline{p}^{''} - \overline{q}^{'} + \overline{r}\right)$$

is called the *formal adjoint* of L.

L is said to be *formally self-adjoint* if

$$L = L^*$$

Theorem 78 *2.14*

Let

$$L : \mathcal{L}^2(a,b) \cap C^2(a,b) \to \mathcal{L}^2(a,b)$$

be a linear differential operator of second order defined by

$$Lu = p(x)u'' + q(x)u' + r(x)u, \qquad x \in (a,b),$$

where $p \in C^2(a,b)$, $q \in C^1(a,b)$, and $r \in C(a,b)$.Then

1. L is formally self-adjoint, that is, $L^* = L$, if the coefficients p, q and r are real and $p' = q$.

2. L is self-adjoint, that is, $L' = L$, if L is formally self-adjoint and

$$p\left(f^{'}\overline{g} - f\overline{g}^{'}\right)\Big|_a^b = 0$$

for all $f, g \in \mathcal{L}^2(a,b) \cap C^2(a,b)$.

3. If L is self adjoint, then the eigenvalues of the equation

$$Lu + \lambda u = 0$$

are all real and any pair of eigenfunctions associated with distinct eigenvalues are orthogonal in $\mathcal{L}^2(a,b)$.

Proof

1. The formal adjoint of L is given by

$$L^* = \overline{p}\frac{d^2}{dx^2} + (2\overline{p}' - \overline{q})\frac{d}{dx} + \left(\overline{p}^{''} - \overline{q}^{'} + \overline{r}\right)$$

   L is *formally self-adjoint* if

$$L = L^*$$

   That is,

$$\begin{aligned} \overline{p} &= p, \\ 2\overline{p}' - \overline{q} &= q, \\ \overline{p}^{''} - \overline{q}^{'} + \overline{r} &= r. \end{aligned}$$

   which are satisfied if and only if p, q and r are real functions and $p' = q$.

2. If L is formally self-adjoint, then

$$\begin{aligned} Lg &= pg^{''} + p^{'}g^{'} + rg \\ &= (pg')' + rg \end{aligned}$$

   That is,

$$L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + r$$

   and (2.24) becomes

$$\langle Lf, g\rangle = p\left(f^{'}\overline{g} - f\overline{g}^{'}\right)\Big|_a^b + \langle f, Lg\rangle$$

   Hence, L is *self-adjoint* if

$$p\left(f^{'}\overline{g} - f\overline{g}^{'}\right)\Big|_a^b = 0$$

   for all $f, g \in \mathcal{L}^2(a,b) \cap C^2(a,b)$.

3. Suppose $\lambda \in \mathbb{C}$ is an eigenvalue of $-L$, then $\exists f \in \mathcal{L}^2(a,b) \cap C^2(a,b)$, $f \neq 0$ such that

$$-Lf = \lambda f$$

   Thus,

$$-\langle Lf, f\rangle = \langle -Lf, f\rangle = \langle \lambda f, f\rangle = \lambda \|f\|^2$$

   But L is self-adjoint implies that

$$-\langle Lf, f\rangle = -\langle f, Lf\rangle = \langle f, -Lf\rangle = \langle f, \lambda f\rangle = \overline{\lambda}\|f\|^2$$

   Thus,

$$\lambda\|f\|^2 = \overline{\lambda}\|f\|^2$$

Since $f \neq 0$, then we can divide the above equation by $\|f\|^2$, which gives

$$\lambda = \overline{\lambda}$$

That is, $\lambda \in \mathbb{R}$.

Second, we want to prove that if f and g are eigenfunctions of $-L$ associated with the eigenvalues λ and μ, respectively where $\lambda \neq \mu$, then f and g are orthogonal.

$$\begin{aligned} \lambda \langle f, g \rangle &= \langle \lambda f, g \rangle \\ &= \langle -Lf, g \rangle \\ &= -\langle Lf, g \rangle \\ &= -\langle f, Lg \rangle \\ &= -\langle f, -\mu g \rangle \\ &= \mu \langle f, g \rangle \end{aligned}$$

Thus,

$$\lambda \langle f, g \rangle - \mu \langle f, g \rangle = 0$$

or

$$(\lambda - \mu) \langle f, g \rangle = 0$$

but $\lambda - \mu \neq 0$, hence $\langle f, g \rangle = 0$.

Remark 79 *2.15*

If $p' = q$, then the continuity of p'' and q' are no longer required. That is, the above theorem is valid under the weaker condition that p' is continuous.

Example 80 *2.16*

Determine the eigenvalues and eigenfunctions of

$$u'' + \lambda u = 0$$

on $(0, \pi)$ subject to the homogenous boundary conditions

$$\begin{aligned} u(0) &= 0, \\ u(\pi) &= 0. \end{aligned}$$

In the differential operator $-L = -\frac{d^2}{dx^2}$, we have $p = -1, q = 0, r = 0$ all are real and $p' = 0 = q$. Thus, $-L$ is formally self-adjoint.

The auxiliary equation is given by

$$m^2 + \lambda = 0$$
$$\Rightarrow m = \pm\sqrt{-\lambda}$$

CASE1: If $\lambda > 0$, then $m = \pm\sqrt{\lambda}i$ and the general solution is given by

$$u(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$$

Using the boundary conditions

$$0 = u(0) = c_1$$

$$\begin{aligned} 0 &= u(\pi) = c_2 \sin\sqrt{\lambda}\pi \\ &\Rightarrow \sqrt{\lambda}\pi = n\pi \\ &\Rightarrow \lambda = n^2, n \in \mathbb{N} \end{aligned}$$

The eigenvalues are

$$\{n^2 : n \in \mathbb{N}\} = \{1, 4, 9, ...\} \subset \mathbb{R}$$

and the corresponding eigenvectors are

$$\{\sin n\pi : n \in \mathbb{N}\} = \{\sin x, \sin 2x, \sin 3x, ...\}$$

where we have chosen $c_2 = 1$.[Verify that the eigenfunctions are orthogonal!]

CASE2: If $\lambda < 0$,then $m = \pm\sqrt{-\lambda}$ and the general solution is given by

$$u(x) = c_1 \cosh\sqrt{-\lambda}x + c_2 \sinh\sqrt{-\lambda}x$$

Using the boundary conditions

$$0 = u(0) = c_1$$

$$\begin{aligned} 0 &= u(\pi) = c_2 \sinh\sqrt{-\lambda}\pi \\ &\Rightarrow c_2 = 0 \end{aligned}$$

But the eigenfunction cannot be zero. Hence, there is no negative eigenvalues.

CASE3: If $\lambda = 0$, then $m = 0$ the general solution is given by

$$u(x) = c_1 + c_2 x$$

Using the boundary conditions

$$0 = u(0) = c_1$$

$$\begin{aligned} 0 &= u(\pi) = c_2\pi \\ &\Rightarrow c_2 = 0 \end{aligned}$$

Again the eigenfunction cannot be zero. So we have no eigenvalues on $(-\infty, 0]$.

Example 81 *2.17*

Determine the eigenvalue and eigenfunctions of

$$u'' + \lambda u = 0$$

on $(0, l)$ subject to the separated boundary conditions

$$\begin{aligned} u(0) &= 0, \\ hu(l) + u'(l) &= 0 \end{aligned}$$

where $h > 0$.

As done in the previous example

$$\begin{gathered} m^2 + \lambda = 0 \\ \Rightarrow m = \pm\sqrt{-\lambda} \end{gathered}$$

CASE1: If $\lambda > 0$, then $m = \pm\sqrt{\lambda}i$ and the general solution is given by

$$u(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$$

Using the boundary conditions

$$0 = u(0) = c_1$$

$$\begin{aligned} 0 &= hu(l) + u'(l) \\ &= hc_2 \sin\sqrt{\lambda}l + c_2\sqrt{\lambda}\cos\sqrt{\lambda}l \end{aligned}$$

$$hc_2 \sin\sqrt{\lambda}l = -c_2\sqrt{\lambda}\cos\sqrt{\lambda}l$$

or

$$\tan\sqrt{\lambda}l = -\frac{\sqrt{\lambda}l}{hl}$$

where we have divided by $c_2 \cos\sqrt{\lambda}l \neq 0$ [Why?].

If we write $\alpha = \sqrt{\lambda}l$, then the solution of the above equation is the intersections of the graphs of $y = \tan\alpha$ with $y = -\frac{\alpha}{hl}$ as shown in the figure below.

The eigenvalues are λ_n that satisfies

$$\alpha_n = \sqrt{\lambda_n}l$$

that is,

$$\{\lambda_n = \left(\frac{\alpha_n}{l}\right)^2, n \in \mathbb{N}\}$$

and the corresponding eigenvectors are

$$\{\sin\left(\frac{\alpha_n}{l}x\right) : n \in \mathbb{N}\}$$

There are no eigenvalue on $(-\infty, 0]$.

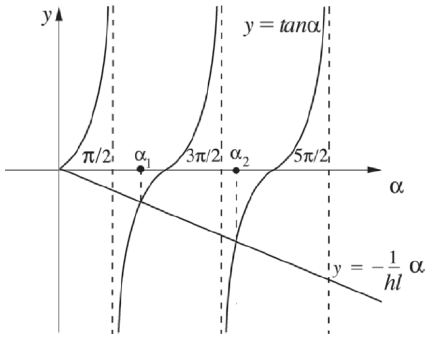


Figure 2.2

8.2 Transforming a second-order differential operator to a formally self-adjoint operator

Recall that in the differential equation

$$Lu = 0,$$

where

$$L : \mathcal{L}^2(a,b) \cap C^2(a,b) \to \mathcal{L}^2(a,b)$$

$$L = p(x)\frac{d^2}{dx^2} + q(x)\frac{d}{dx} + r(x)$$

the operator L is a formally self-adjoint operator iff p, q and r are all real and $p' = q$, but what if last condition is not satisfied!

Theorem 82 *(Transforming the operator L to a formally self-adjoint operator)*

Let

$$L : \mathcal{L}^2(a,b) \cap C^2(a,b) \to \mathcal{L}^2(a,b)$$

be a linear differential operator of second order defined by

$$Lu = p(x)u'' + q(x)u' + r(x)u, \quad x \in (a,b),$$

where $p \in C^2(a,b)$, $q \in C^1(a,b)$, $r \in C(a,b)$ and p, q and r are all real functions, but $p' \neq q$ on (a,b). Then,

1. There exists a strictly positive function $\rho(x) \in C^2(a,b)$ such that

   $$\widetilde{L} = \rho L,$$

   is formally self-adjoint.

2. $\widetilde{L}$ is self-adjoint if $\widetilde{L}$ is formally self-adjoint and

$$\rho p\left(f^{'}\overline{g}-f\overline{g}^{'}\right)\Big|_{a}^{b}=0$$

for all $f, g \in \mathcal{L}^2(a, b) \cap C^2(a, b)$.

3. If $\widetilde{L} = \rho L$ is self-adjoint, then the eigenvalues of the operator L are all real and any pair of eigenfunctions associated with distinct eigenvalues are orthogonal in $\mathcal{L}^2_\rho(a, b)$.

Proof

1. Redoing the algebraic manipulation that was done to find the adjoint operator of L, with L replaced by $\widetilde{L}$, we get

$$\left\langle \widetilde{L}f, g\right\rangle = \rho p\left(f^{'}\overline{g}-f\overline{g}^{'}\right)+\left(\rho q-(\rho p)^{'}\right)f\overline{g}\Big|_{a}^{b}+\left\langle f, \widetilde{L}^{*}g\right\rangle$$

where

$$\widetilde{L}^{*}=\rho p\frac{d^{2}}{dx^{2}}+\left(2(\rho p)'-\rho q\right)\frac{d}{dx}+\left((\rho p)^{''}-\rho q^{'}+\rho r\right)$$

where we have used the fact that ρ, p, q and r are all real functions. Note that $\widetilde{L}$ is formally self-adjoint, i.e. $\widetilde{L}=\widetilde{L}^{*}$ if

$$\begin{aligned}\rho p &= \rho p,\\ 2(\rho p)'-\rho q &= \rho q,\\ (\rho p)^{''}-\rho q^{'}+\rho r &= \rho r,\end{aligned}$$

which is true if

$$(\rho p)'=\rho q$$

which gives

$$\rho' p+\rho p'=\rho q$$

or

$$\rho'+\frac{p'-q}{p}\rho=0$$

This is a first-order homogenous differential equation, with integrating factor

$$\exp\int\left(\frac{p'-q}{p}\right)dx$$

Hence,

$$\exp\left(\int\frac{p'-q}{p}dx\right)\rho'+\exp\left(\int\frac{p'-q}{p}dx\right)\left(\frac{p'-q}{p}\right)\rho=0$$

or

$$\frac{d}{dx}\left(\exp\left(\int\frac{p'-q}{p}dx\right)\rho\right)=0$$

or

$$\exp\left(\int\frac{p'-q}{p}dx\right)\rho=c$$

Assuming without loss of generality that $p(x)>0$ on (a,b), we have

$$\begin{aligned}\rho &= c\exp\left(\int\frac{q-p'}{p}dx\right)\\ &= c\exp\left(\int\frac{q}{p}dx\right)\exp\left(-\int\frac{p'}{p}dx\right)\\ &= c\exp\left(\int\frac{q}{p}dx\right)\exp\left(-\ln p\right)\\ &= c\exp\left(\int\frac{q}{p}dx\right)\exp\left(\ln p^{-1}\right)\\ &= \frac{c}{p}\exp\left(\int\frac{q}{p}dx\right)\end{aligned}$$

where c is a constant. Note that ρ is a strictly positive function and $\rho(x)\in C^2(a,b)$.

2. [This part is homework!]. Hint: use the relation

$$\left\langle\widetilde{L}f,g\right\rangle=\rho p\left(f^{'}\overline{g}-f\overline{g}^{'}\right)+\left(\rho q-(\rho p)^{'}\right)f\overline{g}\Big|_a^b+\left\langle f,\widetilde{L}^{*}g\right\rangle$$

3. Let $u\in\mathcal{L}_\rho^2(a,b)$ be an eigenfunction of the operator L corresponding to the eigenvalue λ, that is

$$Lu+\lambda u=0$$

The above equation is equivalent to

$$\widetilde{L}u+\lambda\rho u=0$$

where $\widetilde{L}=\rho L$ is a self-adjoint operator. Now,

$$\begin{aligned}\lambda\|u\|_\rho^2 &= \lambda\langle\rho u,u\rangle\\ &= \langle\lambda\rho u,u\rangle\\ &= \left\langle-\widetilde{L}u,u\right\rangle\\ &= \left\langle u,-\widetilde{L}u\right\rangle\\ &= \langle u,\lambda\rho u\rangle\\ &= \overline{\lambda}\langle u,\rho u\rangle\\ &= \overline{\lambda}\|u\|_\rho^2\end{aligned}$$

Since $\|u\|_\rho^2 \neq 0$, λ must be a real number.

Now, if $v \in \mathcal{L}_\rho^2(a,b)$ is an other eigenfunction of the operator L corresponding to a different eigenvalue μ, then we have

$$\begin{aligned}(\lambda-\mu)\langle u,v\rangle_\rho &= \lambda\langle u,v\rangle_\rho - \mu\langle u,v\rangle_\rho \\ &= \left\langle -\widetilde{L}u, v\right\rangle - \left\langle u, -\widetilde{L}v\right\rangle \\ &= \left\langle u, -\widetilde{L}v\right\rangle - \left\langle u, -\widetilde{L}v\right\rangle \\ &= 0.\end{aligned}$$

i.e. u and v are orthogonal in $\mathcal{L}_\rho^2(a,b)$.

Corollary 83 *2.19*

If $L : \mathcal{L}^2(a,b) \cap C^2(a,b) \to \mathcal{L}^2(a,b)$ is a self-adjoint linear operator and ρ is a positive and continuous function on $[a,b]$, then the eigenvalues of the equation

$$Lu + \lambda\rho u = 0$$

are all real and any pair of the eigenfunctions associated with distinct eigenvalues are orthogonal in $\mathcal{L}_\rho^2(a,b)$.

Remark 84 *2.20*

1. The eigenvalue problem equivalent to the problem

$$Lu + \lambda\rho u = 0$$

(where L here is a self-adjoint operator) is

$$\frac{1}{\rho}Lu + \lambda u = 0$$

Therefore, the eigenvalues and eigenfunctions obtained by solving the former equation are actually the eigenvalues and eigenfunctions of the operator $-\frac{1}{\rho}L$.

2. If (a,b) is a finite interval, then continuos positive function $\rho(x)$ attains its minimum and maximum on $[a,b]$, that is,

$$0 < \alpha \leq \rho(x) \leq \beta < \infty$$

$\Rightarrow$

$$0 < \int_a^b \alpha |u|^2 \, dx \leq \int_a^b \rho(x)|u|^2 \, dx \leq \int_a^b \beta |u|^2 \, dx < \infty$$

$\Rightarrow$

$$0 < \sqrt{\alpha}\,\|u\| \leq \|u\|_\rho \leq \sqrt{\beta}\,\|u\| < \infty$$

from which we get

$$\|u\| < \infty \Leftrightarrow \|u\|_\rho < \infty$$

i.e. $\mathcal{L}^2(a,b)$ and $\mathcal{L}_\rho^2(a,b)$ are the same, but have different inner product spaces.

3. The operator L in the above theory can be any self-adjoint linear operator on an inner product space.

Example 85 *2.21*

1. Find the eigenfunctions and eigenvalues of the boundary value problem on $[1, b]$

$$x^2y'' + xy' + \lambda y = 0,$$
$$y(1) = y(b) = 0.$$

2. Is the above differential operator self-adjoint? If not transform it to a formally self-adjoint operator.

Solution

1. The above equation is a Cauchy-Euler equation, with $a = 1, b = \lambda$.Thus, the auxiliary equation is given by

$$m^2 + (1-1)\,m + \lambda = 0$$
$$\Rightarrow m^2 + \lambda = 0$$
$$\Rightarrow m = \pm i\sqrt{\lambda}$$

where we have assumed that $\lambda > 0$ [show that there are no eigenvalues in $(-\infty, 0]$].

The general solution is given by

$$\begin{aligned}
y(x) &= d_1 x^{i\sqrt{\lambda}} + d_2 x^{-i\sqrt{\lambda}} \\
&= d_1 e^{\ln x^{i\sqrt{\lambda}}} + d_2 e^{\ln x^{-i\sqrt{\lambda}}} \\
&= d_1 e^{i\sqrt{\lambda}\ln x} + d_2 e^{-i\sqrt{\lambda}\ln x} \\
&= d_1\left[\cos\left(\sqrt{\lambda}\ln x\right) + i\sin\left(\sqrt{\lambda}\ln x\right)\right] \\
&\quad + d_2\left[\cos\left(\sqrt{\lambda}\ln x\right) - i\sin\left(\sqrt{\lambda}\ln x\right)\right] \\
&= (d_1 + d_2)\cos\left(\sqrt{\lambda}\ln x\right) \\
&\quad + (d_1 - d_2)\, i\sin\left(\sqrt{\lambda}\ln x\right) \\
&= c_1\cos\left(\sqrt{\lambda}\ln x\right) + c_2\sin\left(\sqrt{\lambda}\ln x\right)
\end{aligned}$$

Using the boundary conditions

$$\begin{aligned}
0 &= y(1) = c_1, \\
0 &= y(b) = c_2\sin\left(\sqrt{\lambda}\ln b\right)
\end{aligned}$$

but $c_2 \neq 0$ (otherwise, the eigenfunction will be zero!). Thus,

$$\sin\left(\sqrt{\lambda}\ln b\right) = 0$$
$$\Rightarrow \sqrt{\lambda}\ln b = n\pi, n \in \mathbb{N},$$
$$\Rightarrow \lambda_n = \left(\frac{n\pi}{\ln b}\right)^2, n \in \mathbb{N}$$

and the eigenfunction corresponding to these eigenvalues are

$$y_n(x) = \sin\left(\frac{n\pi}{\ln b}\ln x\right)$$

2. In this example,

$$L = x^2\frac{d^2}{dx^2} + x\frac{d}{dx}$$

that is,

$$\begin{aligned} p(x) &= x^2, \\ q(x) &= x, \\ r(x) &= 0, \end{aligned}$$

all are real, but

$$p'(x) = 2x \neq x = q(x)$$

Thus, L is not a formally self-adjoint operator. If we take

$$\begin{aligned} \rho(x) &= \frac{1}{p(x)}\exp\left(\int \frac{q(x)}{p(x)}dx\right) \\ &= \frac{1}{x^2}\exp\left(\int \frac{1}{x}dx\right) \\ &= \frac{1}{x} \end{aligned}$$

then,

$$\begin{aligned} \frac{1}{x}L &= \frac{1}{x}\left(x^2\frac{d^2}{dx^2} + x\frac{d}{dx}\right) \\ &= x\frac{d^2}{dx^2} + \frac{d}{dx} \\ &= \frac{d}{dx}\left(x\frac{d}{dx}\right) \end{aligned}$$

is formally self-adjoint. [Verify!]

According to theory developed above, the eigenfunctions

$$y_n(x) = \sin\left(\frac{n\pi}{\ln b}\ln x\right)$$

are orthogonal in $\mathcal{L}^2_\rho(a,b)$. That is for $n \neq m$,

$$\langle y_n, y_m \rangle_\rho = \int_1^b \frac{1}{x} \sin\left(\frac{n\pi}{\ln b} \ln x\right) \sin\left(\frac{m\pi}{\ln b} \ln x\right) dx = 0$$

[Verify!]

Part III
The Sturm-Liouville Theory

9 The Sturm-Liouville Problem

We are going to learn

- Define The Sturm-Liouville problem.
- Generalize the third part of the eigenvalue theory to an infinite-dimensional space.

Definition 86

Let L be a formally self-adjoint operator of the form

$$L = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + r(x), \tag{(2.33)}$$

The eigenvalue equation

$$Lu + \lambda\rho(x)u = 0, \quad x \in (a,b) \tag{(2.34)}$$

subject to the separated homogenous boundary conditions

$$\begin{aligned} \alpha_1 u(a) + \alpha_2 u'(a) &= 0, \quad |\alpha_1| + |\alpha_2| > 0 \\ \beta_1 u(b) + \beta_2 u'(b) &= 0, \quad |\beta_1| + |\beta_2| > 0 \end{aligned} \tag{(2.35)}$$

where α_i and β_i are real constants, is called a *Sturm-Liouville eigenvalue problem*, or *SL problem* for short.

Definition 87

The SL problem is called *regular* if the interval (a,b) is bounded and $p \neq 0$ on (a,b). Otherwise, the SL problem is called *singular*.

Remark 88 *1*

1. The solution of the SL problem (2.34) with boundary conditions (2.35) are the eigenfunctions of the operator $-\frac{1}{\rho}L$.
2. Under the above boundary conditions, L is a self-adjoint operator [verify!]. Therefore, the eigenvalues in (2.34), if they exist, are real and the corresponding eigenvectors are orthogonal in $\mathcal{L}^2_\rho(a,b)$.
3. In a regular SL problem, we assume that $p(x) > 0$.

The following theorem generalize the third part of the eigenvalue theory, namely "that eigenvectors of a self-adjoint matrix in a finite-dimensional space X form a basis for that space", to an infinite-dimensional inner product space.

Theorem 89 *2.29*

Assuming that $p', r, \rho \in C([a,b])$, and $p, \rho > 0$ on $[a,b]$, the SL eigenvalue problem defined by Equations (2.34) and (2.35) has an infinite sequence of <u>real</u> eigenvalues

$$\lambda_0 < \lambda_1 < \lambda_2 < ...$$

such that $\lambda_n \to \infty$. To each eigenvalue λ_n corresponds a single eigenfunction φ_n, and the sequence of eigenfunctions $\{\varphi_n : n \in \mathbb{N}\}$ forms an orthogonal <u>basis</u> of $\mathcal{L}^2_\rho(a,b)$.

Remark 90 *2.30*

1. For all λ,
$$\lambda \geq -\max\{|r(x)| : a \leq x \leq b\}.$$

2. If the SL problem (2.34) is considered under the periodic boundary conditions
$$\begin{aligned} u(a) &= u(b), \\ u'(a) &= u'(b), \end{aligned}$$
then,

   (a) The operator L defined by (2.33) will be self-adjoint if $p(a) = p(b)$ [verify!].

   (b) Theorem (2.29) holds in this case, except that the uniqueness of the eigenfunctions for each eigenvalues is not guaranteed.

Example 91 *2.31*

Find the eigenvalues and eigenfunctions of the equation
$$u'' + \lambda u = 0, \qquad 0 \leq x \leq l$$
subject to the boundary condition
$$\begin{aligned} u'(0) &= 0, \\ u'(l) &= 0, \end{aligned}$$

1. Find the eigenvalues and eigenfunctions of the above problem.

2. Let $f \in \mathcal{L}^2(0, l)$, write f as a linear combination of the computed eigenfunctions.

3. Write $f(x) = 1$ as a linear combination of the computed eigenfunctions.

4. Do the above problem using the boundary conditions
$$\begin{aligned} u(0) &= 0, \\ u(l) &= 0, \end{aligned}$$
and compare the series representation for $f(x) = 1$ in the current case with that in the previous case.

Solution
The above eigenvalue problem can be written as
$$Lu + \lambda u = 0$$

with

$$L = \frac{d}{dx}\left(\frac{d}{dx}\right),$$
$$\rho(x) = 1$$

and under separated homogenous boundary conditions. Thus, we have an SL problem and Theorem 2.29 holds.

1. From example 2.16, we know that the roots of the auxiliary equation are $m = \pm\sqrt{-\lambda}$.

   (a) According to remark 2.30, the eigenvalues must satisfy

   $$\lambda \geq \max\{|r(x)| : 0 \leq x \leq l\} = 0$$

   hence, there are no negative eigenvalues.

   (b) For $\lambda = 0$, the general solution is given by

   $$u(x) = c_1 x + c_2$$

   and

   $$u'(x) = c_1$$

   using the boundary conditions

   $$\begin{aligned} 0 &= u'(0) = c_1, \\ 0 &= u'(l) = c_1. \end{aligned}$$

   Thus, the eigenfunction corresponding to $\lambda_0 = 0$ is $u_0(x) = 1$.

   (c) For $\lambda > 0$, we have the general solution

   $$u(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$$

   from which we get

   $$u'(x) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$$

   using the boundary conditions

   $$0 = u'(0) = c_2\sqrt{\lambda}$$

   and since $\sqrt{\lambda} \neq 0$, we have $c_2 = 0$.

   $$0 = u'(l) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}l$$

   and since $-c_1\sqrt{\lambda} \neq 0$, we have

   $$\begin{aligned} 0 &= \sin\sqrt{\lambda}l \\ &\Rightarrow \sqrt{\lambda}l = n\pi, n \in \mathbb{N}_0 \\ &\Rightarrow \lambda_n = \frac{n^2\pi^2}{l^2}, n \in \mathbb{N}_0 \end{aligned}$$

Note that we have an infinite sequence of eigenvalues $\{\frac{n^2\pi^2}{l^2} : n \in \mathbb{N}_0\}$ with

$$\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \frac{n^2\pi^2}{l^2} = \infty,$$

as stated in the theorem.

The eigenfunction u_n corresponding to $\lambda_n = \frac{n^2\pi^2}{l^2}$ is given by

$$u_n(x) = \cos\left(\frac{n\pi}{l}x\right),$$

where we have used $c_2 = 1$.

2. According the theorem 2.29, the set of eigenfunctions $\{\cos\left(\frac{n\pi}{l}x\right) : n \in \mathbb{N}_0\}$ is orthogonal in $\mathcal{L}^2(0,l)$ [verify!]. Moreover, the set $\{\cos\left(\frac{n\pi}{l}x\right) : n \in \mathbb{N}_0\}$ form a basis for $\mathcal{L}^2(0,l)$.

That is, we can write any function $f \in \mathcal{L}^2(0,l)$ in the form

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right)$$

where

$$a_n = \frac{\left\langle f, \cos\left(\frac{n\pi}{l}x\right)\right\rangle}{\left\|\cos\left(\frac{n\pi}{l}x\right)\right\|^2}$$

but

$$\begin{aligned}\left\langle f, \cos\left(\frac{n\pi}{l}x\right)\right\rangle &= \int_0^l f(x)\cos\left(\frac{n\pi}{l}x\right)dx,\\ \left\|\cos\left(\frac{n\pi}{l}x\right)\right\|^2 &= \int_0^l \cos^2\left(\frac{n\pi}{l}x\right)dx = \left\{\begin{array}{ll} l & n=0 \\ \frac{l}{2} & n \in \mathbb{N}\end{array}\right.\end{aligned}$$

Therefore, For $f \in \mathcal{L}^2(0,l)$, we have

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right)$$

where

$$\begin{aligned}a_0 &= \frac{1}{l}\int_0^l f(x)\,dx\\ a_n &= \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi}{l}x\right)dx\end{aligned}$$

3. Let $f(x) = 1$, then

$$\begin{aligned} a_0 &= \frac{1}{l}\int_0^l 1dx = 1 \\ a_n &= \frac{2}{l}\int_0^l \cos\left(\frac{n\pi}{l}x\right)dx = 0 \end{aligned}$$

Hence, the function $f(x) = 1$ is represented by a single term, namely, the first eigenfunction $u_0(x)$.

4. For the second pair of boundary conditions, we have

$$\begin{aligned} \lambda_n &= \frac{n^2\pi^2}{l^2} \\ u_n(x) &= \sin\left(\frac{n\pi}{l}x\right) \end{aligned}$$

where $n \in \mathbb{N}$. Any $f \in \mathcal{L}^2(0,l)$ can be written in the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right)$$

where

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi}{l}x\right)dx$$

In this case, if we consider $f(x) = 1$, we get

$$\begin{aligned} b_n &= \frac{2}{l}\int_0^l \sin\left(\frac{n\pi}{l}x\right)dx \\ &= \frac{2}{n\pi}(1-\cos n\pi) \\ &= \frac{2}{n\pi}(1-(-1)^n) \\ &= \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

Therefore,

$$\begin{aligned} 1 &= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{n}\sin\left(\frac{n\pi}{l}x\right) \\ &= \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\left(\frac{(2n+1)\pi}{l}x\right) \\ &= \frac{2}{\pi}\left(\sin\frac{\pi}{l}x + \frac{1}{3}\sin\frac{3\pi}{l}x + \frac{1}{5}\sin\frac{5\pi}{l}x + ...\right) \end{aligned}$$

Remember that the above equality hold in $\mathcal{L}^2(0,l)$, i.e. in the sense that

$$\left\| 1 - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin\left(\frac{(2n+1)\pi}{l} x \right) \right\| = 0$$

or

$$\frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin\left(\frac{(2n+1)\pi}{l} x \right) \overset{\mathcal{L}^2}{\to} 1 \text{ as } k \to \infty.$$

Part IV

Fourier Series

10 Fourier Series in $\mathcal{L}^2$

We are going to learn

- Using "The Sturm-Liouville Theory" to derive "The Fundamental Theorem of Fourier Series".

Theorem 92 *3.2 (Fundamental Theorem of Fourier Series)*

The orthogonal set of functions

$$\left\{1, \cos\frac{n\pi}{l}x, \sin\frac{n\pi}{l}x : n \in \mathbb{N}\right\}$$

is complete in $\mathcal{L}^2(-l,l)$, in the sense that any function $f \in \mathcal{L}^2(-l,l)$ can be represented by the series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right), \qquad -l \le x \le l, \qquad ((3.7))$$

where

$$\begin{aligned} a_0 &= \frac{\langle f, 1\rangle}{\|1\|^2} = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx, \\ a_n &= \frac{\langle f, \cos\frac{n\pi}{l}x\rangle}{\|\cos\frac{n\pi}{l}x\|^2} = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi}{l}x dx, \quad n \in \mathbb{N}, \\ b_n &= \frac{\langle f, \sin\frac{n\pi}{l}x\rangle}{\|\sin\frac{n\pi}{l}x\|^2} = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi}{l}x dx, \quad n \in \mathbb{N}, \end{aligned}$$

The right-hand side of Equation (3.7) is called the Fourier series expansion of f, and the coefficients a_n and b_n in the expansion are the Fourier coefficients of f.

Proof:

Consider the eigenvalue problem

$$u'' + \lambda u = 0, \qquad -l \le x \le l$$

with the periodic boundary conditions

$$\begin{aligned} u(-l) &= u(l), \\ u'(-l) &= u'(l) \end{aligned}$$

1. The above problem is an SL problem because $L = \frac{-d^2}{dx^2}$ is formally self-adjoint. In fact, $L = \frac{-d^2}{dx^2}$ is self-adjoint under the above periodic boundary conditions [verify!].
2. The results of Theorem 2.29 therefore hold (except for the uniqueness of the eigenfunctions as previously mentioned in Remark 2.30). In particular, the eigenfunctions of L are orthogonal and complete in $\mathcal{L}^2(-l,l)$.
3. There are no negative eigenvalues of L since

$$\lambda \ge -\max\{|r(x)| : -l \le x \le l\} = 0$$

(see Remark 2.30).

4. For $\lambda = 0$, the general solution is given by

$$u(x) = c_1 x + c_2$$

Using the boundary condition, we have

$$-c_1 l + c_2 = u(-l) = u(l) = c_1 l + c_2$$

or

$$c_1 = 0$$

and

$$0 = u'(-l) = u'(l) = 0$$

Therefore, the eigenfunction corresponding to $\lambda_0 = 0$ is $u_0(x) = 1$.

5. For $\lambda > 0$, the general solution is given by

$$u(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x$$

Using the boundary conditions gives

$$c_1 \cos \sqrt{\lambda} l - c_2 \sin \sqrt{\lambda} l = u(-l) = u(l) = c_1 \cos \sqrt{\lambda} l + c_2 \sin \sqrt{\lambda} l$$

$\Rightarrow$

$$c_2 \sin \sqrt{\lambda} l = 0$$

and

$$c_1 \sin \sqrt{\lambda} \sqrt{\lambda} l + c_2 \sqrt{\lambda} \cos \sqrt{\lambda} l = u'(-l) = u'(l) = -c_1 \sqrt{\lambda} \sin \sqrt{\lambda} l + c_2 \sqrt{\lambda} \cos \sqrt{\lambda} l$$

$\Rightarrow$

$$c_1 \sqrt{\lambda} \sin \sqrt{\lambda} l = 0$$

Now, since $\lambda > 0$ and c_1 and c_2 cannot both be zero, we have

$$\sin \sqrt{\lambda} l = 0$$
$$\Rightarrow \sqrt{\lambda_n} l = n\pi$$
$$\Rightarrow \lambda_n = \frac{n^2 \pi^2}{l^2}, n \in \mathbb{N}$$

Note that for each eigenvalue $\lambda_n = \frac{n^2 \pi^2}{l^2}$, we have two eigenfunctions, namely,

(a) If we choose $c_1 = 0$, we get the eigenfunction

$$u_n(x) = \sin \frac{n\pi}{l} x$$

(b) If we choose $c_2 = 0$, we get the eigenfunction

$$u_n(x) = \cos \frac{n\pi}{l} x$$

That is, the eigenfunction corresponding to a particular eigenvalue is not unique, which is due to using a non separated boundary conditions (see Theorem 2.29).

6. The last part of Theorem 2.29 states that the set of eigenfunctions

$$\{1, \cos\frac{n\pi}{l}x, \sin\frac{n\pi}{l}x : n \in \mathbb{N}\}$$

are orthogonal and complete in $\mathcal{L}^2(-l,l)$. Therefore, for each $f \in \mathcal{L}^2(-l,l)$, f is represented by the series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right), \quad -l \le x \le l$$

where

$$\begin{aligned} a_0 &= \frac{\langle f, 1\rangle}{\|1\|^2} \\ &= \frac{1}{\int_{-l}^{l} 1dx}\int_{-l}^{l} f(x)\,dx \\ &= \frac{1}{2l}\int_{-l}^{l} f(x)\,dx \end{aligned}$$

and

$$\begin{aligned} a_n &= \frac{\left\langle f, \cos\left(\frac{n\pi}{l}x\right)\right\rangle}{\left\|\cos\left(\frac{n\pi}{l}x\right)\right\|^2} \\ &= \frac{1}{\int_{-l}^{l}\cos^2\left(\frac{n\pi}{l}x\right)dx}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi}{l}x\right)dx \\ &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi}{l}x\right)dx \end{aligned}$$

and

$$\begin{aligned} b_n &= \frac{\left\langle f, \sin\left(\frac{n\pi}{l}x\right)\right\rangle}{\left\|\sin\left(\frac{n\pi}{l}x\right)\right\|^2} \\ &= \frac{1}{\int_{-l}^{l}\sin^2\left(\frac{n\pi}{l}x\right)dx}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi}{l}x\right)dx \\ &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi}{l}x\right)dx \end{aligned}$$

Remark 93 *3.3*

1. If $f \in \mathcal{L}^2(-l, l)$ is an even function, i.e. f satisfy

$$f(-x) = f(x) \quad \forall x \in [-l, l]$$

then, $b_n = 0$ for all $n \in \mathbb{N}$ and f is represented on $[-l, l]$ by a cosine series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{l} x,$$

where

$$\begin{aligned} a_0 &= \frac{1}{2l} \int_{-l}^{l} f(x)\, dx = \frac{1}{l} \int_{0}^{l} f(x)\, dx, \\ a_n &= \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{n\pi}{l} x\right) dx = \frac{2}{l} \int_{0}^{l} f(x) \cos\left(\frac{n\pi}{l} x\right) dx, \ n \in \mathbb{N}. \end{aligned}$$

2. If $f \in \mathcal{L}^2(-l, l)$ is an odd function, i.e. f satisfy

$$f(-x) = -f(x) \quad \forall x \in [-l, l]$$

then, $a_n = 0$ for all $n \in \mathbb{N}_0$ and f is represented on $[-l, l]$ by a sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x,$$

where

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(\frac{n\pi}{l} x\right) dx = \frac{2}{l} \int_{0}^{l} f(x) \sin\left(\frac{n\pi}{l} x\right) dx, \ n \in \mathbb{N}.$$

3. The equality between f and the Fourier series in Theorem 3.2 is in $\mathcal{L}^2(-l, l)$ and not pointwise. Namely, we have

$$\left\| f(x) - \left[a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{l} x + b_n \sin \frac{n\pi}{l} x \right) \right] \right\|^2 = 0$$

or

$$\left\| f(x) - \left[a_0 + \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi}{l} x + b_n \sin \frac{n\pi}{l} x \right) \right] \right\|^2 \to 0 \quad \text{as } N \to \infty$$

But

$$\left\|f(x)-\left[a_0+\sum_{n=1}^{N}\left(a_n\cos\frac{n\pi}{l}x+b_n\sin\frac{n\pi}{l}x\right)\right]\right\|^2$$

$$\begin{aligned}
&= \int_{-l}^{l}\left|f(x)-\left[a_0+\sum_{n=1}^{N}\left(a_n\cos\frac{n\pi}{l}x+b_n\sin\frac{n\pi}{l}x\right)\right]\right|^2 dx \\
&= \int_{-l}^{l}|f(x)|^2\,dx-2\operatorname{Re}\left(\int_{-l}^{l}f(x)\left[\overline{a_0}+\sum_{n=1}^{N}\left(\overline{a_n}\cos\frac{n\pi}{l}x+\overline{b_n}\sin\frac{n\pi}{l}x\right)\right]\right)dx \\
&\quad+\int_{-l}^{l}\left|a_0+\sum_{n=1}^{N}\left(a_n\cos\frac{n\pi}{l}x+b_n\sin\frac{n\pi}{l}x\right)\right|^2 dx \\
&= \|f\|^2-2\operatorname{Re}\left(\overline{a_0}\int_{-l}^{l}f(x)\,dx+\sum_{n=1}^{N}\left(\overline{a_n}\int_{-l}^{l}f(x)\cos\frac{n\pi}{l}x\,dx+\overline{b_n}\int_{-l}^{l}f(x)\sin\frac{n\pi}{l}x\,dx\right)\right) \\
&\quad+\left\|a_0+\sum_{n=1}^{N}\left(a_n\cos\frac{n\pi}{l}x+b_n\sin\frac{n\pi}{l}x\right)\right\|^2 \\
&= \|f\|^2-2\operatorname{Re}\left(\overline{a_0}(2l)a_0+\sum_{n=1}^{N}\left(\overline{a_n}la_n+\overline{b_n}lb_n\right)\right) \\
&\quad+|a_0|^2\|1\|^2+\sum_{n=1}^{N}\left(|a_n|\left\|\cos\frac{n\pi}{l}x\right\|^2+|b_n|^2\left\|\sin\frac{n\pi}{l}x\right\|^2\right) \\
&= \|f\|^2-2\operatorname{Re}\left(2l|a_0|^2+\sum_{n=1}^{N}\left(l|a_n|^2+l|b_n|^2\right)\right) \\
&\quad+2l|a_0|^2+\sum_{n=1}^{N}\left(l|a_n|^2+l|b_n|^2\right) \\
&= \|f\|^2-2\left(2l|a_0|^2+\sum_{n=1}^{N}\left(l|a_n|^2+l|b_n|^2\right)\right)+2l|a_0|^2+\sum_{n=1}^{N}\left(l|a_n|^2+l|b_n|^2\right) \\
&= \|f\|^2-l\left(2|a_0|^2+\sum_{n=1}^{N}\left(|a_n|^2+|b_n|^2\right)\right)
\end{aligned}$$

Since $f\in\mathcal{L}^2(-l,l)$

$$\|f\|^2-l\left(2|a_0|^2+\sum_{n=1}^{N}\left(|a_n|^2+|b_n|^2\right)\right)\to 0 \text{ as } N\to\infty$$

implies that $\sum_{n=1}^{\infty}\left(|a_n|^2+|b_n|^2\right)$ is convergent and therefore both $\sum_{n=1}^{\infty}|a_n|^2$

and $\sum_{n=1}^{\infty} |b_n|^2$ is convergent. Consequently,

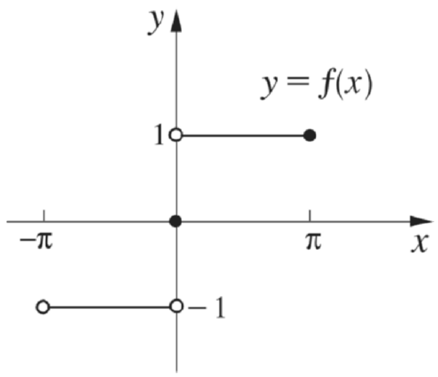
$$\begin{aligned} \lim_{n\to\infty} a_n &= 0, \\ \lim_{n\to\infty} b_n &= 0. \end{aligned}$$

4. If $f \in \mathcal{L}^2(-l, l)$ is continuous on $[-l, l]$ and the Fourier series of f converges uniformly to f, then the equality between f and the Fourier series in Theorem 3.2 is pointwise.

Example 94 *3.4*

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} -1, & -\pi < x < \pi \\ 0, & x = 0 \\ 1, & 0 < x \leq \pi \end{cases}$$



Note that,

1. $f \in \mathcal{L}^2(-\pi, \pi)$ [verify!]

2. f is an odd function [verify!]

Therefore, the Fourier series expansion of f is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx \\ &= -\frac{2}{n\pi}\cos(nx)\,|_0^{\pi} \\ &= \frac{2}{n\pi}\left(1-(-1)^n\right) \\ &= \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4}{n\pi}, & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Note that $\lim_{n\to\infty} b_n = 0$.

The Fourier series can therefore be written in the form

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin nx \\ &= \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)}\sin(2n+1)x \end{aligned}$$

Figure 3.2 below shows the first three terms in the sequence of partial sums of the Fourier series, i.e.

$$S_N(x) = \frac{4}{\pi}\sum_{n=0}^{N} \frac{1}{(2n+1)}\sin(2n+1)x$$

for $N = 0, 1, 2$.

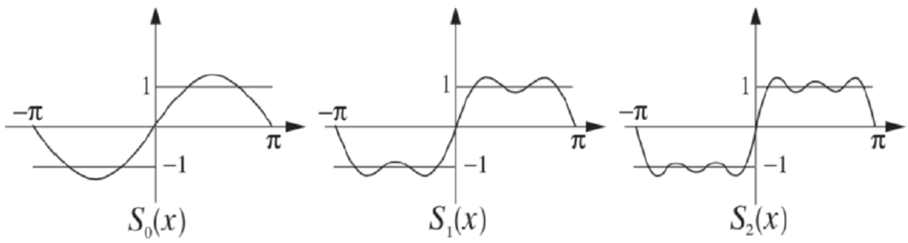


Figure 3.2 The sequence of partial sums S_N.

Note that the larger the N, the better the approximation. Also note that

$$S_N(-\pi) = S_N(\pi) = 0 \quad \text{for } N = 0, 1, 2.$$

In fact, the Fourier series of f at $\pm\pi$ equals to 0 while $f(\pi) = 1$ and $f(-\pi)$ is not defined, which shows that the equality

$$f(x) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \sin(2n+1)x$$

does not hold at every point in $[-\pi, \pi]$.

Corollary 95 *3.5*

Any function $f \in \mathcal{L}^2(-l, l)$ can be represented by the Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l},$$

where

$$c_n = \frac{\langle f, e^{in\pi x/l} \rangle}{\left\| e^{in\pi x/l} \right\|^2} = \frac{1}{2l} \int_{-l}^{l} f(x) e^{-in\pi x/l} dx, \quad n \in \mathbb{Z}.$$

Proof: [Assignment!]

11 Convergence of Fourier Series

We are going to learn about

- Periodic functions and their properties.
- Piecewise continuous and Piecewise smooth functions.
- Pointwise convergence of a Fourier series.
- Uniform and absolute convergence of a Fourier series.

A function $f : \mathbb{R} \to \mathbb{C}$ is *periodic* in p, where $p > 0$ if

$$f(x+p) = f(x) \text{ for all } x \in \mathbb{R},$$

and p is then called a *period* of f.

Properties of periodic functions

1. If f is periodic in p, then f is also periodic in np where $n \in \mathbb{Z}$ [Why?]
2. If a periodic function f in p is integrable on $[0, p]$, then f is integrable over any finite interval, and its integral have the same value over all intervals of length p, that is

$$\int_x^{x+p} f(t)\,dt = \int_0^p f(t)\,dt \quad \text{for all } x \in \mathbb{R}.$$

   [Justify?]

Example 96 *(Periodic functions)*

Determine the period of the following periodic functions:

1. $\cos x$, $\sin x$.
2. $\cos(ax)$, $\sin(ax)$, where $a > 0$.
3. A constant function.

Definition 97 *3.6*

1. A function f defined on a bounded interval I, where $(a, b) \subseteq I \subseteq [a, b]$, is said to be *piecewise continuous* if
   (a) f is continuous on (a, b) except for a finite number of points $\{x_1, x_2, ..., x_n\}$.
   (b) The right-hand and left-hand limits

$$\lim_{x \to x_i^+} f(x) = f\left(x_i^+\right), \qquad \lim_{x \to x_i^-} f(x) = f\left(x_i^-\right)$$

   for all $i \in \{1, 2, ...n\}$.
   (c) The limits at the endpoints exist, that is

$$\lim_{x \to a^+} f(x) = f\left(a^+\right), \qquad \lim_{x \to b^-} f(x) = f\left(b^-\right)$$

2. f is said to be *piecewise smooth* if f and f' are both piecewise continuous.
3. If the interval I is unbounded, then f is *piecewise continuous (smooth)* if it is piecewise continuous (smooth) on every bounded subinterval of I.

Remark 98 *(*piecewise continuous and piecewise smooth functions)

1. The discontinuities in the graph of a piecewise continuous functions results from *jumps* in its values and occur at a finite number of points.
2. The non-smoothness in the graph of a piecewise smooth function results from jumps in its values and/or sharp corners at some points, which occur at a finite number of points.

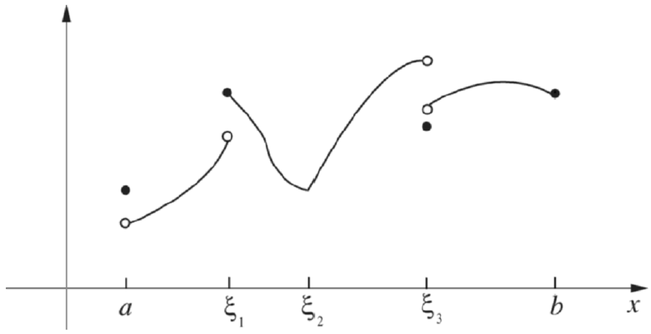


Figure 3.3 A piecewise smooth function.

3. A continuous function is always piecewise continuous, but may not be piecewise smooth. [Example?]
4. A differentiable function may not be piecewise smooth. [Example?] . Note that the right-hand (left-hand) limit of a derivative f' is not the same as the right-hand (left-hand) derivative of f. In other words,

$$\begin{gathered} f \text{ is differentiable at a point } x_0 \Rightarrow \\ \text{right- and left-hand derivatives exits at that point} \\ \text{(or one of them if its an endpoint))} \equiv \\ \lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h}, \quad \lim_{h\to 0^+} \frac{f(x_0)-f(x_0-h)}{h} \text{ exist.} \end{gathered}$$

but

$$\begin{gathered} \text{right- and left hand limit of } f' \text{ exist at a point } x_0 \equiv \\ \lim_{x\to x_0^+} f'(x) = \lim_{h\to 0^+} \frac{f(x_0+h)-f\left(x_0^+\right)}{h} = f'\left(x_0^+\right), \\ \lim_{x\to x_0^-} f'(x) = \lim_{h\to 0^+} \frac{f\left(x_0^-\right)-f(x_0-h)}{h} = f'\left(x_0^-\right), \end{gathered}$$

We already know that any $f \in \mathcal{L}^2(-\pi,\pi)$ can be represented by a Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \quad -\pi \le x \le \pi,$$

where a_0, a_n, b_n are the Fourier coefficients. We also know that the above equality holds in $\mathcal{L}^2(-\pi, \pi)$, but not necessarily pointwise. In the following, we study the pointwise and uniform convergence of Fourier series.

11.1 Pointwise Convergence of Fourier Series

The following theorem discusses the pointwise convergence of the Fourier series to the periodic function defined by extending a function f from $[-\pi, \pi]$ to $\mathbb{R}$ using the equation

$$f(x + 2\pi) = f(x), \quad \forall x \in \mathbb{R}.$$

Theorem 99 *3.9*

Let f be a piecewise smooth function on $[-\pi, \pi]$ which is periodic in 2π. If

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \quad n \in \mathbb{N} \end{aligned}$$

then the Fourier series

$$S(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

converges at every x in $\mathbb{R}$ to $\frac{1}{2}[f(x^+) + f(x^-)]$.

Remark 100 *3.10*

1. If f is continuous on $[-\pi, \pi]$, then the Fourier series converges pointwise to f on $\mathbb{R}$.[Why?]

2. If f is discontinuous at a point x, then the Fourier series converges to the average of the "jump" at x, namely

$$S(x) = \frac{1}{2}\left[f\left(x^+\right) + f\left(x^-\right)\right]$$

regardless of the value of $f(x)$.

3. We can redefine the function f at the points of discontinuities to achieve a pointwise convergence of the Fourier series to f on $\mathbb{R}$. [How?]

4. The conditions on f in Theorem 3.9 are sufficient but not necessary. For example, $f(x) = x^{\frac{1}{3}}$ is not piecewise smooth on $[-\pi, \pi]$, but its Fourier series expansion converges to f on $[-\pi, \pi]$.[Exercise 3.26].

5. Theorem 3.9 holds if the interval $[-\pi, \pi]$ is replaced by any other interval $[-l, l]$. Namely, if f is a piecewise smooth function on $[-l, l]$ which is periodic in $2l$, then the Fourier series

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right) \right),$$

where

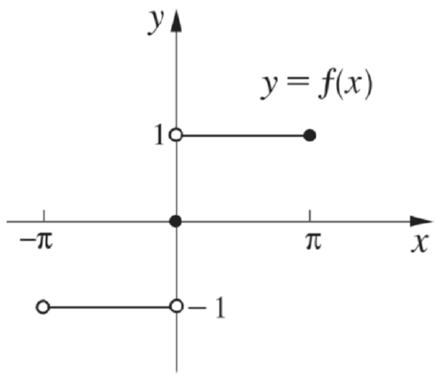
$$\begin{aligned} a_0 &= \frac{1}{2l} \int_{-l}^{l} f(x)\, dx, \\ a_n &= \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{n\pi}{l}x\right) dx, \\ b_n &= \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(\frac{n\pi}{l}x\right) dx, \quad n \in \mathbb{N} \end{aligned}$$

converges at every $x \in \mathbb{R}$ to $\frac{1}{2}\left[f\left(x^+\right) + f\left(x^-\right)\right]$.

Exercise 101 *3.4 (revisited)*

Consider the function

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 0, & x = 0 \\ 1, & 0 < x \leq \pi \end{cases}$$



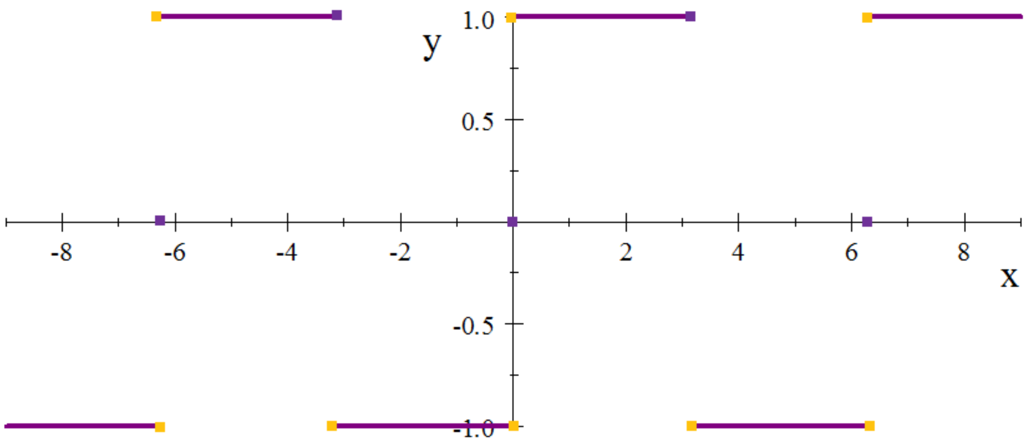
1. Sketch the periodic function that results from extending f from $(-\pi, \pi]$ to $\mathbb{R}$. Is the extension of f piecewise smooth on $[-\pi, \pi]$?

2. Determine the Fourier series expansion of the extension of f . Does it converge to $f(x)$ at every $x \in \mathbb{R}$?

3. Redefine the function f so that the Fourier series converges to f on $\mathbb{R}$.

4. Find a series representation for π.

Solution:



1.

2. We already know that the Fourier series expansion of f is given by

$$S(x) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)}\sin(2n+1)x$$

The graph of the extension of f is clearly piecewise continuous. Also, f has a zero derivative at all points $x \in (-\pi, \pi) - \{0\}$, and $\lim_{x\to x_0^{\pm}} f'(x) = 0$, where x_0 is a point at which f is discontinuous. Therefore, f is piecewise smooth on $[-\pi, \pi]$. Therefore, Theorem 3.9 holds. That is,

$$S(x) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)}\sin(2n+1)x = \frac{1}{2}\left[f\left(x^{+}\right) + f\left(x^{-}\right)\right]$$

(a) At $x = 0$,

$$S(0) = \frac{1}{2}\left[f\left(0^{+}\right) + f\left(0^{-}\right)\right] = \frac{1}{2}\left[1 + (-1)\right] = 0 = f(0)$$

By periodicity of f, the Fourier series converges to f at all points $x = 0 + n(2\pi) = 2n\pi$, where $n \in \mathbb{Z}$.

(b) At $x = \pi$,

$$S(\pi) = \frac{1}{2}\left[f\left(\pi^{+}\right) + f\left(\pi^{-}\right)\right] = \frac{1}{2}\left[-1 + 1\right] = 0 \neq 1 = f(\pi)$$

By periodicity of f, the Fourier series does not converge to f at all points $x = \pi + n(2\pi) = (2n+1)\pi$, where $n \in \mathbb{Z}$.

3. If we redefine f at all points $x = (2n+1)\pi$, $n \in \mathbb{Z}$ as

$$f(x) = \frac{1}{2}\left[f\left(x^{+}\right) + f\left(x^{-}\right)\right] = \frac{1}{2}[-1+1] = 0$$

then the Fourier series becomes convergent to f on $\mathbb{R}$.

4. If we take the point $x = \frac{\pi}{2}$, then we know that the Fourier series does converge to f at this point since f is continuous there. Therefore,

$$\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)}\sin(2n+1)\frac{\pi}{2} = f\left(\frac{\pi}{2}\right),$$

or

$$\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)}(-1)^{n} = 1,$$

or

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - ...\right).$$

11.2 Uniform Convergence of Fourier Series

Lemma 102 *3.13*

If f is a continuous function on the interval $[-\pi,\pi]$ such that $f(-\pi) = f(\pi)$ and f' is piecewise continuous on $(-\pi,\pi)$, then the series

$$\sum_{n=1}^{\infty}\sqrt{|a_n|^2 + |b_n|^2}$$

is convergent, where a_n and b_n are the Fourier coefficients of f defined by

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx. \end{aligned}$$

Proof:

If f' is piecewise continuous on $[-\pi,\pi]$, then $f' \in \mathcal{L}^2(-\pi,\pi)$. [Why?]. The Fourier coefficients of f' is therefore given by

$$\begin{aligned} a_0^{'} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f^{'}(x)\,dx, \\ a_n^{'} &= \frac{1}{\pi}\int_{-\pi}^{\pi} f^{'}(x)\cos nx dx, \\ b_n^{'} &= \frac{1}{\pi}\int_{-\pi}^{\pi} f^{'}(x)\sin nx dx. \end{aligned}$$

Since $f(-\pi) = f(\pi)$, we have

$$a_0^{'} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^{'}(x)\,dx = \frac{1}{2\pi}\left[f(\pi) - f(-\pi)\right] = 0,$$

Integrating by parts,

$$\begin{array}{cc} u = \cos nx & dv = f^{'}(x)\,dx \\ du = -n\sin nx dx & v = f(x) \end{array}, \quad \begin{array}{cc} u = \sin nx & dv = f^{'}(x)\,dx \\ du = n\cos nx dx & v = f(x) \end{array}$$

we have

$$\begin{aligned} a_n^{'} &= \frac{1}{\pi}\left[f(x)\cos nx|_{-\pi}^{\pi} + n\int_{-\pi}^{\pi} f(x)\sin nx dx\right] = nb_n, \\ b_n^{'} &= \frac{1}{\pi}\left[f(x)\sin nx|_{-\pi}^{\pi} - n\int_{-\pi}^{\pi} f(x)\cos nx dx\right] = -na_n. \end{aligned}$$

from which we get

$$a_n = -\frac{1}{n}b_n^{'}, \quad b_n = \frac{1}{n}a_n^{'}$$

Therefore, the nth partial sum

$$\begin{aligned} S_N &= \sum_{n=1}^{N}\sqrt{|a_n|^2 + |b_n|^2} \\ &= \sum_{n=1}^{N}\frac{1}{n}\sqrt{|a_n^{'}|^2 + |b_n^{'}|^2} \end{aligned}$$

Recall that the CBS inequality states that

$$|\langle x, y\rangle| \leq \|x\|\,\|y\|$$

Thus, by taking $x = \left(1, ..., \frac{1}{N}\right)$, $y = \left(\sqrt{|a_1^{'}|^2 + |b_1^{'}|^2}, ..., \sqrt{|a_N^{'}|^2 + |b_N^{'}|^2}\right)$ we have

$$\sum_{n=1}^{N}\frac{1}{n}\sqrt{|a_n^{'}|^2 + |b_n^{'}|^2} \leq \left(\sum_{n=1}^{N}\frac{1}{n^2}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{N}\left(\left|a_n^{'}\right|^2 + \left|b_n^{'}\right|^2\right)\right)^{\frac{1}{2}}$$

But $\sum_{n=1}^{N}\frac{1}{n^2}$ is a convergent $p-$series and is therefore bounded above by its sum. Next we use Bessel's inequality, namely

$$\sum_{n=1}^{\infty}\frac{|\langle g, \varphi_n\rangle|^2}{\|\varphi_n\|^2} \leq \|g\|^2$$

for any orthogonal set $\{\varphi_n : n \in \mathbb{N}\}$ in $\mathcal{L}^2$ and any $g \in \mathcal{L}^2$, to show that $\sum_{n=1}^{N}\left(\left|a_n^{'}\right|^2 + \left|b_n^{'}\right|^2\right)$ is bounded for every $N \in \mathbb{N}$.

$$\begin{aligned}
\sum_{n=1}^{N}\left(\left|a_n^{'}\right|^2 + \left|b_n^{'}\right|^2\right) &= \sum_{n=1}^{N}\frac{\left|\left\langle f^{'}, \cos nx\right\rangle\right|^2}{\|\cos nx\|^4} + \sum_{n=1}^{N}\frac{\left|\left\langle f^{'}, \sin nx\right\rangle\right|^2}{\|\sin nx\|^4} \\
&= \frac{1}{\pi}\sum_{n=1}^{N}\frac{\left|\left\langle f^{'}, \cos nx\right\rangle\right|^2}{\|\cos nx\|^2} + \frac{1}{\pi}\sum_{n=1}^{N}\frac{\left|\left\langle f^{'}, \sin nx\right\rangle\right|^2}{\|\sin nx\|^2} \\
&\leq \frac{1}{\pi}\left\|f^{'}\right\|^2 + \frac{1}{\pi}\left\|f^{'}\right\|^2 = \frac{2}{\pi}\left\|f^{'}\right\|^2 < \infty
\end{aligned}$$

because $f^{'} \in \mathcal{L}^2(-\pi, \pi)$. We conclude that the sequence of partial sums $S_N = \sum_{n=1}^{N}\sqrt{|a_n|^2 + |b_n|^2}$ is bounded for every N and is therefore convergent.

Theorem 103 *3.14*

If f is a continuous function on the interval $[-\pi, \pi]$ such that $f(-\pi) = f(\pi)$ and f' is piecewise continuous on $(-\pi, \pi)$, then the Fourier series

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

converges uniformly and absolutely to f on $[-\pi, \pi]$.

Proof:

Consider the extension of f from $[-\pi, \pi]$ to $\mathbb{R}$ by the relation

$$f(x + 2\pi) = f(x) \quad \text{for all } x \in \mathbb{R},$$

then since f is continuous on $[-\pi, \pi]$ and $f(-\pi) = f(\pi)$, the extension of f is a continuous function on $\mathbb{R}$.Therefore, the Fourier series

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

converges to $f(x)$ for all $x \in \mathbb{R}$.

To prove that the convergence is uniform and absolute, we use the M-Test

$$|a_n \cos nx + b_n \sin nx| \leq |a_n| + |b_n| \leq \sqrt{2}\sqrt{|a_n|^2 + |b_n|^2}$$

[why?] , but $\sum_{n=1}^{\infty}\sqrt{|a_n|^2 + |b_n|^2}$ converges by Lemma 3.13. Therefore, the Fourier series converges uniformly and absolutely.

Remark 104 *(Comparing sufficient conditions for pointwise and uniform convergence of a Fourier series)*

The conditions imposed on f in the above theorem are the same as those of Theorem 3.9 with piecewise continuity replaced by continuity on $[-\pi, \pi]$.

Corollary 105 *3.15*

If f is piecewise smooth on $[-\pi, \pi]$ and periodic in 2π, its Fourier series is uniformly convergent if, and only if, f is continuous.

Remark 106 *3.16*

Corollary 3.15 holds if the interval $[-\pi, \pi]$ is replaced by any other interval $[-l, l]$.

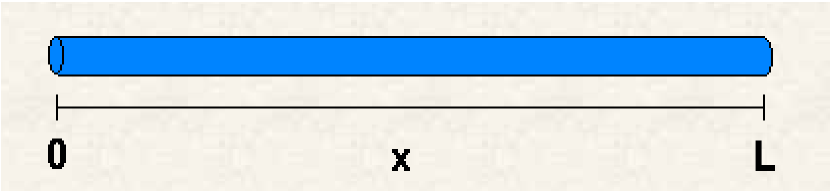
12 Boundary-Value Problems

We are going to learn

- Applications of Fourier series to real physical problems:
  - The One-dimensional Heat Equation.
  - The One-dimensional Wave Equation.

12.1 The One-dimensional Heat Equation

Consider a thin bar of length l as shown in the figure below.



The bar is given an initial temperature distribution $f(x)$ for each $x \in [0, l]$, then it is insulated everywhere except at the two ends of bar where the initial temperature is kept fixed. If we let $u(x, t)$ represents the temperature at the point x meters along the bar at time t (in seconds), then fixing the temperature at the bar ends is represented by the two equations

$$u(0, t) = u_L, \quad u(l, t) = u_R, \quad t > 0$$

We are interested in how the temperatures along the rod vary with time, that is, we want to find $u(x, t)$ for all $x \in (0, l)$ and all $t \in (0, \infty)$. The second-order partial differential equation

$$u_t = ku_{xx}, \ , \qquad 0 < x < l, \ t > 0$$

is used to model one-dimensional temperature evolution and is called the *one-dimensional heat equation.* The positive constant k represents the thermal diffusivity of the bar. It depends on the thermal conductivity of the material composing the bar, the density of the bar, and the specific heat of the bar.

In our sample problem, we will assume that both ends are kept at 0 degrees Celsius. Therefore, the boundary conditions are given by

$$u(0, t) = u(l, t) = 0, \quad t > 0$$

To summarize, we want to solve the following boundary-value problem

$$\begin{aligned} u_t &= ku_{xx}, \qquad 0 < x < l, \ t > 0 \\ u(0, t) &= u(l, t) = 0, \quad t > 0 \\ u(x, 0) &= f(x), \quad 0 < x < l. \end{aligned}$$

We assume that the solution to the heat equation can be expressed as a product of a function of x and a function of t, that is

$$u(x, t) = v(x) w(t)$$

Substituting in the heat equation gives

$$v(x) w'(t) = kv''(x) w(t)$$

or

$$\frac{v''(x)}{v(x)} = \frac{w'(t)}{kw(t)}$$

Therefore, there is a constant $-\lambda^2$ such that

$$\frac{v''(x)}{v(x)} = \frac{w'(t)}{kw(t)} = -\lambda^2$$

for all $(x,t) \in (0,l) \times (0,\infty)$. In other words,

$$\begin{aligned} v''(x) + \lambda^2 v(x) &= 0, \\ w'(x) + \lambda^2 kw(x) &= 0. \end{aligned}$$

which have the solutions

$$\begin{aligned} v(x) &= a\cos\lambda x + b\sin\lambda x, \\ w(t) &= ce^{-\lambda^2 kt} \end{aligned}$$

where a, b, c are constants. This gives

$$u_\lambda(x,t) = ce^{-\lambda^2 kt}(a\cos\lambda x + b\sin\lambda x)$$

Using the boundary conditions,

$$0 = u_\lambda(0,t) = ce^{-\lambda^2 kt}a$$

but $ce^{-\lambda^2 kt} \neq 0$, thus

$$a = 0,$$

Moreover,

$$0 = u_\lambda(l,t) = ce^{-\lambda^2 kt}(b\sin\lambda l)$$

but $ce^{-\lambda^2 kt}b \neq 0$, thus

$$0 = \sin\lambda l,$$

from which we get

$$\lambda_n = \frac{n\pi}{l}, \quad n \in \mathbb{N}$$

where for the case $n = 0$, we have the trivial solution which does not satisfy the initial condition if $f \neq 0$. For $n \in \mathbb{Z}^-$, we have

$$\sin\frac{n\pi}{l}x = -\sin\left(-\frac{n\pi}{l}x\right), \quad -n \in \mathbb{N}$$

Therefore, the solution that satisfies the two boundary conditions can be written in the form

$$u_n(x,t) = b_n e^{-(n\pi/l)^2 kt}\sin\frac{n\pi}{l}x, \quad n \in \mathbb{N}$$

Since the heat equation is linear and homogeneous with homogenous boundary conditions, the general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(n\pi/l)^2 kt}\sin\frac{n\pi}{l}x.$$

Using the initial condition, that is,

$$u(x,0)=\sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x,$$

or

$$f(x)=\sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x,$$

which gives

$$b_n=\frac{2}{\left\|\sin \frac{n\pi}{l}\right\|^2} \int_0^l f(x) \sin \frac{n\pi}{l} x dx=\frac{2}{l} \int_0^l f(x) \sin \frac{n\pi}{l} x dx$$

If we assume that f is piecewise smooth on $[0,l]$, then

$$u(x,0)=\sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x,$$
$$b_n=\frac{2}{l} \int_0^l f(x) \sin \frac{n\pi}{l} x dx$$

is the Fourier series expansion of the odd extension of f to $\mathbb{R}$. Thus, if we moreover assume that

$$f(x)=\frac{1}{2}\left[f\left(x^{+}\right)+f\left(x^{-}\right)\right], \ \forall x \in \mathbb{R},$$

then the Fourier series is the solution of the heat equation with the given boundary and initial conditions.

12.2 The One-dimensional Wave Equation

Consider a thin, flexible and weightless string of length l stretched between two fixed point. The string is given an initial vertical displacement $f(x)$ for each $x \in [0,l]$, then it is released with an initial velocity $g(x)$, with the string ends kept fixed. If we let $u(x,t)$ represents the vertical displacement at the point x meters along the string at time t (in seconds), then fixing the string ends is represented by the two equations

$$u(0,t)=u(l,t)=0, \quad t>0$$

We are interested in how the vertical displacement along the string vary with time, that is, we want to find $u(x,t)$ for all $x \in (0,l)$ and all $t \in (0,\infty)$. The second-order partial differential equation

$$u_{tt}=c^2 u_{xx}, \ , \qquad 0<x<l, \ t>0$$

is used to model one-dimensional transverse vibration of the string and is called the *one-dimensional wave equation.* The positive constant c^2 is determined by the material of the string. It depends on the thermal conductivity of the material composing the bar, the density of the bar, and the specific heat of the bar.

To summarize, we want to solve the following boundary-value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \qquad 0 < x < l,\ t > 0 \\ u(0,t) &= u(l,t) = 0, \quad t > 0 \\ u(x,0) &= f(x), \quad u_t(x,0) = g(x), \quad 0 < x < l. \end{aligned}$$

Using the separation of variables, we assume that the solution to the wave equation can be expressed as a product of a function of x and a function of t, that is

$$u(x,t) = v(x)\, w(t)$$

Substituting into the wave equation gives

$$v(x)\, w''(t) = c^2 v''(x)\, w(t)$$

or

$$\frac{v''(x)}{v(x)} = \frac{w''(t)}{c^2 w(t)}$$

Therefore, there is a constant $-\lambda^2$ such that

$$\frac{v''(x)}{v(x)} = \frac{w''(t)}{c^2 w(t)} = -\lambda^2$$

for all $(x,t) \in (0,l) \times (0,\infty)$. In other words,

$$\begin{aligned} v''(x) + \lambda^2 v(x) &= 0, \\ w''(x) + \lambda^2 c^2 w(x) &= 0. \end{aligned}$$

which have the solutions

$$\begin{aligned} v(x) &= a\cos\lambda x + b\sin\lambda x, \\ w(t) &= a'\cos c\lambda t + b'\sin c\lambda t \end{aligned}$$

where a, b, a', c' are constants. This gives

$$u_\lambda(x,t) = (a\cos\lambda x + b\sin\lambda x)\,(a'\cos c\lambda t + b'\sin c\lambda t)$$

Using the boundary conditions,

$$0 = u_\lambda(0,t) = a\,(a'\cos c\lambda t + b'\sin c\lambda t)$$

which gives

$$a = 0,$$

Moreover,

$$0 = u_\lambda(l,t) = (b\sin\lambda l)(a'\cos c\lambda t + b'\sin c\lambda t)$$

thus,

$$0 = \sin\lambda l,$$

from which we get

$$\lambda_n = \frac{n\pi}{l}, \quad n \in \mathbb{N}$$

Therefore, the solution that satisfies the two boundary conditions can be written in the form

$$u_n(x,t) = (a_n\cos c\lambda t + b_n\sin c\lambda t)\sin\frac{n\pi}{l}x, \quad n \in \mathbb{N}$$

Since the wave equation is linear and homogeneous with homogenous boundary conditions, the general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos c\frac{n\pi}{l}t + b_n\sin c\frac{n\pi}{l}t\right)\sin\frac{n\pi}{l}x.$$

Using the first initial condition

$$u(x,0) = 0, \quad 0 < x < l$$

gives

$$f(x) = \sum_{n=1}^{\infty} a_n\sin\frac{n\pi}{l}x, \quad 0 < x < l$$

If f is piecewise smooth on $[0,l]$, then the above equation is the Fourier expansion of its odd extension to $[-l,l]$ and therefore, we have

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}x\,dx$$

The velocity is given by

$$u_t(x,t) = \sum_{n=1}^{\infty} c\frac{n\pi}{l}\left(-a_n\sin c\frac{n\pi}{l}t + b_n\cos c\frac{n\pi}{l}t\right)\sin\frac{n\pi}{l}x$$

Thus, using second initial condition

$$u_t(x,0) = g(x), \quad 0 < x < l$$

gives

$$\sum_{n=1}^{\infty} c\frac{n\pi}{l}b_n\sin\frac{n\pi}{l}x = g(x),$$

from which we deduce that

$$c\frac{n\pi}{l}b_n = \frac{2}{l}\int_0^l g\left(x\right)\sin\frac{n\pi}{l}xdx$$

or

$$b_n = \frac{2}{cn\pi}\int_0^l g\left(x\right)\sin\frac{n\pi}{l}xdx$$

Thus, the solution to the wave equation with the given initial and boundary conditions is given by

$$u\left(x,t\right) = \sum_{n=1}^{\infty}\left(a_n\cos c\frac{n\pi}{l}t + b_n\sin c\frac{n\pi}{l}t\right)\sin\frac{n\pi}{l}x.$$

where

$$\begin{aligned} a_n &= \frac{2}{l}\int_0^l f\left(x\right)\sin\frac{n\pi}{l}xdx, \\ b_n &= \frac{2}{cn\pi}\int_0^l g\left(x\right)\sin\frac{n\pi}{l}xdx. \end{aligned}$$

Part V

Orthogonal Polynomials

13 The Singular SL Problem

We are going to learn

- The types of singular SL problem considered in this course.
- The boundary conditions that must hold for a formally self-adjoint operator in the singular SL problem to be self-adjoint.
- Extension of Theorem 2.29 to the singular SL problem.
- The Generalized Fourier Series.

We consider the singular SL problem

$$Lu + \lambda \rho u = 0, \quad x \in (a, b),$$
$$L = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + r(x)$$

that results from one or both of the following situations:

1. $p(x) = 0$ at $x = a$ and/or $x = b$.
2. The interval (a, b) is infinite.

Remark 107 *1*

1. Recall that the formally self-adjoint operator becomes self-adjoint if
$$\rho p(u'v - uv')|_a^b = 0$$
for all $f, g \in \mathcal{L}^2(a, b) \cap C^2(a, b)$.
2. If $p(a) = p(b) = 0$ and $\lim u$ at a and b exist, then L is self adjoint. In this case, the conclusions of Theorem 2.29 hold.

The solution of singular SL problems provide important examples of the so-called spacial functions of mathematical physics. In this chapter we consider singular SL problems whose eigenfunctions are polynomials, namely: Legendre polynomials, Hermit polynomials, Laguerre polynomials.

13.1 The Generalized Fourier Series

Consider the singular SL problem

$$Lu + \lambda \rho u = 0, \quad x \in (a, b),$$
$$L = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + r(x)$$

where
$$\rho p(u'v - uv')|_a^b = 0$$
is satisfied for any pair of eigenfunctions u and v.

If $\{\varphi_n : n \in \mathbb{N}_0\}$ is the set of eigenfunction of the above SL problem, then its orthogonal and complete in $\mathcal{L}^2_\rho(a, b)$. Therefore, any function $f \in \mathcal{L}^2_\rho(a, b)$ can be represented by the formula

$$f(x) = \sum_{n=0}^{\infty} \frac{\langle f, \varphi_n \rangle_\rho}{\|\varphi_n\|_\rho^2} \varphi_n(x)$$

The series above is called the *generalized Fourier series of* f, and

$$c_n = \frac{\langle f, \varphi_n \rangle_\rho}{\|\varphi_n\|_\rho^2}, \quad n \in \mathbb{N}_0,$$

are the *generalized Fourier coefficients of* f.

Theorem 108 *(Convergence of the generalized Fourier series)*

If f is piecewise smooth on (a, b), and

$$c_n = \frac{1}{\|\varphi_n\|_\rho^2} \int_a^b f(x)\, \varphi_n(x)\, \rho(x)\, dx,$$

then the series

$$S(x) = \sum_{n=0}^{\infty} c_n \varphi_n(x)$$

converges at every $x \in (a, b)$ to $\frac{1}{2}\left[f\left(x^{+}\right) + f\left(x^{-}\right)\right]$.

14 Legendre Polynomials

We are going to learn

- The Legendre Equation.
- Derivation of Legendre Polynomials and Legendre Functions.
- Properties of Legendre Polynomials.

The *Legendre equation* is given by

$$\left(1-x^2\right)u''-2xu'+\lambda u=0, \quad x\in(-1,1) \tag{(4.4)}$$

which is a singular SL problem.[Why?]

14.1 Derivation of Legendre Polynomials and Legendre Functions

Equation (4.4) is equivalent to

$$u''-\frac{2x}{1-x^2}u'+\frac{\lambda}{1-x^2}u=0 \tag{(4.5)}$$

Since the coefficients are rational functions on $(-1,1)$, they are analytic functions. Therefore, the solution $u(x)$ of the above differential equation can be represented by a power series about $x=0$, that is,

$$u(x)=\sum_{k=0}^{\infty}c_kx^k, \quad k\in(-1,1) \tag{(4.6)}$$

Substituting into the differential equation (4.4) gives

$$\left(1-x^2\right)\sum_{k=2}^{\infty}k(k-1)c_kx^{k-2}-2x\sum_{k=1}^{\infty}kc_kx^{k-1}+\lambda\sum_{k=0}^{\infty}c_kx^k=0,$$

or

$$\sum_{k=2}^{\infty}k(k-1)c_kx^{k-2}-\sum_{k=2}^{\infty}k(k-1)c_kx^k-2\sum_{k=1}^{\infty}kc_kx^k+\lambda\sum_{k=0}^{\infty}c_kx^k=0,$$

or

$$2c_2+6c_3x+\sum_{k=4}^{\infty}k(k-1)c_kx^{k-2}-\sum_{k=2}^{\infty}k(k-1)c_kx^k-2c_1x-2\sum_{k=2}^{\infty}kc_kx^k+\lambda(c_0+c_1x)+\lambda\sum_{k=2}^{\infty}c_kx^k=0,$$

or

$$2c_2+6c_3x-2c_1x+\lambda(c_0+c_1x)+\sum_{k=2}^{\infty}(k+2)(k+1)c_{k+2}x^k-\sum_{k=2}^{\infty}k(k-1)c_kx^k-2\sum_{k=2}^{\infty}kc_kx^k+\lambda\sum_{k=2}^{\infty}c_kx^k=0,$$

or

$$2c_2+\lambda c_0+[6c_3-2c_1+\lambda c_1]x+\sum_{k=2}^{\infty}[(k+2)(k+1)c_{k+2}-k(k-1)c_k-2kc_k+\lambda c_k]x^k=0$$

or

$$2c_2+\lambda c_0+[6c_3+(-2+\lambda)c_1]x+\sum_{k=2}^{\infty}\left[(k+2)(k+1)c_{k+2}+\left(-k^2-k+\lambda\right)c_k\right]x^k=0$$

or

$$\sum_{k=0}^{\infty}\left[(k+2)(k+1)c_{k+2}+\left(-k^2-k+\lambda\right)c_k\right]x^k=0$$

from which we get

$$(k+2)(k+1)c_{k+2}+\left(-k^2-k+\lambda\right)c_k=0,\quad \forall k\in\mathbb{N}_0$$

or

$$c_{k+2}=\frac{k(k+1)-\lambda}{(k+2)(k+1)}c_k,\quad \forall k\in\mathbb{N}_0 \qquad ((4.7))$$

Equation (4.7) is a *recursion formula* for the coefficients of the power series (4.6).

If we choose the eigenvalues as follows

$$\lambda=n(n+1),\quad n\in\mathbb{N}_0,$$

then

$$\begin{aligned} c_{k+2} &= \frac{k(k+1)-n(n+1)}{(k+2)(k+1)}c_k \\ &= \frac{(k-n)(k+n+1)}{(k+2)(k+1)}c_k \end{aligned}$$

from which we get

$$\begin{aligned} c_2 &= -\frac{n(n+1)}{2!}c_0, \\ c_3 &= -\frac{(n-1)(n+2)}{3!}c_1 \\ c_4 &= -\frac{(n-2)(n+3)}{12}c_2=\left(-\frac{(n-2)(n+3)}{12}\right)\left(-\frac{n(n+1)}{2!}c_0\right)=\frac{n(n-2)(n+1)(n+3)}{4!}c_0, \\ c_5 &= \frac{(n-3)(n-1)(n+2)(n+4)}{5!}c_1, \\ &\vdots \end{aligned}$$

Therefore, the solution of the Legendre equation is given by

$$\begin{aligned} u(x) &= \sum_{k=0}^{\infty}c_kx^k \\ &= c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+... \\ &= c_0\left[1-\frac{n(n+1)}{2!}x^2+\frac{n(n-2)(n+1)(n+3)}{4!}x^4+...\right] \\ &\quad +c_1\left[x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5+...\right] \\ &= c_0u_0(x)+c_1u_1(x), \end{aligned}$$

where both $u_0(x)$ and $u_1(x)$ converge in $(-1,1)$ and are linearly independent because one is in even powers of x and the other is in odd powers of x.

We conclude that for each $n \in \mathbb{N}_0$, we have two linearly independent solutions, namely

1. For $n = 0$, we have

$$\begin{aligned} u_0(x) &= 1, \\ u_1(x) &= x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + ... \end{aligned}$$

2. For $n = 1$, we have

$$\begin{aligned} u_0(x) &= 1 - x^2 - \frac{1}{3}x^4 + ..., \\ u_1(x) &= x. \end{aligned}$$

3. For $n = 2$, we have

$$\begin{aligned} u_0(x) &= 1 - 3x^2, \\ u_1(x) &= x - \frac{2}{3}x^3 - \frac{1}{5}x^5 + ... \end{aligned}$$

4. For $n = 3$, we have

$$\begin{aligned} u_0(x) &= 1 - 6x^2 + 3x^4 + ..., \\ u_1(x) &= x - \frac{5}{3}x^3. \end{aligned}$$

Remark 109 *1*

1. For each $n \in \mathbb{N}_0$, one of the two solutions is a polynomial, which can be proved from the recursion formula

$$c_{k+2} = \frac{(k-n)(k+n+1)}{(k+2)(k+1)} c_k$$

Note that for any $n \in \mathbb{N}_0$, $c_{n+2} = 0$, and consequently

$$... = c_{n+6} = c_{n+4} = 0.$$

[why?]. That is in one of the two series, all but a finite number of terms of the series vanish. In other words, one of the two series is a polynomial.

Definition 110 *(Legendre polynomial)*

A *Legendre polynomial* of degree n, denoted by $P_n(x)$, is a scalar multiple of the polynomial solution of the singular SL problem (4.4) with $\lambda = n(n+1)$. In particular, the coefficient of the highest power in a Legendre polynomial is given by

$$a_n = \frac{(2n)!}{2^n (n!)^2}$$

Lower coefficients in $P_n(x)$ are therefore determined using the relation

$$c_{k+2} = \frac{(k-n)(k+n+1)}{(k+2)(k+1)} c_k$$

which gives

$$c_k = \frac{(k+2)(k+1)}{(k-n)(k+n+1)} c_{k+2}$$

That is,

$$\begin{aligned} a_{n-2} &= \frac{n(n-1)}{(n-2-n)(n-2+n+1)} a_n \\ &= -\frac{n(n-1)}{2(2n-1)} a_n \\ &= -\frac{n(n-1)}{2(2n-1)} \frac{(2n)!}{2^n (n!)^2} \\ &= \frac{(2n-2)!}{2^n (n-1)!(n-2)!} \end{aligned}$$

and so on. In general, we have

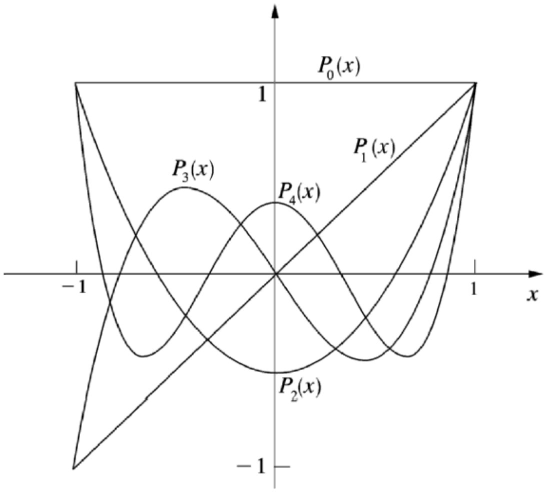
$$a_{n-2k} = (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!}, n \geq 2k$$

Therefore, the Legendre polynomial is given by

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{\left[\frac{n}{2}\right]} a_{n-2k} x^{n-2k} \\ &= \frac{1}{2^n} \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{(2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k} \end{aligned}$$

where $\left[\frac{n}{2}\right]$ is the integral part of $\frac{n}{2}$.

$P_0(x) = 1,$	$P_1(x) = x,$
$P_2(x) = \frac{1}{2}(3x^2 - 1),$	$P_3(x) = \frac{1}{2}(5x^3 - 3x),$
$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3),$	$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x).$



Definition 111 *(Legendre Functions)*

A *Legendre function*, denoted by $Q_n(x)$, is the infinite series solution of the above singular SL problem (4.4) with $\lambda = n(n+1)$. For example,

$$Q_0(x) = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + ... = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),$$

[why?]. Also,

$$Q_1(x) = 1 - x^2 - \frac{1}{3}x^4 + ... = 1 - \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right),$$

[see exercise 4.4].

Remark 112 *2*

1. The Legendre polynomials $P_n(x)$ are bounded at $x = \pm 1$, for all $n \in \mathbb{N}_0$.
2. The Legendre functions $Q_n(x)$ are unbounded at $x = \pm 1$, for all $n \in \mathbb{N}_0$.
3. (1) and (2) above shows that the operator L in the singular SL problem (4.4) is self-adjoint only in the first case, namely, when the Legendre polynomials P_n are taken to be solutions of the singular SL problem. In this case theorem 2.29 holds, that is,
   (a) The eigenvalues of L, $\lambda_n = n(n+1)$, tend to ∞.
   (b) The set of eigenfunctions of L, $\{P_n(x) : n \in \mathbb{N}_0\}$, is orthogonal and complete in $\mathcal{L}^2(-1,1)$.

14.2 Properties of the Legendre Polynomials

1. Rodrigues Formula for Legendre polynomials

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n$$

2. Using formula (4.13) , one can prove the following identities

$$\begin{aligned} P'_{n+1}(x) - P'_{n-1}(x) &= (2n+1) P_n(x), \\ (n+1) P_{n+1}(x) - n P_{n-1}(x) &= (2n+1) x P_n(x), \quad n \in \mathbb{N}. \end{aligned}$$

3. The Legendre polynomials are orthogonal in $\mathcal{L}^2(-1,1)$. That is,

$$\langle P_n, P_m \rangle = 0 \qquad \text{for all } m \neq n$$

4. The norm of a Legendre polynomial tends to 0 as $n \to \infty$

$$\|P_n\| = \sqrt{\frac{2}{2n+1}}, \quad n \in \mathbb{N}_0,$$

and therefore, $\{\frac{1}{\sqrt{2}} P_0(x), \sqrt{\frac{3}{2}} P_1(x), \sqrt{\frac{5}{2}} P_2(x), ..., \sqrt{\frac{2n+1}{2}} P_n(x), ...\}$ is a complete orthonormal set in $\mathcal{L}^2(-1,1)$.

5. Legendre polynomials satisfy

$$P_n(1) = 1, \quad P_n(-1) = (-1)^n \qquad \text{for all } n \in \mathbb{N}_0.$$

Example 113 *4.1*

Consider the function

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$$

1. Show that $f \in \mathcal{L}^2(-1,1)$.
2. Give the Legendre series expansion of f.
3. Show that
$$S(0) = \frac{1}{2}\left[f\left(0^+\right) + f\left(0^-\right)\right].$$

Solution:

1.
$$\int_{-1}^{1} |f(x)|^2 \, dx = \int_0^1 1 dx = 1 < \infty \Rightarrow f \in \mathcal{L}^2(-1,1).$$

2. Since $f \in \mathcal{L}^2(-1,1)$, we can write

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x),$$

where

$$\begin{aligned} c_n &= \frac{\langle f, P_n \rangle}{\|P_n\|^2} \\ &= \frac{1}{\left(\sqrt{\frac{2}{2n+1}}\right)^2} \int_{-1}^{1} f(x) P_n(x) \, dx \\ &= \frac{2n+1}{2} \int_0^1 P_n(x) \, dx \\ &= \frac{2n+1}{2} \int_0^1 \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2-1\right)^n dx \\ &= \frac{2n+1}{2^{n+1} n!} \int_0^1 \frac{d^n}{dx^n} \left(x^2-1\right)^n dx \end{aligned}$$

Thus,

$$\begin{aligned} c_0 &= \frac{1}{2} \int_0^1 \frac{d^0}{dx^0} \left(x^2-1\right)^0 dx = \frac{1}{2} \int_0^1 1 dx = \frac{1}{2}, \\ c_1 &= \frac{3}{2^2} \int_0^1 \frac{d^1}{dx^1} \left(x^2-1\right)^1 dx = \frac{3}{4} \int_0^1 2x dx = \frac{3}{4}, \\ c_2 &= \frac{5}{(2^3)(2)} \int_0^1 \frac{d^2}{dx^2} \left(x^2-1\right)^2 dx = 0, \\ c_3 &= \frac{7}{(2^4)(6)} \int_0^1 \frac{d^3}{dx^3} \left(x^2-1\right)^3 dx = -\frac{7}{16}, \\ &\vdots \end{aligned}$$

from which we get

$$f(x) = \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) - \frac{7}{16} P_3(x) + ...$$

3.

$$S(0) = \frac{1}{2} P_0(0) + \frac{3}{4} P_1(0) - \frac{7}{16} P_3(0) + ...$$

but $P_0(0) = 1$. Next, we prove that

$$P_{2n+1}(0) = 0 \quad \forall n \in \mathbb{N}_0$$

Using the identity

$$(m+1) P_{m+1}(x) + m P_{m-1}(x) = (2m+1) x P_m(x), \ m \in \mathbb{N}$$

at $x = 0$ gives

$$(m+1)\,P_{m+1}(0) + mP_{m-1}(0) = 0$$

or

$$P_{m+1}(0) = -\frac{m}{m+1}P_{m-1}(0)$$

Now, using $m = 2n$, $n = 1, 2, 3...$ in the above relation gives

$$P_{2n+1}(0) = -\frac{2n}{2n+1}P_{2n-1}(0)$$

Starting from $n = 1$, we have

$$P_3(0) = -\frac{2}{3}P_1(0) = -\frac{2}{3}(0) = 0$$

and consequently,

$$P_{2n+1}(0) = 0 \;\; \forall n \in \mathbb{N}_0$$

Therefore,

$$S(0) = \frac{1}{2}(1) + \frac{3}{4}(0) - \frac{7}{16}(0) + .. = \frac{1}{2}$$

Now,

$$\frac{1}{2}\left[f\left(0^+\right) + f\left(0^-\right)\right] = \frac{1}{2}[1+0] = \frac{1}{2}$$

That is,

$$S(0) = \frac{1}{2}\left[f\left(0^+\right) + f\left(0^-\right)\right]$$

as expected.

15 Hermite Polynomials

Definition 114 *(Hermite Polynomial)*

For each $n \in \mathbb{N}_0$, the *Hermite polynomial* $H_n : \mathbb{R} \to \mathbb{R}$ is defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Example 115 *1*

1. For $n = 0$,

$$H_0(x) = (-1)^0 e^{x^2} \frac{d^0}{dx^0} e^{-x^2} = e^{x^2} e^{-x^2} = e^0 = 1.$$

2. For $n = 1$,

$$H_1(x) = (-1)^1 e^{x^2} \frac{d^1}{dx^1} e^{-x^2} = -e^{x^2} e^{-x^2} (-2x) = 2x.$$

3. For $n = 2$,

$$H_2(x) = (-1)^2 e^{x^2} \frac{d^2}{dx^2} e^{-x^2} = e^{x^2} \frac{d}{dx} \left(-2xe^{-x^2}\right) = e^{x^2} \left(-2e^{-x^2} + 4x^2 e^{-x^2}\right) = -2+4x^2.$$

15.1 Properties of Hermite Polynomials

1. H_n is a polynomial of degree n.
2. The set $\{H_n : n \in \mathbb{N}_0\}$ is orthogonal in $\mathcal{L}^2_{e^{-x^2}}(\mathbb{R})$.
3. The norm of H_n is given by

$$\|H_n\|_{e^{-x^2}} = \left(2^n n! \sqrt{\pi}\right)^{\frac{1}{2}}.$$

4. For every $x \in \mathbb{R}$,

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^n.$$

   In other words, e^{2xt-t^2} is a generating function for the Hermite polynomials.

Theorem 116 *4.2*

H_n satisfies the second-order differential equation

$$u'' - 2xu' + 2nu = 0, \quad x \in \mathbb{R},$$

which is called the Hermite equation.

Remark 117 *1*

1. The other solution of the Hermite equation is an analytical function that can be represented by a series in x [see exercise 4.21].

2. The differential operator in the Hermite equation is not formally self-adjoint, but can be transformed to one by multiplying the equation by

$$\rho(x) = e^{\int -2x dx} = e^{-x^2}.$$

   In which case, we have the SL problem

$$e^{-x^2} u'' - 2xe^{-x^2} u' + 2nu = 0, \quad x \in \mathbb{R} \qquad ((4.31))$$

   where

$$L = \frac{d}{dx}\left(e^{-x^2} \frac{d}{dx}\right)$$

3. The Hermite polynomials $\{H_n : n \in \mathbb{N}_0\}$ are the eigenfunctions of the singular SL problem (4.31) associated with the eigenvalues $\lambda_n = 2n$. Therefore, the set $\{H_n : n \in \mathbb{N}_0\}$ is complete orthogonal set in $\mathcal{L}^2_{e^{-x^2}}(\mathbb{R})$.

16 Laguerre Polynomials

Definition 118 *(Laguerre Polynomial)*

For each $n \in \mathbb{N}_0$, the *Laguerre polynomial* $H_n : (0, \infty) \to (0, \infty)$ is defined by

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} \left(x^n e^{-x} \right).$$

Example 119 *1*

1. For $n = 0$,
$$L_0(x) = \frac{e^x}{0!} \frac{d^0}{dx^0} \left(x^0 e^{-x} \right) = e^x e^{-x} = e^0 = 1.$$

2. For $n = 1$,
$$L_1(x) = \frac{e^x}{1!} \frac{d^1}{dx^1} \left(x^1 e^{-x} \right) = e^x \left(e^{-x} - x e^{-x} \right) = 1 - x.$$

3. For $n = 2$,
$$\begin{aligned} L_2(x) &= \frac{e^x}{2!} \frac{d^2}{dx^2} \left(x^2 e^{-x} \right) \\ &= \frac{e^x}{2!} \frac{d}{dx} \left(2x e^{-x} - x^2 e^{-x} \right) \\ &= \frac{e^x}{2!} \left(2e^{-x} - 2x e^{-x} - 2x e^{-x} + x^2 e^{-x} \right) \\ &= \frac{e^x}{2} \left(2e^{-x} - 4x e^{-x} + x^2 e^{-x} \right) \\ &= 1 - 2x + \frac{1}{2} x^2. \end{aligned}$$

16.1 Properties of Laguerre Polynomials

1. L_n is a polynomial of degree n.
2. The set $\{L_n : n \in \mathbb{N}_0\}$ is orthogonal in $\mathcal{L}^2_{e^{-x}}(0, \infty)$.
3. The norm of L_n is given by
$$\|L_n\| = 1.$$

Theorem 120 *4.3*

L_n satisfies the second-order differential equation

$$x u'' + (1 - x) u' + n u = 0, \quad x \in (0, \infty).$$

which is called the Laguerre equation.

Remark 121 *1*

1. The differential operator in the Laguerre equation is not formally self-adjoint, but can be transformed to one by multiplying the equation by

$$\rho(x) = \frac{1}{x} e^{\int \frac{1-x}{x} dx} = \frac{1}{x} e^{\int \frac{1}{x} - 1 dx} = \frac{1}{x} \left(e^{\ln x} e^{-x} \right) = e^{-x}$$

   In which case, we have the SL problem

$$xe^{-x}u'' + (1-x)e^{-x}u' + ne^{-x}u = 0, \quad x \in (0, \infty). \qquad ((4.33))$$

   where

$$L = \frac{d}{dx}\left(xe^{-x} \frac{d}{dx} \right)$$

2. The Laguerre polynomials $\{L_n : n \in \mathbb{N}_0\}$ are the eigenfunctions of the singular SL problem (4.33) associated with the eigenvalues $\lambda_n = n$. Therefore, the set $\{L_n : n \in \mathbb{N}_0\}$ is complete orthogonal set in $\mathcal{L}^2_{e^{-x}}(0, \infty)$.

Part VI
Bessel Functions

17 The Gamma Function

Definition 122 *(Gamma Function)*

The *gamma function* is given by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt,$$

for $x > 0$.

17.1 Properties of the Gamma Function

1. Γ is of class C^∞ on $(0, \infty)$.
2. For any $x > 0$,
$$\Gamma(x+1) = x\Gamma(x)$$
3. $\Gamma(1) = 1$.
4. For any $n \in \mathbb{N}$
$$\Gamma(n+1) = n!$$
That is, the gamma function is an extension of the factorial mapping $f(n+1) = n!$ from $\mathbb{N}$ to $(0, \infty)$.
5. The domain of the gamma function can be extended from $(0, \infty)$ to $\mathbb{R} - \{0, -1, -2, ...\}$ as follows
$$\begin{aligned} \Gamma(x) &= \frac{\Gamma(x+1)}{x} \\ &= \frac{\Gamma(x+2)}{x(x+1)} \\ &= \frac{\Gamma(x+3)}{x(x+1)(x+2)} \\ &\vdots \\ &= \frac{\Gamma(x+n)}{x(x+1)\ldots(x+n-1)}, \quad n \in \mathbb{N} \end{aligned}$$
Note that for any $n \in \mathbb{N}$,
$$\lim_{x \to -(n-1)} \Gamma(x+n) = \Gamma(-(n-1)+n) = \Gamma(1) = 1$$

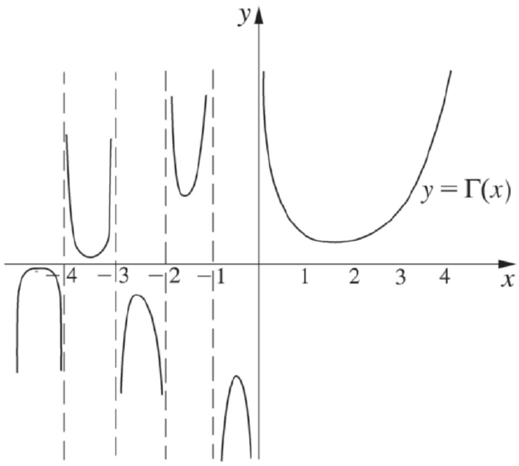


Figure 5.1 The gamma function.

Therefore,

$$\lim_{x \to -(n-1)^{\pm}} \left| \frac{\Gamma(x+n)}{x(x+1)\dots(x+n-1)} \right| = \infty$$

Note that $-(n-1)$ is a simple pole of the gamma function.

6.
$$\lim_{x \to \infty} \Gamma(x) = \infty.$$

18 Bessel Functions of the First Kind

The *Bessel's equation* is the second-order differential equation

$$x^2y'' + xy' + \left(x^2 - \nu^2\right) y = 0, \quad x \in (0, \infty) \qquad ((5.3))$$

where ν is a nonnegative parameter.

18.1 Derivation of the Bessel Functions of the First Kind

The Bessel equation is a singular SL problem. If we write the equation in the form

$$y'' + \frac{1}{x}y' + \frac{\left(x^2 - \nu^2\right)}{x^2}y = 0,$$

then clearly the solution cannot be represented by a power series about $x = 0$ [why?], but since the numerators 1 and $x^2 - \nu^2$ are both analytic at $x = 0$ we can seek a solution of the from

$$y(x) = \sum_{k=0}^{\infty} c_k x^{k+t},$$

where $t \in \mathbb{C}$ and $c_0 \neq 0$.

Substituting the above series in (5.3) gives

$$x^2 \sum_{k=0}^{\infty} c_k (k+t)(k+t-1) x^{k+t-2} + x \sum_{k=0}^{\infty} c_k (k+t) x^{k+t-1} + \left(x^2 - \nu^2\right) \sum_{k=0}^{\infty} c_k x^{k+t} = 0$$

or

$$\sum_{k=0}^{\infty} c_k (k+t)(k+t-1) x^{k+t} + \sum_{k=0}^{\infty} c_k (k+t) x^{k+t} + \sum_{k=0}^{\infty} c_k x^{k+t+2} - \nu^2 \sum_{k=0}^{\infty} c_k x^{k+t} = 0$$

or

$$\sum_{k=0}^{\infty} \left[(k+t)(k+t-1) + (k+t) - \nu^2\right] c_k x^{k+t} + \sum_{k=0}^{\infty} c_k x^{k+t+2} = 0$$

or

$$\sum_{k=0}^{\infty} \left[(k+t)^2 - \nu^2\right] c_k x^{k+t} + \sum_{k=0}^{\infty} c_k x^{k+t+2} = 0$$

or

$$\left[t^2 - \nu^2\right] c_0 x^t + \left[(1+t)^2 - \nu^2\right] c_1 x^{1+t} + \sum_{k=2}^{\infty} \left[(k+t)^2 - \nu^2\right] c_k x^{k+t} + \sum_{k=2}^{\infty} c_{k-2} x^{k+t} = 0$$

or

$$\left[t^2 - \nu^2\right] c_0 x^t + \left[(1+t)^2 - \nu^2\right] c_1 x^{1+t} + \sum_{k=2}^{\infty} \left\{ \left[(k+t)^2 - \nu^2\right] c_k + c_{k-2} \right\} x^{k+t} = 0$$

from which we get

$$\begin{aligned}
\left[t^2-\nu^2\right]c_0 &= 0,\\
\left[(1+t)^2-\nu^2\right]c_1 &= 0,\\
\left[(k+t)^2-\nu^2\right]c_k+c_{k-2} &= 0, \quad k\in\{2,3,...\}
\end{aligned}$$

Since $c_0\neq 0$, we have

$$t^2-\nu^2=0$$

or

$$t=\pm\nu.$$

Case 1: Let $t=\nu$, then

1.
$$\left[(1+t)^2-\nu^2\right]c_1=0$$
leads to
$$(2\nu+1)\,c_1=0$$
[why?]. But, $2\nu+1\geq 1$ [why?], and thus c_1 must be zero.

2.
$$\left[(k+t)^2-\nu^2\right]c_k+c_{k-2}=0$$
gives
$$k\,(2\nu+k)\,c_k+c_{k-2}=0$$
[why?], which can be written in the form
$$c_k=-\frac{1}{k\,(2\nu+k)}c_{k-2}$$

   (a) If $k=2m+1$ where $m\in\mathbb{N}$, then
   $$c_{2m+1}=-\frac{1}{(2m+1)\,(2\nu+2m+1)}c_{2m-1}$$
   and hence $c_{2m+1}=0$ [why?].

   (b) If $k=2m$ where $m\in\mathbb{N}$, then
   $$c_{2m}=-\frac{1}{(2m)\,(2\nu+2m)}c_{2m-2}=-\frac{1}{2^2m\,(\nu+m)}c_{2m-2}$$
   Therefore,
   $$\begin{aligned}
   c_2 &= \frac{(-1)^1}{2^2\,(\nu+1)}c_0,\\
   c_4 &= -\frac{1}{2^2\,(2)\,(\nu+2)}c_2=\left(-\frac{1}{2^2\,(2)\,(\nu+2)}\right)\left(-\frac{1}{2^2\,(\nu+1)}\right)c_0=\frac{(-1)^2}{2^4 2!\,(\nu+1)\,(\nu+2)}c_0,
   \end{aligned}$$

In general, we have

$$c_{2m} = \frac{(-1)^m}{2^{2m} m! (\nu + 1) (\nu + 2) \dots (\nu + m)} c_0$$

so if we choose

$$c_0 = \frac{1}{2^\nu \Gamma (\nu + 1)},$$

then we get

$$\begin{aligned} c_{2m} &= \frac{(-1)^m}{2^{2m+\nu} m! \Gamma (\nu + 1) (\nu + 1) (\nu + 2) \dots (\nu + m)} \\ &= \frac{(-1)^m}{2^{2m+\nu} m! \Gamma (\nu + m + 1)}. \end{aligned}$$

The solution of the Bessel equation is therefore given by

$$x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m+\nu} m! \Gamma (\nu + m + 1)} x^{2m}$$

Definition 123 *(Bessel function of the first kind)*

The Bessel function of the first kind of order ν is denoted by $J_\nu (x)$, and is given by

$$J_\nu (x) = \left(\frac{x}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma (\nu + m + 1)} \left(\frac{x}{2}\right)^{2m}, \quad x > 0$$

Remark 124 *1*

1. $J_\nu (x)$ is defined for all $x \in (0, \infty)$ because
   (a) The series
   $$\sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma (\nu + m + 1)} \left(\frac{x}{2}\right)^{2m}$$
   is convergent for all $x \in \mathbb{R}$.[why?].
   (b) The power x^ν is defined for all $x > 0$.
2. The domain of $J_\nu (x)$ can be extended from $(0, \infty)$ to $[0, \infty)$ by defining
   $$J_v (0) = \lim_{x \to 0^+} J_\nu (x) = \begin{cases} 1, & \nu = 0 \\ 0, & \nu > 0 \end{cases}$$

<u>**Case 2:**</u> If $t = -\nu < 0$, then following the same steps as above leads to

$$J_{-\nu} (x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma (-\nu + m + 1)} \left(\frac{x}{2}\right)^{2m}, \quad x > 0$$

Example 125 *(The first two Bessel functions of first kind of integral order)*

1. Find Bessel function of order 0.
2. Find Bessel function of order 1.
3. Show that
$$J_0^{'}(x) = -J_1(x)$$
4. Prove that
$$\int_0^x tJ_0(x)\,dt = xJ_1(x)$$
for all $x > 0$.

Solution:

1.

$$\begin{aligned} J_0(x) &= \left(\frac{x}{2}\right)^0 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+1)} \left(\frac{x}{2}\right)^{2m} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!m!} \left(\frac{x}{2}\right)^{2m} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2}\right)^{2m} \\ &= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4(2!)^2} - ... \end{aligned}$$

2.

$$\begin{aligned} J_1(x) &= \left(\frac{x}{2}\right) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+2)} \left(\frac{x}{2}\right)^{2m} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!} \left(\frac{x}{2}\right)^{2m+1} \\ &= \frac{x}{2} - \frac{x^3}{2^3 2!} + \frac{x^5}{2^5 2! 3!} - ... \end{aligned}$$

3.

$$
\begin{aligned}
J_0^{'}(x) &= \frac{d}{dx}\left(\sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}\right)\\
&= -\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{2(m!)^2}(2m)\left(\frac{x}{2}\right)^{2m-1}\\
&= -\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m!(m-1)!}\left(\frac{x}{2}\right)^{2m-1}\\
&= -\sum_{m=0}^{\infty}\frac{(-1)^{m}}{(m+1)!(m)!}\left(\frac{x}{2}\right)^{2m+1}\\
&= -J_1(x)
\end{aligned}
$$

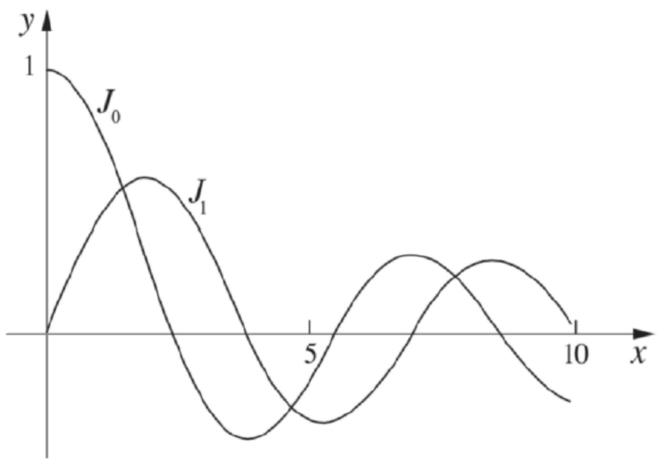


Figure 5.2 Bessel functions J_0 and J_1.

4.

$$\begin{aligned}
\int_0^x tJ_0(t)\,dt &= \int_0^x t\sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{t}{2}\right)^{2m} dt \\
&= \sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2\,2^{2m}}\int_0^x t^{2m+1}dt \\
&= \sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2\,2^{2m}}\left(\left.\frac{t^{2m+2}}{2m+2}\right|_0^x\right) \\
&= \sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2\,2^{2m}}\left(\frac{x^{2m+2}}{2m+2}\right) \\
&= \sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2\,2^{2m+1}}\left(\frac{x^{2m+2}}{m+1}\right) \\
&= x\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,(m+1)!}\left(\frac{x}{2}\right)^{2m+1} \\
&= xJ_1(x)
\end{aligned}$$

Theorem 126 *5.1*

The Bessel functions J_ν and $J_{-\nu}$ are linearly independent if and only if ν is not an integer.

Proof:

1. If $\nu = n \in \mathbb{N}_0$, then

$$J_{-n}(x) = \left(\frac{x}{2}\right)^{-n}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(-n+m+1)}\left(\frac{x}{2}\right)^{2m}, \quad x>0$$

but since $-n+m+1 \in \mathbb{Z}$, we have

$$|\Gamma(-n+m+1)| = \infty$$

if $-n+m+1 \leq 0$. That is,

$$\frac{1}{\Gamma(-n+m+1)} = 0$$

for all $m \leq n-1$. Therefore, the first n terms of the above series vanish, i.e,

$$\begin{aligned}
J_{-n}(x) &= \left(\frac{x}{2}\right)^{-n} \sum_{m=n}^{\infty} \frac{(-1)^m}{m!\Gamma(-n+m+1)} \left(\frac{x}{2}\right)^{2m} \\
&= \left(\frac{x}{2}\right)^{-n} \sum_{m=0}^{\infty} \frac{(-1)^{m+n}}{(m+n)!\Gamma(m+1)} \left(\frac{x}{2}\right)^{2m+2n} \\
&= (-1)^n \left(\frac{x}{2}\right)^{n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)(m+2)\ldots(m+n)\Gamma(m+1)} \left(\frac{x}{2}\right)^{2m} \\
&= (-1)^n \left(\frac{x}{2}\right)^{n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m} \\
&= (-1)^n J_n(x)
\end{aligned}$$

That is, $J_{-n}(x)$ and $J_n(x)$ are linearly dependent.

2. If $\nu \notin \mathbb{N}$ and $\nu > 0$, then let

$$aJ_{\nu}(x) + bJ_{-\nu}(x) = 0. \qquad ((5.14))$$

We Know that for $\nu > 0$, $\lim_{x\to 0^+} J_{\nu}(x) = 0$, but

$$\lim_{x\to 0^+} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(-\nu+m+1)} \left(\frac{x}{2}\right)^{2m} = \frac{1}{\Gamma(-\nu+1)} \in \mathbb{R},$$

$$\lim_{x\to 0^+} \left(\frac{x}{2}\right)^{-\nu} = \lim_{x\to 0^+} \left(\frac{2}{x}\right)^{\nu} = \infty,$$

from which we deduce, $\lim_{x\to 0^+} |J_{-\nu}(x)| = \infty$.

Therefore, equation (5.14) can hold only if $b = 0$. But, $aJ_{\nu}(x) = 0$ for all $x > 0$ only if $a = 0$. We conclude that $J_{-\nu}(x)$ and $J_{\nu}(x)$ are linearly independent in this case.

Remark 127 *2*

Theorem 5.1 leads to the conclusion that the general solution of the Bessel equation with a non-integer parameter ν is given by

$$y(x) = c_1 J_{\nu}(x) + c_2 J_{-\nu}(x),$$

where $x \in (0, \infty)$.

19 Bessel Functions of the Second Kind

Definition 128 *(Bessel function of the second kind)*

The Bessel function of the second kind of order ν is denoted by $Y_\nu(x)$, and is given by

$$Y_\nu(x) = \begin{cases} \frac{1}{\sin \nu\pi}\left[J_\nu(x)\cos\nu\pi - J_{-\nu}(x)\right], & \nu \neq 0,1,2,... \\ \lim_{\nu\to n} Y_n & n = 0,1,2,... \end{cases}$$

Remark 129 *2*

1. If $\nu = n \in \mathbb{N}_0$, then Bessel function of the second kind of order ν can be expressed in terms of the Bessel function of the first kind as follows

$$Y_n(x) = \frac{2}{\pi}\left(\ln\frac{x}{2}+\gamma\right)J_n(x) - \frac{1}{\pi}\left(\frac{x}{2}\right)^n \sum_{m=0}^{\infty}\frac{(-1)^m(h_m+h_{n+m})}{m!\,(n+m)!}\left(\frac{x}{2}\right)^{2m}$$
$$-\frac{1}{\pi}\left(\frac{x}{2}\right)^{-n}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{m!}\left(\frac{x}{2}\right)^{2m}, \quad x>0$$

where

$$\begin{aligned} h_0 &= 0, \\ h_m &= 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{m}, \\ \gamma &\simeq 0.577215 \end{aligned}$$

2. $Y_n(x)$ is also a solution of Bessel equation.

3. The Bessel functions J_ν and Y_ν are linearly independent for all $\nu \geq 0$.

4. The general solution of Bessel equation with a parameter $\nu \geq 0$ is given by

$$y(x) = c_1 J_\nu(x) + c_2 Y_n(x),$$

where $x \in (0,\infty)$.

Example 130 *(Bessel functions of second kind of integral order)*

Find the asymptotic behaviour of $Y_n(x)$ as $x \to 0$.
Solution:
We say that a function f is asymptotic to a function g as $x \to c$, and write $f \sim g$, if $\lim_{x\to c}\frac{f(x)}{g(x)} = 1$.

1. For $\nu = 0$, we have

$$\begin{aligned} Y_0(x) &= \frac{2}{\pi}\left(\ln\frac{x}{2}+\gamma\right)J_0(x) - \frac{1}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m(2h_m)}{(m!)^2}\left(\frac{x}{2}\right)^{2m} \\ &= \frac{2}{\pi}\left(\ln\frac{x}{2}+\gamma\right)\sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m} - \frac{1}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m(2h_m)}{(m!)^2}\left(\frac{x}{2}\right)^{2m} \\ &= \frac{2}{\pi}\left(\ln\frac{x}{2}+\gamma\right)\left(1-\frac{x^2}{2^2}+...\right) - \frac{1}{\pi}\left(-\frac{x^2}{2^2}+...\right) \end{aligned}$$

Therefore,

$$Y_0(x) \sim \frac{2}{\pi}\ln\frac{x}{2}$$

2. For $\nu \in \mathbb{N}$, we have

$$\begin{aligned} Y_n(x) &= \frac{2}{\pi}\left(\ln\frac{x}{2}+\gamma\right)J_n(x) - \frac{1}{\pi}\left(\frac{x}{2}\right)^n\sum_{m=0}^{\infty}\frac{(-1)^m(h_m+h_{n+m})}{m!(n+m)!}\left(\frac{x}{2}\right)^{2m} \\ &\quad -\frac{1}{\pi}\left(\frac{x}{2}\right)^{-n}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{m!}\left(\frac{x}{2}\right)^{2m} \\ &= \frac{2}{\pi}\left(\ln\frac{x}{2}+\gamma\right)\left(\frac{x}{2}\right)^n\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{2m} - \frac{1}{\pi}\left(\frac{x}{2}\right)^n\left(\frac{1+\frac{1}{2}+...+\frac{1}{n}}{n!}+...\right) \\ &\quad -\frac{1}{\pi}\left(\frac{x}{2}\right)^{-n}((n-1)!+...) \\ &= \frac{2}{\pi}\left(\ln\frac{x}{2}+\gamma\right)\left(\frac{x}{2}\right)^n\left(\frac{1}{\Gamma(n+1)}-...\right) - \frac{1}{\pi}\left(\frac{x}{2}\right)^n\left(\frac{1+\frac{1}{2}+...+\frac{1}{n}}{n!}+...\right) \\ &\quad -\frac{1}{\pi}\left(\frac{x}{2}\right)^{-n}((n-1)!+...) \end{aligned}$$

Therefore, ,

$$Y_n(x) \sim -\frac{1}{\pi}\left(\frac{x}{2}\right)^{-n}(n-1)!,\ n \in \mathbb{N}.$$

Note that for all $n \in \mathbb{N}_0$,

$$\lim_{x\to 0^+} Y_n(x) = -\infty.$$

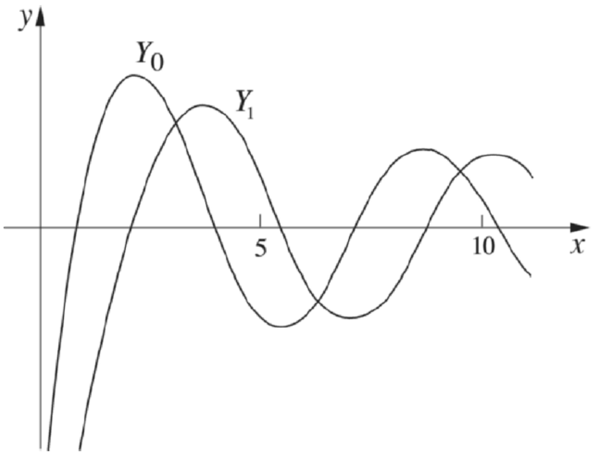


Figure 5.3 Bessel functions Y_0 and Y_1.

20 Orthogonality Properties

Lemma 131 *1*

The Bessel equation

$$x^2y'' + xy' + \left(x - \nu^2\right) y = 0, \qquad ((5.26))$$

is equivalent to the equation

$$xu'' + u' + \left(\mu^2 x - \frac{\nu^2}{x}\right) u = 0, \qquad ((5.27))$$

where $\mu \neq 0$.

Proof:

Dividing equation (5.26) by x gives

$$xy'' + y' + \left(x - \frac{\nu^2}{x}\right) y = 0 \qquad ((5.28))$$

Using the change of variables

$$x \to \mu x, \quad y(x) \to y(\mu x) = u(x)$$

where $\mu \neq 0$, which gives

$$\begin{aligned} u'(x) &= \mu y'(\mu x), \\ u''(x) &= \mu^2 y''(\mu x) \end{aligned}$$

Equation (5.28) becomes

$$x\mu\frac{u''}{\mu^2}+\frac{u'}{\mu}+\left(\mu x-\frac{\nu^2}{\mu x}\right)u=0$$

or

$$xu''+u'+\left(\mu^2 x-\frac{\nu^2}{x}\right)u=0$$

Theorem 132 *1*

The eigenvalue problem

$$\begin{aligned} xu''+u'+\left(\mu^2 x-\frac{n^2}{x}\right)u &= 0, \quad x\in(0,b), \qquad ((5.29)) \\ \beta_1 u(b)+\beta_2 u'(b) &= 0, \end{aligned}$$

where $b<\infty$, and $n\in\mathbb{N}_0$ is a singular SL problem, and its general solution is given by

$$u(x)=c_n J_n(\mu x).$$

Proof:
Problem (5.27) can be written as the eigenvalue problem

$$Lu+\lambda\rho(x)u=0, \quad x\in(0,b)$$
$$\beta_1 u(b)+\beta_2 u'(b)=0,$$

where

$$L=x\frac{d^2}{dx^2}+\frac{d}{dx}-\frac{\nu^2}{x},$$
$$\lambda=\mu^2,$$
$$\rho(x)=x.$$

Now,

1. L is a self-adjoint operator because:
   (a) $p(x)=x$, $q(x)=1$, $r(x)=-\frac{\nu^2}{x}$ are all real functions.
   (b) $p'(x)=1=q(x)$.
   (c) For any two solution u,v of the problem, we have

$$\begin{aligned} p(uv'-vu')\big|_0^b &= p(b)(u(b)v'(b)-v(b)u'(b))-p(0)(u(0)v'(0)-v(0)u'(0)) \\ &= p(b)\left(\frac{-\beta_2}{\beta_1}u'(b)v'(b)-\frac{-\beta_2}{\beta_1}v'(b)u'(b)\right) \\ &= 0 \end{aligned}$$

   where we have assumed that $\beta_1\neq 0$. A similar result occurs if $\beta_2\neq 0$.

2. $p(0) = 0$.
3. Since the differential equation in eigenvalue problem (5.29) is equivalent to the Bessel equation, its general solution is given by

$$u(x) = c_n J_n(\mu x) + d_n Y_n(\mu x)$$

but,

$$\lim_{x \to 0^+} u(x) \text{ exists} \quad \Leftrightarrow \quad d_n = 0$$

Therefore, (5.29) is a singular SL problem with the general solution

$$u(x) = c_n J_n(\mu x).$$

Theorem 133 *2*

If $\beta_2 = 0$ in problem (5.29), then

1. The eigenvalues of (5.29) are

$$\lambda_k = \mu_k^2 = \left(\frac{\xi_{nk}}{b}\right)^2$$

where ξ_{nk} are the solution of

$$J_n(\mu_k b) = 0$$

and the corresponding eigenfunctions are

$$u_k(x) = J_n(\mu_k x)$$

2. The set $\{J_n(\mu_k x) : k \in \mathbb{N}\}$ is orthogonal. That is,

$$\left\langle J_n(\mu_k x), J_n(\mu_j x)\right\rangle_x = 0, \quad k \neq j$$

3. The norm of $J_n(\mu_k x)$ is given by

$$\|J_n(\mu_k x)\|_x = \left(\frac{b^2}{2} J_{n+1}^2(\mu_k b)\right)^{\frac{1}{2}}$$

4. For any $f \in \mathcal{L}_x^2(0, b)$

$$f(x) = \sum_{k=1}^{\infty} \frac{\langle f(x), J_n(\mu_k x)\rangle_x}{\|J_n(\mu_k x)\|_x^2} J_n(\mu_k x) \qquad ((5.32))$$

5. If f is smooth on $(0, b)$, then the above equality holds pointwise provided f is defined by

$$f(x) = \frac{1}{2}\left[f\left(x^-\right) + f\left(x^+\right)\right]$$

at the points of discontinuity.

Proof:

1. If $\beta_2 = 0$, the boundary condition becomes
$$u(b) = 0,$$
Thus, we have
$$0 = u(b) = c_n J_n(\mu b),$$
or
$$J_n(\mu b) = 0$$
which leads to
$$\mu_k b = \xi_{nk},$$
where $k \in \mathbb{N}$ [why?] and ξ_{nk} are the zeros of J_n. Thus,
$$\mu_k = \frac{\xi_{nk}}{b},$$
and the eigenvalues are therefore
$$\lambda_k = \mu_k^2 = \left(\frac{\xi_{nk}}{b}\right)^2$$
The corresponding eigenfunction are
$$u_k(x) = J_n(\mu_k x).$$

2. Theorem 2.29 holds for a singular SL problem (5.29) (See the subsection "The singular SL problem"). In particular, the eigenfunctions of the problem are orthogonal and form a basis for $\mathcal{L}_x^2(0,b)$. Therefore,
$$\left\langle J_n(\mu_k x), J_n(\mu_j x)\right\rangle_x = 0, \qquad k \neq j$$

3. Multiplying equation (5.27) by $2xu'$ gives
$$2x^2 u'' u' + 2x(u')^2 + 2(\mu^2 x^2 - n^2) u' u = 0,$$
or
$$\left((xu')^2\right)' + (\mu^2 x^2 - n^2)(u^2)' = 0$$
Integrating the above equation on $(0,b)$ gives
$$(xu')^2\Big|_0^b + \mu^2 \int_0^b x^2 (u^2)' \, dx - n^2 \, u^2\big|_0^b = 0$$
$$\Rightarrow (xu')^2\Big|_0^b + \mu^2 x^2 u^2\big|_0^b - 2\mu^2 \int_0^b x u^2 dx - n^2 \, u^2\big|_0^b = 0$$
$$\Rightarrow 2\mu^2 \int_0^b x u^2 dx = (xu')^2\Big|_0^b + \mu^2 x^2 u^2\big|_0^b - n^2 \, u^2\big|_0^b$$
$$\Rightarrow \|u\|_x^2 = \frac{1}{2\mu^2}\left[(xu')^2 + (\mu^2 x^2 - n^2) u^2\right]\Big|_0^b$$

Using the solution $u(x) = J_n(\mu x)$ in the above equation gives

$$\begin{aligned}\|J_n(\mu x)\|_x^2 &= \frac{1}{2\mu^2}\left[\left(\mu x J_n'(\mu x)\right)^2 + \left(\mu^2 x^2 - n^2\right) J_n^2(\mu x)\right]\Big|_0^b \\ &= \frac{1}{2\mu^2}\left[\left(\mu b J_n'(\mu b)\right)^2 + \left(\mu^2 b^2 - n^2\right) J_n^2(\mu b) + n^2 J_n^2(0)\right] \\ &= \frac{1}{2\mu^2}\left[\left(\mu b J_n'(\mu b)\right)^2 + \left(\mu^2 b^2 - n^2\right) J_n^2(\mu b)\right]\end{aligned}$$

since $n^2 J_n^2(0) = 0 \; \forall n \in \mathbb{N}_0$.

Now,

$$\begin{aligned}\|J_n(\mu_k x)\|_x^2 &= \frac{1}{2\mu_k^2}\left[\left(\mu b J_n'(\mu_k b)\right)^2 + \left(\mu_k^2 b^2 - n^2\right) J_n^2(\mu_k b)\right] \\ &= \frac{b^2}{2}\left[J_n'(\mu_k b)\right]^2\end{aligned}$$

but using the identity (exercise 5.9)

$$x J_\nu'(x) = \nu J_\nu(x) - x J_{\nu+1}(x)$$

gives

$$\begin{aligned}\mu_k b J_n'(\mu_k b) &= n J_n(\mu_k b) - \mu_k b J_{n+1}(\mu_k b) \\ \Rightarrow \mu_k b J_n'(\mu_k b) &= -\mu_k b J_{n+1}(\mu_k b) \\ \Rightarrow J_n'(\mu_k b) &= -J_{n+1}(\mu_k b)\end{aligned}$$

Thus, we have

$$\|J_n(\mu_k x)\|_x^2 = \frac{b^2}{2} J_{n+1}^2(\mu_k b)$$

4. Follows form Theorem 2.29.
5. Generalized version of Theorem 3.9.

Example 134 *5.5*

Consider the function

$$f(x) = \begin{cases} 1, & 0 \le x < 2 \\ 0, & 2 < x \le 4 \end{cases}$$

with

$$J_0(4\mu) = 0$$

1. Expand f in a Fourier-Bessel series.

2. Find the sum of the series at $x = 1$.

3. Find the sum of the series at $x = 2$.

Solution:

1. Clearly $f \in \mathcal{L}_x^2(0, 4)$, therefore

$$f(x) = \sum_{k=1}^{\infty} \frac{\langle f(x), J_0(\mu_k x)\rangle_x}{\|J_0(\mu_k x)\|_x^2} J_0(\mu_k x)$$

Now,

(a)

$$\begin{aligned} \langle f(x), J_0(\mu_k x)\rangle_x &= \int_0^4 f(x) J_0(\mu_k x) x dx \\ &= \int_0^2 J_0(\mu_k x) x dx \\ &= \frac{1}{\mu_k^2} \int_0^2 J_0(\mu_k x) \mu_k x \mu_k dx \\ &= \frac{1}{\mu_k^2} \int_0^{2\mu_k} J_0(y) y dy \end{aligned}$$

but

$$\int_0^x t J_0(t) dt = x J_1(x)$$

hence

$$\begin{aligned} \langle f(x), J_0(\mu_k x)\rangle_x &= \frac{1}{\mu_k^2} 2\mu_k J_1(2\mu_k) \\ &= \frac{2}{\mu_k} J_1(2\mu_k) \end{aligned}$$

(b)

$$\|J_0(\mu_k x)\|_x^2 = \frac{16}{2} J_1^2(4\mu_k) = 8J_1^2(4\mu_k)$$

So we have

$$\begin{aligned} f(x) &= \sum_{k=1}^{\infty} \frac{\frac{2}{\mu_k} J_1(2\mu_k)}{8J_1^2(4\mu_k)} J_0(\mu_k x) \\ &= \frac{1}{4} \sum_{k=1}^{\infty} \frac{J_1(2\mu_k)}{\mu_k J_1^2(4\mu_k)} J_0(\mu_k x) \end{aligned}$$

for $0 < x < 4$.

2. Since f is continuous at $x = 1$, we have

$$\frac{1}{4}\sum_{k=1}^{\infty}\frac{J_1(2\mu_k)}{\mu_k J_1^2(4\mu_k)}J_0(\mu_k) = f(1) = 1.$$

$$\Rightarrow \sum_{k=1}^{\infty}\frac{J_1(2\mu_k)}{\mu_k J_1^2(4\mu_k)}J_0(\mu_k) = 4$$

3. Since f is discontinuous at $x = 2$, we have

$$\frac{1}{4}\sum_{k=1}^{\infty}\frac{J_1(2\mu_k)}{\mu_k J_1^2(4\mu_k)}J_0(2\mu_k) = \frac{1}{2}\left(f\left(2^-\right) + f\left(2^+\right)\right) = \frac{1}{2}(1+0) = \frac{1}{2}$$

$$\Rightarrow \sum_{k=1}^{\infty}\frac{J_1(2\mu_k)}{\mu_k J_1^2(4\mu_k)}J_0(2\mu_k) = 2$$

Theorem 135 *3*

If $\beta_1 = 0$ in problem (5.29), then

1. For $n = 0$, the eigenvalues of (5.29) are

$$\lambda_k = \mu_k^2 = \left(\frac{\xi_{1k}}{b}\right)^2, \ k \in \mathbb{N}_0$$

where ξ_{1k} are the zeros of J_1, and the corresponding eigenfunctions are

$$u_k(x) = J_0(\mu_k x)$$

2. For $n \in \mathbb{N}$, the eigenvalues are

$$\lambda_k = \mu_k^2 = \left(\frac{\xi_{nk}}{b}\right)^2, \quad k \in \mathbb{N}$$

where μ_k are the solution of

$$J_n^{'}(\mu b) = 0,$$

The corresponding eigenfunctions are

$$u_k(x) = J_n(\mu_k x)$$

3. The set $\{J_n(\mu_k x) : k \in \mathbb{N}\}$ is orthogonal in $\mathcal{L}_x^2(0,b)$. That is,

$$\left\langle J_n(\mu_k x), J_n\left(\mu_j x\right)\right\rangle_x = 0, \quad k \neq j$$

4. The norm of $J_n(\mu_k x)$ is given by

$$\|J_n(\mu_k x)\|_x = \left[\frac{1}{2\mu_k^2}\left(\mu_k^2 b^2 - n^2\right) J_n^2(\mu_k b)\right]^{\frac{1}{2}}$$

5. For any $f \in \mathcal{L}_x^2(0,b)$

$$f(x) = \sum_{k=1}^{\infty} \frac{\langle f(x), J_n(\mu_k x)\rangle_x}{\|J_n(\mu_k x)\|_x^2} J_n(\mu_k x) \qquad ((5.32))$$

6. If f is smooth on $(0,b)$, then the above equality holds pointwise provided f is defined by

$$f(x) = \frac{1}{2}\left[f\left(x^-\right) + f\left(x^+\right)\right]$$

at the points of discontinuity.

Proof:

1. If $\beta_1 = 0$, the boundary condition becomes

$$u'(b) = 0,$$

but,

$$u'(x) = \mu J_n'(\mu x)$$

and thus we have the boundary condition,

$$0 = u'(b) = \mu J_n'(\mu b)$$

(a) Now, or $n = 0$ we know that

$$J_0'(x) = -J_1(x),$$

which leads to get

$$0 = \mu J_0'(\mu b) \Rightarrow -\mu J_1(\mu b) = 0$$

Now, if $\mu = 0$, then $\lambda = 0$ and the corresponding eigenvector

$$u_0(x) = J_0(\mu x) = J_0(0) = 1$$

If $\mu \neq 0$

$$J_1(\mu b) = 0$$

then

$$\mu_k = \frac{\xi_{1k}}{b}$$

where ξ_{1k} are the positive zeros of J_1 and the eigenvalues are

$$u_0(x) = J_0(\mu_k x), \quad k \in \mathbb{N}$$

Summarizing, for $n = 0$, the eigenvalues are given by

$$\lambda_k = \mu_k^2 = \left(\frac{\xi_{1k}}{b}\right)^2,$$

and the corresponding eigenfunctions are given by

$$u_k(x) = J_0(\mu_k x),$$

where $k \in \mathbb{N}_0$.

(b) For $n \in \mathbb{N}$, the boundary condition leads to

$$0 = u'(b) = \mu J_n'(\mu b)$$

now, If $\mu = 0$, then $\lambda = \mu^2 = 0$ is an eigenvalue of $J_n(\mu x)$, and the corresponding eigenfunction is

$$J_n(\mu x) = J_n(0) = 0$$

which cannot be true. Therefore,

$$J_n'(\mu b) = 0$$

which if solved gives the eigenvalues

$$\lambda_k = \mu_k^2$$

and the corresponding eigenfunctions

$$u_k(x) = J_n(\mu_k x).$$

Part VII
The Fourier Transformation

21 The Fourier Transform

We are going to learn

- The space $\mathcal{L}^1(I)$.
- The Fourier transform of a function $f \in \mathcal{L}^1(I)$.
- Properties of the Fourier transform.

Definition 136 *(The Space $\mathcal{L}^1(I)$)*

For any real interval I, we say that $f : I \to \mathbb{C}$ is absolutely integrable on I, and write $f \in \mathcal{L}^1(I)$ if

$$\int_I |f(x)|\, dx < \infty.$$

Remark 137 *1*

1. $\mathcal{L}^1(I)$ is a vector space.

2. If I is bounded, then any integrable function f is in $\mathcal{L}^1(I)$.

3. If I is unbounded, then a function f may be integrable, but not in $\mathcal{L}^1(I)$, for example, take $f(x) = \frac{\sin x}{x}$ on $(0, \infty)$.

Definition 138 *6.1*

For any $f \in \mathcal{L}^1(\mathbb{R})$ we define the *Fourier transform* of f as the function $\hat{f} : \mathbb{R} \to \mathbb{C}$ defined by the improper integral

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-i\ \xi x} dx,$$

In the book, the symbol $\mathcal{F}(f)$ is used instead of $\hat{f}$ to denote the Fourier transform of f.

Example 139 *6.2*

For $a \in \mathbb{R}$, consider the function $f_a : \mathbb{R} \to \mathbb{R}$ defined by

$$f_a(x) = \begin{cases} 1, & |x| \leq a \\ 0, & |x| > a \end{cases}.$$

1. Show that $f_a \in \mathcal{L}^1(\mathbb{R})$.

2. Find the Fourier transform $\hat{f}_a$ of f_a.

3. Find $f(x) = \lim_{a\to\infty} f_a(x)$. Does $f \in \mathcal{L}^1(\mathbb{R})$?

4. Does $\lim_{a\to\infty} \hat{f}_a(\xi)$ exist?

Solution:

1. $f_a \in \mathcal{L}^1(\mathbb{R})$ because

$$\int_{-\infty}^{\infty} |f_a(x)|\, dx = \int_{-a}^{a} 1 dx = x|_{-a}^{a} = 2a < \infty.$$

2. The Fourier transform is given by

$$\begin{aligned}
\hat{f}_a(\xi) &= \int_{-\infty}^{\infty} f_a(x)\, e^{-i\,\xi x} dx \\
&= \int_{-a}^{a} e^{-i\,\xi x} dx \\
&= \left.\frac{e^{-i\,\xi x}}{-i\xi}\right|_{-a}^{a} \\
&= \frac{1}{-i\xi}\left(e^{-i\,\xi a} - e^{i\,\xi a}\right) \\
&= \frac{2}{\xi}\frac{\left(e^{i\,\xi a} - e^{-i\,\xi a}\right)}{2i} \\
&= \frac{2}{\xi}\sin \xi a
\end{aligned}$$

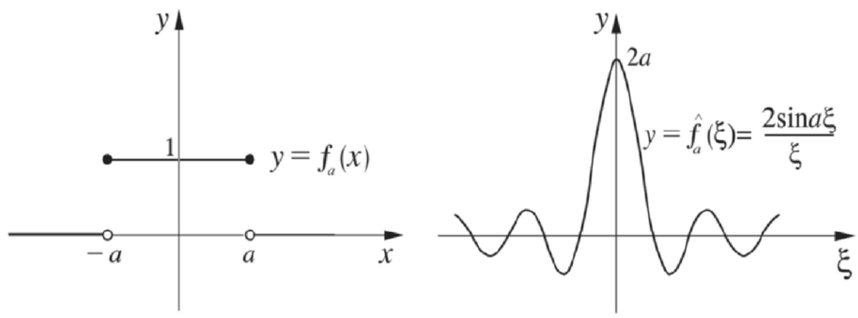


Figure 6.1

3. $f(x) = \lim_{a\to\infty} f_a(x) = 1$, for all $x \in \mathbb{R}$. Now,

$$\int_{-\infty}^{\infty} |f_a(x)|\, dx = \int_{-\infty}^{\infty} 1 dx = \int_{-\infty}^{0} 1 dx + \int_{0}^{\infty} 1 dx$$

but

$$\int_{-\infty}^{0} 1 dx = \lim_{r\to-\infty}\int_{r}^{0} 1 dx = \lim_{r\to-\infty} x|_r^0 = \lim_{r\to-\infty} -r = \infty,$$

Therefore, $\int_{-\infty}^{\infty} |f_a(x)|\, dx$ does not converge. In other words, $f \notin \mathcal{L}^1(\mathbb{R})$.

4.

$$\lim_{a\to\infty} \hat{f}_a(\xi) = \lim_{a\to\infty} \frac{2}{\xi}\sin \xi a = \frac{2}{\xi}\lim_{a\to\infty} \sin \xi a$$

does not exist, because if we take $\xi = \frac{\pi}{2}$ and let $a = 2n+1 \to \infty$ where $n \in \mathbb{N}_0$, then

$$\sin a\xi = \sin(2n+1)\frac{\pi}{2}$$

alternates between -1 and 1 as $n \to \infty$. Since $\hat{f}_a\left(\frac{\pi}{2}\right)$ does not converge as $a \to \infty$, $\hat{f}_a(\xi)$ does not converge.

Example 140 *6.3*

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = e^{-|x|}.$$

1. Show that $f \in \mathcal{L}^1(\mathbb{R})$.
2. Find the Fourier transform $\hat{f}$ of f.

Solution:

1. $f \in \mathcal{L}^1(\mathbb{R})$ because

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(x)|\, dx &= \int_{-\infty}^{\infty} e^{-|x|} dx \\
&= \int_{-\infty}^{0} e^{x} dx + \int_{0}^{\infty} e^{-x} dx \\
&= \lim_{r \to -\infty} \int_{r}^{0} e^{x} dx + \lim_{s \to \infty} \int_{0}^{s} e^{-x} dx \\
&= \lim_{r \to -\infty} e^{x}\Big|_{r}^{0} + \lim_{s \to \infty} -e^{-x}\Big|_{0}^{s} \\
&= \lim_{r \to -\infty} (1 - e^{r}) + \lim_{s \to \infty} \left(-e^{-s} + 1\right) \\
&= 1 + 1 = 2 < \infty
\end{aligned}$$

2. The Fourier transform is given by

$$\begin{aligned}
\hat{f}(\xi) &= \int_{-\infty}^{\infty} f(x)\, e^{-i\,\xi x} dx \\
&= \int_{-\infty}^{\infty} e^{-|x|} e^{-i\,\xi x} dx \\
&= \int_{-\infty}^{0} e^{x} e^{-i\,\xi x} dx + \int_{0}^{\infty} e^{-x} e^{-i\,\xi x} dx \\
&= \lim_{r \to -\infty} \int_{r}^{0} e^{(1-i\,\xi)x} dx + \lim_{s \to \infty} \int_{0}^{s} e^{-(1+i\,\xi)x} dx \\
&= \lim_{r \to -\infty} \frac{e^{(1-i\,\xi)x}}{1 - i\xi}\Bigg|_{r}^{0} + \lim_{s \to \infty} -\frac{e^{-(1+i\,\xi)x}}{1 + i\xi}\Bigg|_{0}^{s} \\
&= \lim_{r \to -\infty} \left(\frac{1}{1 - i\xi} - \frac{e^{(1-i\,\xi)r}}{1 - i\xi}\right) + \lim_{s \to \infty} \left(-\frac{e^{-(1+i\,\xi)s}}{1 + i\xi} + \frac{1}{1 + i\xi}\right) \\
&= \frac{1}{1 - i\xi} + \frac{1}{1 + i\xi} \\
&= \frac{2}{1 + \xi^2}.
\end{aligned}$$

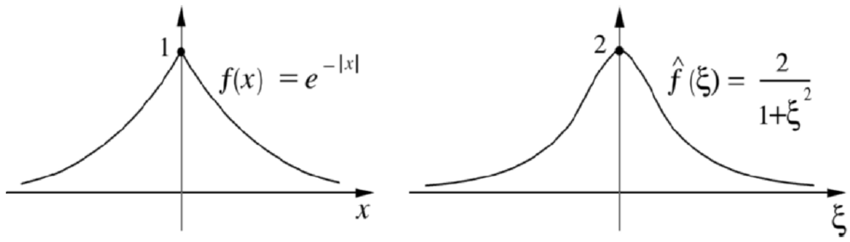


Figure 6.2

21.1 Properties of the Fourier Transform

1. The Fourier transformation $\mathcal{F} : f \to \hat{f}$ is a linear function, that is

$$\mathcal{F}(c_1 f_1 + c_2 f_2) = c_1 \mathcal{F}(f_1) + c_2 \mathcal{F}(f_2)$$

for any $c_1, c_2 \in \mathbb{C}$ and any $f_1, f_2 \in \mathcal{L}^1(\mathbb{R})$ because for any $\xi \in \mathbb{R}$ we have

$$\begin{aligned} \mathcal{F}(c_1 f_1 + c_2 f_2)(\xi) &= \int_{-\infty}^{\infty} (c_1 f_1 + c_2 f_2)(x) e^{-\xi x i} dx \\ &= c_1 \int_{-\infty}^{\infty} f_1(x) e^{-\xi x i} dx + c_2 \int_{-\infty}^{\infty} f_2(x) e^{-\xi x i} dx \\ &= c_1 \mathcal{F}(f_1)(\xi) + c_2 \mathcal{F}(f_2)(\xi) \\ &= (c_1 \mathcal{F}(f_1) + c_2 \mathcal{F}(f_2))(\xi) \end{aligned}$$

2. $\hat{f}$ is a bounded function on $\mathbb{R}$ because

$$\begin{aligned} \left|\hat{f}(\xi)\right| &\leq \int_{-\infty}^{\infty} \left|f(x) e^{-i\,\xi x}\right| dx \\ &= \int_{-\infty}^{\infty} |f(x)|\, dx < \infty \end{aligned}$$

[why?] .

Lemma 141 *6.4*

Let $(f_n : n \in \mathbb{N})$ be a sequence of functions in $\mathcal{L}^1(I)$, where I is a real interval, and suppose that $f_n \to f$ pointwise on I. If there is a positive function $g \in \mathcal{L}^1(I)$ such that

$$|f_n(x)| \leq g(x) \quad \text{for all } x \in I, \quad n \in \mathbb{N}$$

then $f \in \mathcal{L}^1(I)$ and

$$\lim_{n\to\infty} \int_I f_n(x)\, dx = \int_I f(x)\, dx.$$

Theorem 142 *6.6*

For any $f \in \mathcal{L}^1(\mathbb{R})$, the Fourier transform

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-i\,\xi x} dx$$

is a bounded continuous function on $\mathbb{R}$ and

$$\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0.$$

Proof:

We will only prove the first part of the theorem, namely that the Fourier transform is continuous.

Let ξ be any real number and let (ξ_n) be a sequence such that

$$\lim_{n \to \infty} \xi_n = \xi,$$

To prove that $\hat{f}$ is continuous, we must prove that

$$\lim_{n \to \infty} \hat{f}(\xi_n) = \hat{f}(\xi).$$

Now,

$$\begin{aligned} \left|\hat{f}(\xi_n) - \hat{f}(\xi)\right| &= \left|\int_{-\infty}^{\infty} f(x)\, e^{-i\,\xi_n x} dx - \int_{-\infty}^{\infty} f(x)\, e^{-i\,\xi x} dx\right| \quad ((6.8)) \\ &= \left|\int_{-\infty}^{\infty} f(x) \left[e^{-i\,\xi_n x} - e^{-i\,\xi x}\right] dx\right| \\ &\leq \int_{-\infty}^{\infty} |f(x)| \left|e^{-i\,\xi_n x} - e^{-i\,\xi x}\right| dx \end{aligned}$$

If we take the sequence

$$g_n(x) = \left|f(x)\left(e^{-i\,\xi_n x} - e^{-i\,\xi x}\right)\right|$$

then, $(g_n : n \in \mathbb{N})$ satisfies:

1. $g_n \in \mathcal{L}^1(\mathbb{R})$ for all $n \in \mathbb{N}$ since

$$\begin{aligned} \int_{-\infty}^{\infty} |g_n(x)|\, dx &= \int_{-\infty}^{\infty} |f(x)| \left|e^{-i\,\xi_n x} - e^{-i\,\xi x}\right| dx \\ &\leq 2\int_{-\infty}^{\infty} |f(x)|\, dx < \infty \end{aligned}$$

2. $$\lim_{n \to \infty} g_n(x) = \lim_{n \to \infty} |f(x)| \left|e^{-i\,\xi_n x} - e^{-i\,\xi x}\right| = 0$$

   [why?] .

3.

$$\begin{aligned} |g_n(x)| &= |f(x)| \left| e^{-i\,\xi_n x} - e^{-i\,\xi x} \right| \\ &\leq 2\,|f(x)| \end{aligned}$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

Taking the limit of both sides in equation (6.8) and using lemma 6.4 gives

$$\begin{aligned} \lim_{n\to\infty} \left| \hat{f}(\xi_n) - \hat{f}(\xi) \right| &\leq \lim_{n\to\infty} \int_{-\infty}^{\infty} |f(x)| \left| e^{-i\,\xi_n x} - e^{-i\,\xi x} \right| dx \\ &= \int_{-\infty}^{\infty} \lim_{n\to\infty} \left[|f(x)| \left| e^{-i\,\xi_n x} - e^{-i\,\xi x} \right| \right] dx \\ &= 0 \end{aligned}$$

That is,

$$\lim_{n\to\infty} \hat{f}(\xi_n) = \hat{f}(\xi).$$

22 The Fourier Integral

We are going to learn

- The Fourier Integral of a function $f \in \mathcal{L}^1(I)$.
- The Fundamental Theorem of Fourier integral.
- The Integral in Trigonometric Form.

Recall that any function $f \in \mathcal{L}^2(-\pi, \pi)$ can be represented by the Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx, \quad n \in \mathbb{Z}.$$

Analogous to the Fourier series representation, a function $f \in \mathcal{L}^1(\mathbb{R})$ can be represented by the integral

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} d\xi,$$

where $\hat{f}$ is the Fourier transform of f and is given by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\,\xi x} dx$$

Theorem 143 *6.10*

Let f be a piecewise smooth function in $\mathcal{L}^1(\mathbb{R})$. If

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\,\xi x} dx, \qquad \xi \in \mathbb{R}$$

then

$$\lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} \hat{f}(\xi) e^{ix\xi} d\xi = \frac{1}{2} \left[f\left(x^+\right) + f\left(x^-\right) \right],$$

for all $x \in \mathbb{R}$.

Remark 144 *6.11*

If f is defined by

$$f(x) = \frac{1}{2} \left[f\left(x^+\right) + f\left(x^-\right) \right],$$

at every point of discontinuity x, then

$$\lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} \hat{f}(\xi) e^{ix\xi} d\xi$$

is called the *Fourier integral* of f or the *inverse Fourier transform* of $\hat{f}$ and we therefore write

$$f(x) = \lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} \hat{f}(\xi) e^{ix\xi} d\xi = \mathcal{F}^{-1}\left(\hat{f}\right)(x).$$

22.1 Fourier Integral in Trigonometric Form

Let $f \in \mathcal{L}^1(\mathbb{R})$ be a real, piecewise smooth function satisfying

$$f(x) = \frac{1}{2}\left[f\left(x^+\right) + f\left(x^-\right)\right],$$

at each point of discontinuity, then the Fourier integral representation of f can be written in the form

$$f(x) = \frac{1}{\pi}\int_0^\infty [A(\xi)\cos x\xi + B(\xi)\sin x\xi]\, d\xi, \quad x \in \mathbb{R}$$

where

$$\begin{aligned} A(\xi) &= \int_{-\infty}^\infty f(x)\cos\xi x dx, \\ B(\xi) &= \int_{-\infty}^\infty f(x)\sin\xi x dx, \quad \xi \in \mathbb{R}. \end{aligned}$$

1. If f is an even function, then $B(\xi) = 0$, and the integral form becomes

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_0^\infty A(\xi)\cos x\xi d\xi, \\ A(\xi) &= 2\int_0^\infty f(x)\cos\xi x dx, \end{aligned}$$

2. If f is an odd function, then $A(\xi) = 0$, and the integral form becomes

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_0^\infty B(\xi)\sin x\xi d\xi, \\ B(\xi) &= 2\int_0^\infty f(x)\sin\xi x dx, \end{aligned}$$

Example 145 *6.12*

Consider the function $f_a : \mathbb{R} \to \mathbb{R}$ defined by

$$f_a(x) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}.$$

where $a \in \mathbb{R}$. Use the Fourier integral theorem to find the value of $\frac{2}{\pi}\int_0^\infty \frac{1}{\xi}\sin a\xi\cos x\xi d\xi$ at every $x \in \mathbb{R}$

Solution:

We already know from example 6.2 that $f_a \in \mathcal{L}^1(\mathbb{R})$. Moreover, f is clearly a piecewise smooth function. Thus, Theorem 6.10 holds, that is,

$$\lim_{L\to\infty}\frac{1}{2\pi}\int_{-L}^L \hat{f}_a(\xi)\, e^{ix\xi}d\xi = \frac{1}{2}\left[f_a\left(x^+\right) + f_a\left(x^-\right)\right]$$

From example 6.2, the Fourier transform of f is given by

$$\hat{f}_a(\xi) = \frac{2}{\xi}\sin a\xi$$

and thus we have,

$$\begin{aligned}
\lim_{L\to\infty}\frac{1}{2\pi}\int_{-L}^{L}\hat{f}_a(\xi)\,e^{ix\xi}d\xi &= \lim_{L\to\infty}\frac{1}{2\pi}\int_{-L}^{L}\frac{2}{\xi}\sin(a\xi)\,e^{ix\xi}d\xi \\
&= \lim_{L\to\infty}\frac{1}{\pi}\int_{-L}^{L}\frac{1}{\xi}\sin a\xi\,[\cos x\xi + i\sin x\xi]\,d\xi \\
&= \lim_{L\to\infty}\frac{2}{\pi}\int_{0}^{L}\frac{1}{\xi}\sin a\xi\cos x\xi d\xi \\
&= \frac{2}{\pi}\int_{0}^{\infty}\frac{1}{\xi}\sin a\xi\cos x\xi d\xi
\end{aligned}$$

[why?].

Therefore,

$$\begin{aligned}
\frac{2}{\pi}\int_{0}^{\infty}\frac{1}{\xi}\sin a\xi\cos x\xi d\xi &= \frac{1}{2}\left[f_a\left(x^{+}\right) + f_a\left(x^{-}\right)\right] \\
&= \begin{cases} 0, & x < -a \\ \frac{1}{2}, & x = a \\ 1, & -a < x < a \\ \frac{1}{2}, & x = a \\ 0, & x > a \end{cases}.
\end{aligned}$$

Example 146 *6.13*

Find Fourier integral representation of the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin x, & |x| < \pi \\ 0, & |x| > \pi \end{cases}$$

Solution:

Note that

$$\int_{-\infty}^{\infty}|f(x)|\,dx = \int_{-\pi}^{\pi}|\sin x|\,dx < \infty$$

that is $f \in \mathcal{L}^1(\mathbb{R})$. Theorem 6.10 holds because f is clearly a piecewise smooth function. Moreover, since f is an odd continuous function, we have

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}B(\xi)\sin x\xi d\xi$$

where

$$
\begin{aligned}
B(\xi) &= 2\int_0^\infty f(x)\sin\xi x dx \\
&= 2\int_0^\pi \sin x \sin\xi x dx \\
&= 2\int_0^\pi \frac{1}{2}\left[\cos(1-\xi)x - \cos(1+\xi)x\right]dx \\
&= \left.\frac{\sin(1-\xi)x}{1-\xi} - \frac{\sin(1+\xi)x}{1+\xi}\right|_0^\pi \\
&= \frac{\sin(1-\xi)\pi}{1-\xi} - \frac{\sin(1+\xi)\pi}{1+\xi} \\
&= \frac{(1+\xi)\sin(1-\xi)\pi - (1-\xi)\sin(1+\xi)\pi}{1-\xi^2} \\
&= \frac{(1+\xi)\left[\sin\pi\cos\xi\pi - \sin\xi\pi\cos\pi\right] - (1-\xi)\left[\sin\pi\cos\xi\pi + \sin\xi\pi\cos\pi\right]}{1-\xi^2} \\
&= \frac{(1+\xi)\left[\sin\xi\pi\right] + (1-\xi)\left[\sin\xi\pi\right]}{1-\xi^2} \\
&= \frac{2\sin\xi\pi}{1-\xi^2}
\end{aligned}
$$

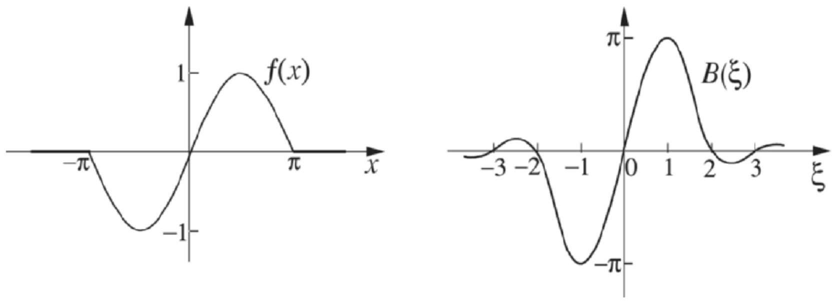


Figure 6.4 f and its sine transform.

Example 147 *6.14*

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = e^{-|x|}$$

1. Fine the Fourier integral representation of f.
2. Deduce that

$$\int_0^\infty \frac{1}{1+\xi^2}d\xi = \frac{\pi}{2}.$$

Solution:

1. We already know from example 6.3 that $f \in \mathcal{L}^1(\mathbb{R})$. Moreover, f is clearly a piecewise smooth function. Thus, Theorem 6.10 holds, that is,

$$\lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} \hat{f}(\xi) e^{ix\xi} d\xi = \frac{1}{2}\left[f\left(x^+\right) + f\left(x^-\right)\right]$$

From example 6.3, the Fourier transform of f is given by

$$\hat{f}(\xi) = \frac{2}{1+\xi^2}$$

and thus we have

$$\begin{aligned} \lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} \hat{f}(\xi) e^{ix\xi} d\xi &= \lim_{L\to\infty} \frac{1}{\pi} \int_{-L}^{L} \frac{1}{1+\xi^2} [\cos x\xi + i \sin x\xi] \, d\xi \\ &= \lim_{L\to\infty} \frac{2}{\pi} \int_{0}^{L} \frac{1}{1+\xi^2} \cos x\xi d\xi \\ &= \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{1+\xi^2} \cos x\xi d\xi \end{aligned}$$

but f is continuous, that is for all $x \in \mathbb{R}$,

$$f(x) = \frac{1}{2}\left[f\left(x^+\right) + f\left(x^-\right)\right]$$

Therefore, we have

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{1+\xi^2} \cos x\xi d\xi, \quad x \in \mathbb{R}$$

2. Using the above equation at $x = 0$, we get

$$f(0) = \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{1+\xi^2} \cos 0 d\xi$$

but $f(0) = e^0 = 1$, and hence we have

$$1 = \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{1+\xi^2} d\xi$$

or

$$\int_{0}^{\infty} \frac{1}{1+\xi^2} d\xi = \frac{\pi}{2}.$$

23 Properties and Applications

We are going to learn about the properties of the Fourier transformations under differentiation.

Theorem 148 *6.15*

Let $f \in \mathcal{L}^1(\mathbb{R})$

1. $f' \in \mathcal{L}^1(\mathbb{R})$ and f is continuous on $\mathbb{R}$, then

$$\mathcal{F}(f')(\xi) = i\xi\mathcal{F}(f)(\xi), \qquad \xi \in \mathbb{R}$$

2. If $xf(x) \in \mathcal{L}^1(\mathbb{R})$, then $\mathcal{F}(f)$ is differentiable and its derivative

$$\frac{d}{d\xi}\mathcal{F}(f)(\xi) = \mathcal{F}(-ixf)(\xi), \qquad \xi \in \mathbb{R}$$

is continuous on $\mathbb{R}$.

Corollary 149 *6.16*

Suppose $f \in \mathcal{L}^1(\mathbb{R})$ and $n \in \mathbb{N}$, then

1. $f^{(k)} \in \mathcal{L}^1(\mathbb{R})$ for $1 \leq k \leq n$, and $f^{(n-1)}$ is continuous on $\mathbb{R}$, then

$$\mathcal{F}\left(f^{(n)}\right)(\xi) = (i\xi)^n \mathcal{F}(f)(\xi), \qquad \xi \in \mathbb{R}$$

2. If $x^n f(x) \in \mathcal{L}^1(\mathbb{R})$, then $\mathcal{F}(f)$ is differentiable and its derivative

$$\frac{d^n}{d\xi^n}\mathcal{F}(f)(\xi) = \mathcal{F}((-ix)^n f)(\xi), \qquad \xi \in \mathbb{R}$$

Example 150 *6.17*

Consider the function

$$f(x) = e^{-x^2}, \qquad x \in \mathbb{R}.$$

1. Show that $f \in \mathcal{L}^1(\mathbb{R})$ and $xf(x) \in \mathcal{L}^1(\mathbb{R})$.
2. Find the derivative of the Fourier transform $\hat{f}$ of f.
3. Find a closed form of the Fourier transform $\hat{f}$.

Solution:

1.
$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} < \infty$$

$$\begin{aligned}
\int_{-\infty}^{\infty} |x| e^{-x^2} dx &= \int_{-\infty}^{0} -xe^{-x^2} dx + \int_{0}^{\infty} xe^{-x^2} dx \\
&= \lim_{r\to-\infty} \int_{r}^{0} -xe^{-x^2} dx + \lim_{r\to\infty} \int_{0}^{r} xe^{-x^2} dx \\
&= \lim_{r\to-\infty} \frac{1}{2} e^{-x^2}\Big|_{r}^{0} + \lim_{r\to\infty} -\frac{1}{2} e^{-x^2}\Big|_{0}^{r} \\
&= \frac{1}{2} \lim_{r\to-\infty} \left[1 - e^{-r^2}\right] + -\frac{1}{2} \lim_{r\to\infty} \left[e^{-r^2} - 1\right] \\
&= 1 < \infty
\end{aligned}$$

2. From 1, we see that both f and xf are in $\mathcal{L}^1(\mathbb{R})$, and therefore we have

$$\begin{aligned}
\frac{d}{d\xi} \hat{f}(\xi) &= \mathcal{F}(-ixf)(\xi) \\
&= \int_{-\infty}^{\infty} -ixf(x) e^{-i\,\xi x} dx \\
&= -i \int_{-\infty}^{\infty} xe^{-x^2} e^{-i\,\xi x} dx \\
&= \frac{i}{2} \int_{-\infty}^{\infty} -2xe^{-x^2} e^{-i\,\xi x} dx \\
&= \frac{i}{2} \int_{-\infty}^{\infty} \frac{d}{dx}\left(e^{-x^2}\right) e^{-i\,\xi x} dx \\
&= \frac{i}{2} \left[\lim_{r\to\infty} e^{-x(x+i\,\xi)}\Big|_{-r}^{r} + i\xi \int_{-\infty}^{\infty} e^{-x^2} e^{-i\,\xi x} dx\right] \\
&= -\frac{\xi}{2} \int_{-\infty}^{\infty} e^{-x^2} e^{-i\,\xi x} dx \\
&= -\frac{\xi}{2} \hat{f}(\xi)
\end{aligned}$$

3. We have
$$\frac{d}{d\xi} \hat{f}(\xi) + \frac{\xi}{2} \hat{f}(\xi) = 0$$

Multiplying the above equation by the integrating factor $\exp\left(\frac{\xi^2}{4}\right)$ gives
$$\frac{d}{d\xi}\left(e^{\frac{\xi^2}{4}} \hat{f}(\xi)\right) = 0$$

which have the solution
$$e^{\frac{\xi^2}{4}} \hat{f}(\xi) = c$$

where c is a constant. In other words, we have

$$\hat{f}(\xi) = ce^{-\frac{\xi^2}{4}}$$

but

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-x^2} e^{-i\ \xi x} dx$$

Therefore, at $\xi = 0$, we have

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2} dx = \hat{f}(0) = c$$

Using the value of c, we have the following closed form of $\hat{f}(\xi)$

$$\hat{f}(\xi) = \sqrt{\pi} e^{-\frac{\xi^2}{4}}.$$

24 Heat Transfer in an Infinite Bar

Consider an infinite thin bar with an initial temperature distribution

$$u(x,0) = f(x), \quad x \in \mathbb{R}$$

where f is a piecewise smooth function in $\mathcal{L}^1(\mathbb{R})$. We are interested in finding the temperature distribution $u(x,t)$ along the bar at time $t > 0$.

To find the temperature function $u(x,t)$, we need to solve the heat equation

$$u_t = ku_{xx}, \quad x \in \mathbb{R},\ t > 0,$$

subject to the initial condition

$$u(x,0) = f(x), \quad x \in \mathbb{R}.$$

Using the method of separation of variables, we assume that

$$u(x,t) = v(x)\, w(t),$$

which if substituted in the heat equation gives

$$\frac{v''}{v} = \frac{1}{k}\frac{w'}{w}$$

and hence we have a constant $-\lambda^2$ such that

$$\begin{aligned} v'' + \lambda^2 v &= 0, \\ w' + \lambda^2 k w &= 0. \end{aligned}$$

The solution to the above equations are given by

$$\begin{aligned} v(x) &= A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x, \\ w(t) &= C(\lambda)\, e^{-\lambda^2 kt}, \end{aligned}$$

where A, B, C are the constants of integration and are function of λ.

The solution of the heat equation corresponding to $\lambda \in \mathbb{R}$ thus becomes

$$u_\lambda(x,t) = [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\, e^{-\lambda^2 kt},$$

where we have assumed that $C(\lambda) = 1$.

The general solution of the heat equation results from taking the integral of $u_\lambda(x,t)$ with respect to λ over $\mathbb{R}$, which gives

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} u_\lambda(x,t)\, d\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\, e^{-\lambda^2 kt} d\lambda \end{aligned}$$

Using the initial condition, we have

$$f(x) = u(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\, d\lambda$$

that is, $A(\lambda)$ and $B(\lambda)$ are the Fourier cosine and sine transform of f, and are therefore given by

$$\begin{aligned} A(\lambda) &= \int_{-\infty}^{\infty} f(y)\cos\lambda y dy, \\ B(\lambda) &= \int_{-\infty}^{\infty} f(y)\sin\lambda y dy. \end{aligned}$$

Substituting in the solution of the heat equation gives

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\, e^{-\lambda^2 kt} d\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(y)\cos\lambda y dy\cos\lambda x + \int_{-\infty}^{\infty} f(y)\sin\lambda y dy\sin\lambda x\right] e^{-\lambda^2 kt} d\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)[\cos\lambda y\cos\lambda x + \sin\lambda y\sin\lambda x]\, e^{-\lambda^2 kt} dy d\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)\cos[\lambda(x-y)]\, e^{-\lambda^2 kt} dy d\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)\cos[\lambda(x-y)]\, e^{-\lambda^2 kt} dy d\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\int_{-\infty}^{\infty}\cos[\lambda(x-y)]\, e^{-\lambda^2 kt} d\lambda dy \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\int_{0}^{\infty}\cos[\lambda(x-y)]\, e^{-\lambda^2 kt} d\lambda dy \end{aligned}$$

but

$$\int_0^{\infty}\cos\lambda z e^{-c\lambda^2} d\lambda = \frac{1}{2}\sqrt{\frac{\pi}{c}} e^{-z^2/4c} \qquad \text{for all } x\in\mathbb{R},\quad c>0,$$

therefore,

$$\begin{aligned} u(x,t) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\int_{0}^{\infty}\cos[\lambda(x-y)]\, e^{-\lambda^2 kt} d\lambda dy \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\left(\frac{1}{2}\sqrt{\frac{\pi}{kt}} e^{-(x-y)^2/4kt}\right) dy \\ &= \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} f(y)\, e^{-(x-y)^2/4kt} dy \end{aligned}$$

25 Heat Transfer in an a Semi-Infinite Bar

Consider a semi-infinite bar that is insulated at one end, and suppose that the initial temperature distribution along the bar length is known. To find the temperature distribution $u(x,t)$ along the bar at time $t > 0$, we need to solve the boundary-value problem

$$u_t = ku_{xx}, \quad x \in (0,\infty), \ t > 0,$$

subject to the conditions

$$\begin{aligned} u(x,0) &= f(x), \quad x \in (0,\infty), \\ u_x(x,0) &= 0, \quad t > 0. \end{aligned}$$

We already know that the solution to the heat equation is given by

$$u_\lambda(x,t) = [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\, e^{-\lambda^2 kt},$$

where $\lambda \in \mathbb{R}$. We can assume, without loss of generality, that $\lambda \geq 0$ [why?].

Using the boundary condition, we have

$$\begin{aligned} 0 &= \frac{\partial u_\lambda}{\partial x}(0,t) \\ &= [-A(\lambda)\sin\lambda(0) + B(\lambda)\cos\lambda(0)]\,\lambda e^{-\lambda^2 kt} \\ &= \lambda B(\lambda)\, e^{-\lambda^2 kt} \end{aligned}$$

So, we have two cases:

1. if $\lambda = 0$, then the solution is

   $$u_0(x,t) = A(0)$$

   i.e. constant.

2. If $\lambda \neq 0$, then $B(\lambda) = 0$ and

   $$u_\lambda(x,t) = A(\lambda)\cos\lambda x e^{-\lambda^2 kt}$$

Integrating the above solution over all $\lambda > 0$ gives the general solution,

$$u(x,t) = \frac{1}{\pi}\int_0^\infty A(\lambda)\cos\lambda x e^{-\lambda^2 kt} d\lambda$$

Applying the initial condition gives

$$\begin{aligned} f(x) &= u(x,0) \\ &= \frac{1}{\pi}\int_0^\infty A(\lambda)\cos\lambda x d\lambda \end{aligned}$$

which is the cosine Fourier transform of the even extension of the piecewise smooth function $f \in \mathcal{L}^1(0,\infty)$ to $(-\infty,\infty)$. That is,

$$A(\lambda) = 2\int_0^\infty f(y)\cos\lambda y dy.$$

Substituting in the solution gives

$$\begin{aligned} u(x,t) &= \frac{1}{\pi}\int_0^\infty 2\int_0^\infty f(y)\cos\lambda y dy \cos\lambda x e^{-\lambda^2 kt} d\lambda \\ &= \frac{2}{\pi}\int_0^\infty \int_0^\infty f(y)\cos\lambda y\cos\lambda x e^{-\lambda^2 kt} dy d\lambda \\ &= \frac{1}{\pi}\int_0^\infty f(y)\int_0^\infty 2\cos\lambda y\cos\lambda x e^{-\lambda^2 kt} d\lambda dy \end{aligned}$$

but

$$2\cos\lambda y\cos\lambda x = \cos\lambda(y+x) + \cos\lambda(y-x)$$

and hence we have

$$\begin{aligned} u(x,t) &= \frac{1}{\pi}\int_0^\infty f(y)\int_0^\infty \left[\cos\lambda(y+x)+\cos\lambda(y-x)\right] e^{-\lambda^2 kt} d\lambda dy \\ &= \frac{1}{\pi}\int_0^\infty f(y)\int_0^\infty \cos\lambda(y+x) e^{-\lambda^2 kt} d\lambda dy + \frac{1}{\pi}\int_0^\infty f(y)\int_0^\infty \cos\lambda(y-x) e^{-\lambda^2 kt} d\lambda dy \\ &= \frac{1}{2\sqrt{\pi kt}}\int_0^\infty f(y) e^{-(x+y)^2/4kt} dy + \frac{1}{2\sqrt{\pi kt}}\int_0^\infty f(y) e^{-(y-x)^2/4kt} dy \\ &= \frac{1}{2\sqrt{\pi kt}}\int_0^\infty f(y)\left[e^{-(x+y)^2/4kt} + e^{-(y-x)^2/4kt}\right] dy \end{aligned}$$

If we take,

$$f(y) = \begin{cases} 1, & 0 < y < a \\ 0, & y > a \end{cases}$$

for some $a > 0$, then we have

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi kt}}\int_0^\infty f(y)\left[e^{-(x+y)^2/4kt} + e^{-(y-x)^2/4kt}\right] dy \\ &= \frac{1}{2\sqrt{\pi kt}}\int_0^a \left[e^{-(x+y)^2/4kt} + e^{-(y-x)^2/4kt}\right] dy \end{aligned}$$

Since the error function is given by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-p^2} dp$$

we write

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}}\left[\int_0^a e^{-(x+y)^2/4kt} dy + \int_0^a e^{-(y-x)^2/4kt} dy\right]$$

and use the substitutions

$$\begin{aligned} p &= \frac{x+y}{2\sqrt{kt}}, \\ q &= \frac{y-x}{2\sqrt{kt}} \end{aligned}$$

in the first and second integrals, respectively which gives

$$\begin{aligned} dp &= \frac{1}{2\sqrt{kt}}dy, \\ dq &= \frac{1}{2\sqrt{kt}}dy, \end{aligned}$$

or

$$dy = 2\sqrt{kt}dp = 2\sqrt{kt}dq$$

and hence

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi kt}}\left[\int_{\frac{x}{2\sqrt{kt}}}^{\frac{x+a}{2\sqrt{kt}}} e^{-p^2}2\sqrt{kt}dp + \int_{\frac{-x}{2\sqrt{kt}}}^{\frac{a-x}{2\sqrt{kt}}} e^{-q^2}2\sqrt{kt}dq\right] \\ &= \frac{1}{\sqrt{\pi}}\left[\int_{\frac{x}{2\sqrt{kt}}}^{\frac{x+a}{2\sqrt{kt}}} e^{-p^2}dp + \int_{\frac{-x}{2\sqrt{kt}}}^{\frac{a-x}{2\sqrt{kt}}} e^{-q^2}dq\right] \\ &= \frac{1}{\sqrt{\pi}}\left[\int_{\frac{x}{2\sqrt{kt}}}^{\frac{x+a}{2\sqrt{kt}}} e^{-z^2}dz + \int_{\frac{-x}{2\sqrt{kt}}}^{\frac{a-x}{2\sqrt{kt}}} e^{-z^2}dz\right] \\ &= \frac{1}{\sqrt{\pi}}\left[\int_{0}^{\frac{x+a}{2\sqrt{kt}}} e^{-z^2}dz + \int_{0}^{\frac{a-x}{2\sqrt{kt}}} e^{-z^2}dz\right] \quad \text{[why?]} \\ &= \frac{1}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\frac{x+a}{2\sqrt{kt}}\right) + \frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\frac{a-x}{2\sqrt{kt}}\right)\right] \\ &= \frac{1}{2}\operatorname{erf}\left(\frac{a+x}{2\sqrt{kt}}\right) + \frac{1}{2}\operatorname{erf}\left(\frac{a-x}{2\sqrt{kt}}\right). \end{aligned}$$

Summarizing, when the initial temperature along a semi-infinite bar that is insulated at one end is given

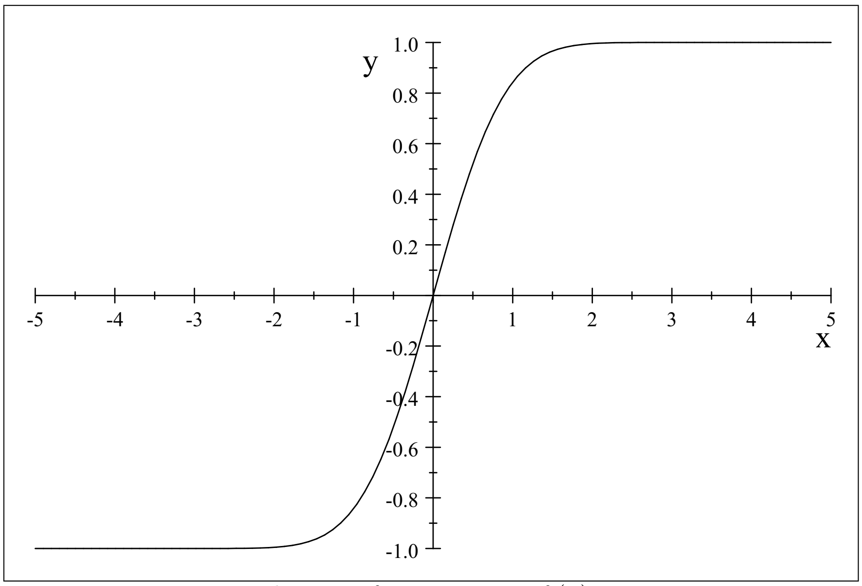
$$f(y) = \begin{cases} 1, & 0 < y < a \\ 0, & y > a \end{cases}$$

for some point a along the bar, then the temperature at at a point x along the bar at time $t > 0$ is given by the function

$$u(x,t) = \frac{1}{2}\operatorname{erf}\left(\frac{a+x}{2\sqrt{kt}}\right) + \frac{1}{2}\operatorname{erf}\left(\frac{a-x}{2\sqrt{kt}}\right).$$

Questions:

1. What is the temperature far from the point a at any time $t > 0$.
2. What is the temperature at the insulated side at any time $t > 0$.
3. What is the temperature distribution along the bar as $t \to \infty$.



The error function $y = \text{erf}\,(x)$

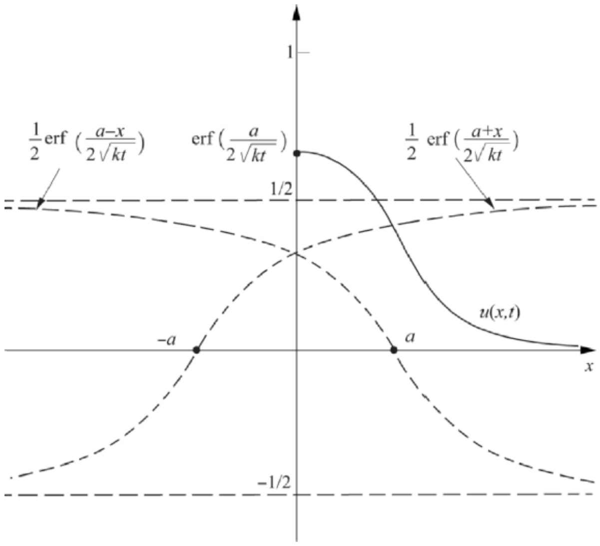


Figure 6.5 Temperature distribution on a semi-infinite rod.